

On the enumeration of partitions with summands in arithmetic progression

M. A. NYBLOM C. EVANS

*Department of Mathematics and Statistics
RMIT University, GPO Box 2476V
Melbourne, Victoria 3001
Australia*

Abstract

Enumerating formulae are constructed which count the number of partitions of a positive integer into positive summands in arithmetic progression with common difference D . These enumerating formulae (denoted $p_D(n)$) which are given in terms of elementary divisor functions together with auxiliary arithmetic functions (to be defined) are then used to establish a known characterisation for an integer to possess a partition of the form in question.

1 Introduction

In recent times there has been some interest in the problem of representing a positive integer as the sum of at least two consecutive terms of an arithmetic progression of positive integers with a prescribed common difference. It is known ([2], [3, p. 85], [4]) that the number n can be expressed as a sum of consecutive positive integers provided it is not a power of 2 and that the number of such representations is one less than the number of odd divisors of n . A more general result in this direction has been found ([1]) which gives a necessary and sufficient condition for a positive integer to possess a partition with summands in arithmetic progression. If $n = 2^h s$ with s odd, and $n > 1$, then n is the sum of positive integers in arithmetic progression with common difference D if and only if

- (1) when D is odd, n is not a power of 2 and either $s > D(2^{h+1} - 1)$ or $n > \frac{1}{2}Dp(p-1)$ where p is the smallest odd prime factor of n ;
- (2) when D is even, either n is even and $n > D$ or n is odd and $n > \frac{1}{2}Dp(p-1)$ where again p is the smallest odd prime factor of n .

In this paper we will show how the above characterisation can, for $D > 2$, be derived as a corollary of two new formulae which count the number of partitions of the desired type and which depend on the parity of D . These enumerating functions, denoted $p_D(n)$, like those of Jacobi for representations of a number as the sum of two, four, six or eight squares, are given in terms of elementary divisor functions, but together with auxiliary arithmetic functions, $f(n)$ and $g(n)$, which are defined later. Although these latter functions do not possess a closed form expression for general n , we are able to find specific conditions under which $f(n)$, $g(n)$ assume the value 0, thereby allowing closed form expressions for $p_D(n)$ in those instances. Before deriving these enumerating functions in §3 we will, for completeness, determine in §2 a closed form expression for $p_2(n)$. Indeed, we shall show that

$$p_2(n) = \frac{1}{2} \left(d(n) - 2 + \frac{(-1)^{d(n)+1} + 1}{2} \right), \quad (1)$$

where $d(n)$ is the number of divisors of n . In addition, as a consequence of (1), we shall derive an enumerating function for the number of representations of n as a difference of two squares.

2 Partition formula for $D = 2$

In what follows $d_i(n)$ denotes the number of divisors d of n with $d \equiv i \pmod{2}$, that is, $d_0(n)$ and $d_1(n)$ are the number of even and odd divisors of n respectively, and $d(n) = d_0(n) + d_1(n)$ is the total number of divisors of n . In addition, let $\mathbb{N}$ denote the set of non-negative integers. We proceed now to establish a closed form expression for $p_2(n)$ via the use of generating functions.

Theorem 2.1 *For any integer $n > 1$, the number of partitions of n with summands in arithmetic progression having common difference 2 is given by*

$$p_2(n) = \frac{1}{2} \left(d(n) - 2 + \frac{(-1)^{d(n)+1} + 1}{2} \right). \quad (2)$$

Proof: Recall that

$$a + (a + 2) + \cdots + (a + 2(n - 1)) = n(n + a - 1)$$

and for the partitions in question $a, n \in \mathbb{N}$ with $a \geq 1$ and $n \geq 2$. Thus we see that the generating function of $p_2(n)$ is given by

$$f(q) = \sum_{n=2}^{\infty} p_2(n) q^n = \sum_{n=2}^{\infty} \frac{q^{n^2}}{1 - q^n} = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} q^{n(n+k)}.$$

