Property Enumerators and a Partial Sum Theorem

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1. Introduction

Suppose we are given a sequence $S = \langle s_i \rangle$, where k ranges over the natural numbers. Associated with S is the function $s(k) = s_k$, that when evaluated for successive natural number values of k produces the sequence S. Now let the sequence $T = \langle t_k \rangle$ (and the corresponding "partial sum function" $t(k) = t_k$) denote the left-partial sums of S:

$$t(n) = \sum_{k=1}^{n} s(k) \tag{1.1}$$

For example, when $s(k) = k^{-1}$ the corresponding sequence defined by $t(n)$ gives the harmonic numbers $H(n) = \sum_{k=1}^{n} k^{-1}$.

The goal of this paper is to investigate transformations on sums having the form

$$\sum_{k=1}^{\infty} t(k) f(k) \tag{1.2}$$

where $t(k)$ is the partial sum function of a known function $s(k)$.

2. The Partial Sum Theorem (PST)

Our fundamental result is:

Theorem 2.1 (Partial Sum Theorem): Let $S = \langle s_k \rangle = s(k)$ be a sequence of complex numbers and let $t(n)$ denote its left partial sums $t(n) = \sum_{i=1}^{n} s(k)$. For any function $f(k)$, denote by $g(n)$ the right partial sum $g(n) = \sum_{k=n}^{\infty} f(k)$. If (1) all the $g(n)$ converge; and (2) $\sum_{k=1}^{\infty} s(k) g(k)$ converges; and (3) $\lim_{n \to \infty} t(n) g(n) = 0$, then:

$$\sum_