It follows that $p_2(N)$ is the number of representations of $N = n(n + k)$ with $n \geq 2$ and $k \geq 0$. As $n \geq 2$ and $n + k \geq n$ our task is reduced to determining the number

of divisors d of N such that $d \neq 1, N$ and $d \leq \frac{N}{d}$. If N is not a square then $d(N)$ is even. Excluding the divisors $1, N$ we see after grouping the remaining $d(N) - 2$ divisors into pairs of the form $(d, \frac{N}{d})$ that there are precisely $(d(N) - 2)/2$ divisors that satisfy the above condition. On the other hand if N is a square then $d(N)$ is odd. After excluding the divisors $1, \sqrt{N}, N$ and pairing, we see that there are $(d(N) - 3)/2$ divisors d with $d < \frac{N}{d}$, and, including $\sqrt{N}$, there are $(d(N) - 1)/2$ divisors d with $d \neq 1, N$ and $d \leq \frac{N}{d}$. So in either case,

$$p_D(N) = \frac{1}{2}(d(N) - 2 + \frac{(-1)^{d(N)+1} + 1}{2}).$$

■

Corollary 2.1 *An integer $n > 1$ is representable as a sum of positive integers in arithmetic progression with common difference 2 if and only if n is not prime.*

Proof: For prime p , $d(p) = 2$, so $p_2(p) = 0$. Conversely, if $p_2(n) = 0$ then

$$d(n) + \frac{(-1)^{d(n)+1} + 1}{2} = 2.$$

However if $n > 1$, $d(n) \geq 2$, so the only solution to the above equation is $d(n) = 2$, and n is prime. ■

We now examine an unexpected consequence of Theorem 2.1.

Corollary 2.2 *The number $s(n)$ of representations of an integer $n > 1$, as a difference of squares of two non-negative integers is given by*

$$s(n) = \frac{1}{2} \left(d_0(n) + (-1)^{n+1} d_1(n) + \frac{(-1)^{d(n)+1} + 1}{2} \right). \quad (3)$$

Proof: We begin by making the simple observation that the partitions of n counted by $p_2(n)$ have summands that are either all odd or all even. If we denote by $\phi(n)$, $\sigma(n)$ the number of partitions with consecutive even and odd summands respectively we have

$$p_2(n) = \phi(n) + \sigma(n).$$

Now for $n > 2$ and even, there are $p_1(\frac{n}{2}) = d_1(\frac{n}{2}) - 1 = d_1(n) - 1$ partitions of $\frac{n}{2}$ of the form $\frac{n}{2} = \sum_{r=m}^p r$ with $p > m$. Consequently there are $d_1(n) - 1$ partitions of n of the form $n = \sum_{r=m}^p 2r$, and so $\phi(n) = d_1(n) - 1$. Of course, when n is odd, $\phi(n) = 0$ so

$$\phi(n) = \frac{(-1)^n + 1}{2} (d_1(n) - 1).$$

Thus from the decomposition of $p_2(n)$ above and (2) we find

$$\begin{aligned} \sigma(n) &= \frac{1}{2} \left(d(n) - 2 + \frac{(-1)^{d(n)+1} + 1}{2} \right) - \frac{(-1)^n + 1}{2} (d_1(n) - 1) \\ &= \frac{1}{2} \left(d_0(n) + (-1)^{n+1} d_1(n) + \frac{(-1)^{d(n)+1} + 1}{2} \right) + \frac{(-1)^n - 1}{2}, \end{aligned} \quad (4)$$

where we have made use of the fact that $d(n) = d_0(n) + d_1(n)$. Recalling that n^2 is equal to the sum of the first n consecutive odd integers, it is clear that each partition counted by $\sigma(n)$ corresponds to a unique representation of n in the form $x^2 - y^2$ with $x, y \in \mathbb{N}$. Since by definition each partition counted by $\sigma(n)$ contains at least two summands, we have $x - y > 1$. However, when $n = 2r + 1$ for some $r \in \mathbb{N}$, one of the representations counted by $s(n)$ is $n = (r + 1)^2 - r^2$, and so $s(n) = \sigma(n) + 1$. On the other hand, if $n = 2r$ then n is not the difference of consecutive squares and $s(n) = \sigma(n)$. Thus we may set

$$s(n) = \sigma(n) + \frac{(-1)^{n+1} + 1}{2}.$$

This together with (4) yields (3). Finally, observe that (3) also holds for $n = 2$. ■

Remark 2.1 Clearly for any positive integer n , $s(n^2) - 1$ gives the number of Pythagorean triads with n as a side.

3 Partition formulae for $D > 2$

So far we have managed to produce a closed form expression for $p_2(n)$ in terms of the number of divisors $d(n)$, while it is well-known that $p_1(n) = d_1(n) - 1$. In this section we shall derive two further formulae for $p_D(n)$ based on the parity of D . We shall establish these enumerating formulae via purely combinatorial arguments. In what follows we need only consider integers $n \geq D + 2$, since clearly $n = 1 + (1 + D)$ is the smallest number with a partition of the desired form. We begin with case D odd.

Theorem 3.1 Suppose $D > 1 \in \mathbb{N}$ is odd with $n \geq D + 2$. Then the number of partitions of n into positive integers in arithmetic progression with common difference D is given by

$$p_D(n) = \begin{cases} d_1(n) - 2 - f(n) & \text{if } n = D \frac{m(m+1)}{2} \text{ for some } m > 1 \\ d_1(n) - 1 - f(n) & \text{otherwise} \end{cases}$$

where $f(n) = |A_n|$ with $A_n = \{d | n : d \text{ odd}, d^2 < D(2n - d), 2n < Dd(d - 1)\}$.

Proof: The argument will be split into two main steps. In the first step, we demonstrate that the number of ways of expressing n as a finite sum of integers, some possibly negative, in arithmetic progression with the required common difference, is $2d_1(n)$. In the second step, we show how to count those arithmetic progressions with positive terms only, which will lead to the construction of the desired enumerating functions.

Step 1:

Suppose that n is representable as a sum of integers in arithmetic progression with common difference D ,

$$n = a + (a + D) + (a + 2D) + \cdots + (a + rD),$$

for some pair $(a, r) \in \mathbb{Z} \times \mathbb{Z}$. Then clearly we have

$$2n = (r + 1)(2a + Dr). \quad (5)$$

For the given n and D consider the set

$$S_D(n) = \{(a, r) \in \mathbb{Z} \times \mathbb{Z} : 2n = (r + 1)(2a + Dr)\},$$

which we now show contains exactly $2d_1(n)$ distinct elements. By recalling that D is odd, observe from the equality

$$(r + 1) + (2a - 1 + (D - 1)r) = 2a + Dr,$$

that the terms $r + 1$ and $2a + Dr$ are of opposite parity. Thus to solve the Diophantine equation in (5) it suffices to consider the system of simultaneous equations

$$\begin{aligned} r + 1 &= x \\ 2a + Dr &= y \end{aligned}$$

where $(x, y) = (d, \frac{2n}{d})$ or $(\frac{2n}{d}, d)$ for a positive odd divisor d of n . If we denote the solutions (a, r) arising from these right hand sides by $(a_1(d), r_1(d))$ and $(a_2(d), r_2(d))$ respectively, we find that

$$(a_1(d), r_1(d)) = \left(\frac{1}{2} \left(\frac{2n}{d} - D(d - 1) \right), d - 1 \right)$$

and

$$(a_2(d), r_2(d)) = \left(\frac{1}{2} \left(d - D \left(\frac{2n}{d} - 1 \right) \right), \frac{2n}{d} - 1 \right).$$

As $d|n$ and both d and D are odd, a simple parity check establishes that both solutions are ordered pairs of integers. Thus the set of integer solutions (a, r) to (5) can be recast in the form

$$S_D(n) = \bigcup_{d \text{ odd}, d|n} I_d,$$

where $I_d = \{(a_1(d), r_1(d)), (a_2(d), r_2(d))\}$. To show that there is no repetition (or duplication) of any ordered pairs, it will suffice to demonstrate that the second components of all ordered pairs in $S_D(n)$ are distinct. Now as r_1 and r_2 are clearly of opposite parity we have $r_1(d) \neq r_2(d')$ for any two odd, possibly equal, divisors d, d' of n . Moreover, $r_i(d) = r_i(d')$ for $i = 1, 2$ if and only if $d = d'$. Consequently $I_d \cap I_{d'}$ is empty when $d \neq d'$ and so $S_D(n)$ is a finite union of mutually disjoint

sets, each containing two different elements. Thus $S_D(n)$ contains $2d_1(n)$ distinct elements, which is the number of integer arithmetic progressions, as required.

Step 2:

Clearly the partitions we seek correspond to those arithmetic progressions of n in Step 1 which consist of at least two terms, all of which are strictly positive. Consequently we wish to count those ordered pairs $(a, r) \in S_D(n)$ where $a \geq 1$ and $r \geq 1$. With this in mind it is convenient to consider the following two cases separately.

Case 1: $n \neq D \frac{m(m+1)}{2}$ for all $m > 1$.

In this instance, no ordered pair $(a, r) \in S_D(n)$ has $a = 0$, since otherwise as $n \geq D+2$ we would have $n = \sum_{i=1}^r iD = D \frac{r(r+1)}{2}$ for some $r > 1$. Now to determine the number of ordered pairs $(a, r) \in S_D(n)$ with $a \geq 1$ and $r \geq 1$, we examine the elements in I_d for every odd divisor d of n . Clearly I_1 contributes no such ordered pairs as $r_1(1) = 0$, while $a_2(1) = 1 - D(2n - 1) < 0$. In the remaining solution set $S_D(n) \setminus I_1$, observe that since $d \geq 3$, $r_1(d) = d - 1 \geq 2$ and $r_2(d) = \frac{2n}{d} - 1 \geq 1$ as $\frac{n}{d} \geq 1$. Thus we need only concentrate on finding those ordered pairs $(a, r) \in S_D(n) \setminus I_1$ with $a > 0$. To this end, consider the sum

$$\begin{aligned} 2(a_1(d) + a_2(d)) &= (1 - D) \left(\frac{2n}{d} + d \right) + 2D \\ &\leq (1 - D)5 + 2D \\ &= 5 - 3D, \end{aligned}$$

noting here that the inequality holds since $\frac{n}{d} \geq 1$ and $d \geq 3$. Now, $5 - 3D \leq -4$ as $D \geq 3$ and so $a_1(d) + a_2(d) < 0$. Consequently, in each set I_d for $d \geq 3$, $a_1(d)$ and $a_2(d)$ are not both positive. That is, $a_1(d)$ and $a_2(d)$ are both negative or are of opposite sign. Thus if we extract from $S_D(n) \setminus I_1$ those sets I_d with both $a_1(d)$ and $a_2(d)$ negative, exactly half the remaining ordered pairs (a, r) have $a > 0$. By definition, A_n is the set of odd divisors d of n for which both $a_1(d) < 0$ and $a_2(d) < 0$ and so after extracting the $2f(n)$ ordered pairs (a, r) with $a < 0$ from $S_D(n) \setminus I_1$ (noting here that $1 \notin A_n$) we find

$$\begin{aligned} p_D(n) &= \frac{1}{2}(2d_1(n) - 2 - 2f(n)) \\ &= d_1(n) - 1 - f(n). \end{aligned}$$

Case 2: $n = D \frac{m(m+1)}{2}$ for some $m > 1$.

In this case, one representation of n is $n = 0 + D + \cdots + mD$ and so there exists an odd divisor $d' > 1$ of n such that either $a_1(d') = 0$ or $a_2(d') = 0$ (noting here that $d' > 1$ since again I_1 contributes no partition of the required form). Furthermore we have

$$n = D + \cdots + (D + (m - 1)D),$$

that is, $(D, m - 1) \in S_D(N) \setminus I_1$ and this ordered pair rather than $(0, m)$ can be considered as corresponding to one of the required partitions of n . Moreover as $a_1(d') + a_2(d') < 0$ we see that the remaining ordered pairs $(a, r) \in I_{d'}$ have $a < 0$, and so $(D, m - 1) \notin I_{d'}$, since $D > 0$. Consequently the number of desired partitions

of n is equal to the number of ordered pairs $(a, r) \in S_D(n) \setminus (I_1 \cup I_{a'})$ with $a > 0$. Thus as in Case 1, after extracting from this set the $2f(n)$ ordered pairs (a, r) with $a < 0$, precisely half the remaining ordered pairs have $a > 0$ (noting here that $1, d' \notin A_n$). Hence

$$\begin{aligned} p_D(n) &= \frac{1}{2}(2d_1(n) - 4 - 2f(n)) \\ &= d_1(n) - 2 - f(n), \end{aligned}$$

as required. ■

Using the above formulation for $p_D(n)$, we can now establish the characterisation, proved in [1], for a number to be representable as a sum of positive integers in arithmetic progression with odd common difference $D > 1$.

Corollary 3.1 *A number $n = 2^r s \geq D + 2$ with s odd is a sum of positive integers in arithmetic progression with odd common difference $D > 1$ if and only if n is not a power of 2 and either $s > D(2^{r+1} - 1)$ or $n > \frac{1}{2}Dp(p - 1)$ where p is the smallest odd prime factor of n .*

Proof: Suppose n satisfies the above condition. It suffices to show that $p_D(n) \geq 1$ when $n \neq D\frac{m(m+1)}{2}$, since if $n = D\frac{m(m+1)}{2}$ for some $m > 1$ then $n = D + 2D + \dots + mD$ and $p_D(n) \geq 1$. We note first that $1 \notin A_n$ as $n > 0$ and so $0 \leq f(n) \leq d_1(n) - 1$, since A_n has at most $d_1(n) - 1$ elements. Now if $s > D(2^{r+1} - 1)$ it is clear that the inequality $d^2 < D(2n - d)$ fails for $d = s$ while if $n > \frac{1}{2}Dp(p - 1)$ it is clear that the inequality $2n < Dd(d - 1)$ fails for $d = p$ (noting that $s, p > 1$). So A_n fails to contain another odd divisor of n . Thus A_n has at most $d_1(n) - 2$ elements. Hence the function $f(n)$ does not attain its maximum value, $d_1(n) - 1$, and so $p_D(n) \geq 1$.

Establishing the converse is equivalent to showing that if n is a power of 2 or if both $s \leq D(2^{r+1} - 1)$ and $n \leq \frac{1}{2}Dp(p - 1)$ then $p_D(n) = 0$. Now if $n = 2^r$ then the only odd divisor of n is 1, and as $1 \notin A_n$, clearly A_n is empty and $p_D(n) = 1 - 1 - 0 = 0$. Now suppose n is not a power of 2. If $n \neq D\frac{m(m+1)}{2}$ then for any odd divisor $d > 1$ of n we have $n < \frac{1}{2}Dp(p - 1) \leq \frac{1}{2}Dd(d - 1)$ (noting here that the strict inequality holds since $n \neq D\frac{p(p-1)}{2}$). Furthermore, $s < D(2^{r+1} - 1)$, since if $s = D(2^{r+1} - 1)$ then $(a_2(s), r_2(s)) = (0, 1)$ and so $n < D + 2$, a contradiction. Consequently, for any odd divisor $d > 1$ of n we have $d \leq s < D(2^{r+1} - 1) \leq D(\frac{2n}{d} - 1)$ as $\frac{n}{d} \geq 2^r$. That is, $d^2 < D(2n - d)$. Thus there are $d_1(n) - 1$ odd divisors of n contained in A_n , and so $f(n) = d_1(n) - 1$ and $p_D(n) = 0$. If $n = D\frac{m(m+1)}{2}$ then since $n \leq \frac{1}{2}Dp(p - 1)$ we have $m \leq p - 1$. However, from the minimality of p we have $m = p - 1$. So for any odd divisor $d > p$ of n we have $n = \frac{1}{2}Dp(p - 1) < \frac{1}{2}Dd(d - 1)$. That is, precisely $d_1(n) - 2$ odd divisors of n satisfy the inequality $2n < Dd(d - 1)$. Moreover, since $s < D(2^{r+1} - 1)$ we see that all odd divisors $d > 1$ of n satisfy the inequality $d^2 < D(2n - d)$. Thus in this case A_n has exactly $d_1(n) - 2$ elements and so $f(n) = d_1(n) - 2$ and again $p_D(n) = 0$. ■

Clearly for an arbitrary positive integer n it may not be easy to evaluate $f(n)$. In the following corollary we provide an example where $f(n)$ can be determined explicitly, thereby giving a closed form expression for $p_D(n)$.

Corollary 3.2 *If $n = 2^r s \geq D + 2$ where s is odd and $2^{r+1} > D(s - 1)$ then*

$$p_D(n) = \begin{cases} d_1(n) - 2 & \text{if } n = D \frac{m(m+1)}{2} \text{ for some } m > 1 \\ d_1(n) - 1 & \text{otherwise.} \end{cases}$$

Proof: The assumed inequality can be recast in the form of $\frac{2n}{s} > D(s - 1)$. Then for any odd divisor d of n we have $\frac{2n}{d} \geq \frac{2n}{s} > D(s - 1) \geq D(d - 1)$. That is, $2n > Dd(d - 1)$, so A_n is empty and $f(n) = 0$. ■

By applying a somewhat analogous argument to that used in Theorem 3.1 we can now obtain a formulation for $p_D(n)$ in the case D even, which is given in terms of the number of divisors of n and another auxiliary function.

Theorem 3.2 *Suppose $D > 2 \in \mathbb{N}$ is even and $n \geq D + 2$. Then the number of partitions of n into positive integers in arithmetic progression with common difference D is given by*

$$p_D(n) = \begin{cases} \frac{1}{2}(d(n) - 4 + \frac{(-1)^{d(n)+1}+1}{2} - 2g(n)) & \text{if } n = D \frac{m(m+1)}{2} \text{ for some } m > 1 \\ \frac{1}{2}(d(n) - 2 + \frac{(-1)^{d(n)+1}+1}{2} - 2g(n)) & \text{otherwise} \end{cases}$$

where $g(n) = |B_n|$ with $B_n = \{d|n : d \leq \sqrt{n}, 2n < Dd(d - 1), 2d^2 < D(n - d)\}$.

Proof: Once again we split the argument into two main steps. In the first step we demonstrate that the number of ways of expressing n as a finite sum of integers, some possibly negative, in arithmetic progression with the required common difference, is $d(n)$. In the second step we show how to count those arithmetic progressions with positive terms only, by examining the solution set of a Diophantine equation in two cases based on the parity of $d(n)$.

Step 1:

Suppose that n is representable as a sum of integers in arithmetic progression with common difference D ,

$$n = a + (a + D) + (a + 2D) + \cdots + (a + rD),$$

for some pair $(a, r) \in \mathbb{Z} \times \mathbb{Z}$. Denoting $S_D(n) = \{(a, r) \in \mathbb{Z} \times \mathbb{Z} : 2n = (r + 1)(2a + Dr)\}$ it is clear, since $2a + Dr$ is even, that to solve the Diophantine equation, it suffices to consider the system of simultaneous equations

$$\begin{aligned} 2a + Dr &= 2d \\ r + 1 &= \frac{n}{d} \end{aligned}$$

where d is a positive divisor of n . Denoting for each such d the resulting solution (a, r) by $(a(d), r(d))$, we have $(a(d), r(d)) = (\frac{1}{2}(2d - D(\frac{n}{d} - 1)), \frac{n}{d} - 1)$. A simple parity check establishes that $(a(d), r(d))$ is an ordered pair of integers. Thus for every divisor d of n there is an integer pair (a, r) corresponding to the equation $2n = (r + 1)(2a + Dr)$. Moreover, there are exactly $d(n)$ ordered pairs, since the second components are distinct for distinct divisors. Consequently $S_D(n)$ has $d(n)$ distinct elements which correspond to the integer arithmetic progressions, as required.

Step 2:

As before, in order to determine $p_D(n)$ it suffices to count those ordered pairs $(a, r) \in S_D(n)$ with $a \geq 1$ and $r \geq 1$. To this end it is convenient to consider the following two cases.

Case 1: $d(n)$ even

In this case n is not a square and so $\frac{n}{d} \neq d$ for every divisor d of n . Hence $S_D(n)$ can be recast in the form

$$S_D(n) = \bigcup_{d|n, 1 \leq d < \sqrt{n}} I_d,$$

where $I_d = \{(a(d), r(d)), (a(\frac{n}{d}), r(\frac{n}{d}))\}$. Now if $n \neq D\frac{m(m+1)}{2}$ then as no ordered pair $(a, r) \in S_D(n)$ has $a = 0$, it suffices to determine the number of such pairs with $a \geq 1$ and $r \geq 1$. Clearly I_1 contributes no such ordered pairs as $a(1) = \frac{1}{2}(2 - D(n - 1)) < 0$ and $r(n) = 0$. In the remaining solution set $S_D(n) \setminus I_1$, observe that since $d, \frac{n}{d} > 1$, $r(d), r(\frac{n}{d}) \geq 1$. Thus we need only concentrate on finding those ordered pairs $(a, r) \in S_D(n) \setminus I_1$ with $a > 0$. To this end it will be necessary to examine the sign of $a(d) + a(\frac{n}{d})$. First observe from the arithmetic-geometric mean inequality that $d + \frac{n}{d} \geq 2\sqrt{n} \geq 2\sqrt{D+2} \geq 2\sqrt{6}$, and since $d + \frac{n}{d}$ is a positive integer, $d + \frac{n}{d} \geq 5$. Consequently

$$\begin{aligned} 2(a(d) + a(\frac{n}{d})) &= (2 - D)(d + \frac{n}{d}) + 2D \\ &\leq (2 - D)5 + 2D \\ &= 10 - 3D \\ &\leq -2 \\ &< 0. \end{aligned}$$

Thus in each set I_d with $1 < d < \sqrt{n}$, $a(d)$ and $a(\frac{n}{d})$ aren't both positive. That is, either $a(d)$ and $a(\frac{n}{d})$ are both negative or they are of opposite sign. If we extract from $S_D(n) \setminus I_1$ those sets I_d with both $a(d)$ and $a(\frac{n}{d})$ negative, exactly half the remaining ordered pairs (a, r) have $a > 0$. By definition, B_n is the set of divisors d of n with both $a(d) < 0$ and $a(\frac{n}{d}) < 0$, and so after extracting these $2g(n)$ ordered pairs (a, r) with $a < 0$ from $S_D(n) \setminus I_1$ (noting here that $i \notin B_n$), we find

$$p_D(n) = \frac{1}{2}(d(n) - 2 - 2g(n)).$$

Suppose now $n = D\frac{m(m+1)}{2}$ for some $m > 1$. Then one of the representations of n is

of the form $n = 0 + D + \cdots + mD$ and so there is a divisor $1 < d' < \sqrt{n}$ such that either $a(d') = 0$ or $a(\frac{n}{d'}) = 0$. Furthermore, we also have

$$n = D + \cdots + (D + (m-1)D),$$

that is, $(D, m-1) \in S_D(n) \setminus I_1$ and this ordered pair rather than $(0, m)$ can be considered to correspond to one of the required partitions of n . Moreover as $a(d') + a(\frac{n}{d'}) < 0$ we see that the remaining $(a, r) \in I_{d'}$ have $a < 0$ and so $(D, m-1) \notin I_{d'}$ since $D > 0$. Consequently the number of desired partitions of n equals the number of ordered pairs $(a, r) \in S_D(n) \setminus (I_1 \cup I_{d'})$ with $a > 0$. Thus after extracting from this set the $2g(n)$ ordered pairs (a, r) with $a < 0$, exactly half the remainder have $a > 0$ (noting here that $1, d' \notin B_n$). Hence

$$p_D(n) = \frac{1}{2}(d(n) - 4 - 2g(n)).$$

Case 2: $d(n)$ odd

In this case n is a square and $S_D(n)$ is of the form

$$S_D(n) = \bigcup_{d|n, 1 \leq d \leq \sqrt{n}} I_d,$$

where we note that $I_{\sqrt{n}} = \{(a(\sqrt{n}), r(\sqrt{n}))\}$. Now if $n \neq D\frac{m(m+1)}{2}$ then as above we need only count those ordered pairs $(a, r) \in S_D(n) \setminus I_1$ with $a > 0$. If $d' = \sqrt{n}$ observe that $2a(d') = 2a(\frac{n}{d'}) = 2d' - D(d' - 1) < 0$ as $D \geq 4$, so as $I_{d'}$ contains only one element, there are $2g(n) - 1$ ordered pairs $(a, r) \in S_D(n) \setminus I_1$ with $a < 0$. After extracting these ordered pairs, exactly half the remainder have $a > 0$ (noting here that $1 \notin B_n$). Hence

$$\begin{aligned} p_D(n) &= \frac{1}{2}(d(n) - 2 - (2g(n) - 1)) \\ &= \frac{1}{2}(d(n) - 1 - 2g(n)). \end{aligned}$$

However if $n = D\frac{m(m+1)}{2}$ for some $m > 1$ then again there is a divisor $1 < d'' < \sqrt{n}$ such that either $a(d'') = 0$ or $a(\frac{n}{d''}) = 0$. Arguing as in Case 1, we deduce that the number of desired partitions of n equals the number of ordered pairs $(a, r) \in S_D(n) \setminus (I_1 \cup I_{d''})$ with $a > 0$. After extracting from this set the $2g(n) - 1$ ordered pairs with $a < 0$, exactly half the remaining have $a > 0$ (noting here that $1, d'' \notin B_n$). Hence

$$\begin{aligned} p_D(n) &= \frac{1}{2}(d(n) - 4 - (2g(n) - 1)) \\ &= \frac{1}{2}(d(n) - 3 - 2g(n)). \end{aligned}$$

Thus Theorem 3.2 is proven. ■

Using the above formulation for $p_D(n)$, we can now establish the characterisation, proved in [1], for a number to be representable as a sum of positive integers in arithmetic progression with an even common difference $D > 1$.

Corollary 3.3 *A number $n \geq D + 2$ is a sum of positive integers in arithmetic progression with even common difference $D > 2$ if and only if either n is even or n is odd and $n > \frac{1}{2}Dp(p-1)$ where p is the smallest odd prime factor of n .*

Proof: Suppose n satisfies the above condition. It suffices to show that $p_D(n) \geq 1$ when $n \neq D\frac{m(m+1)}{2}$, since if $n = D\frac{m(m+1)}{2}$ for some $m > 1$, $n = D + 2D + \cdots + mD$ and $p_D(n) \geq 1$. Recall that the number of divisors d of n with $1 < d \leq \sqrt{n}$ is $\frac{d(n)-2}{2}$ when $d(n)$ is even, and $\frac{d(n)-1}{2}$ when $d(n)$ is odd. Consequently since $1 \notin B_n$, as $n > 0$, we deduce that B_n contains at most $\frac{d(n)-2}{2}$ elements when $d(n)$ is even, at most $\frac{d(n)-1}{2}$ elements when $d(n)$ is odd. If n is even, then the inequality $2n < Dd(d-1)$ fails to hold for $d = 2$ since $n \geq D + 2$, while if n is odd and $n > \frac{1}{2}Dp(p-1)$ then the same inequality will fail for $d = p$. So B_n fails to contain one of $2, p$. In either case $g(n)$ doesn't attain its maximum value and $p_D(n) \geq 1$.

Conversely, assume $p_D(n) \geq 1$. If n is even then $n > D$, since $D + 2$ is the smallest value of n for which $p_D(n) \neq 0$. If n is odd, suppose $n < \frac{1}{2}Dp(p-1)$ (noting here that $n \neq \frac{1}{2}Dp(p-1)$ as n is odd). If $d > 1$ is a divisor of n then d is odd and $d \geq p$. Consequently $n < \frac{1}{2}Dp(p-1) \leq \frac{1}{2}Dd(d-1)$ and the inequality $2n < Dd(d-1)$ holds for every divisor $d > 1$ of n . However provided $d \neq n$, as $\frac{n}{d}$ is also a divisor of n with $\frac{n}{d} > 1$ we find, on substituting $\frac{n}{d}$ for d in the inequality $2n < Dd(d-1)$ that $2d^2 < D(n-d)$. Thus all divisors $1 < d \leq \sqrt{n}$, must be contained in B_n and the function $g(n)$ attains its maximum value, and $p_D(n) = 0$, a contradiction. Hence $n > \frac{1}{2}Dp(p-1)$, as required. ■

Acknowledgement. The authors are indebted to the anonymous referee for the many valuable comments and suggestions made which helped to improve the presentation of this paper.

References

- [1] R. Cook and D. Sharp, Sums of arithmetic progressions, *Fibonacci Quarterly* **33** (1995), 218–221.
- [2] E. E. Guerin, Consecutive integer partitions, *Ars Combinatoria* **39** (1995), 255–260.
- [3] W. J. LeVeque, *Topics in Number Theory*, Addison-Wesley, (1965).
- [4] M. A. Nyblom, On the representation of integers as a difference of non-consecutive triangular numbers, *Fibonacci Quarterly* **39** (2001), 256–263.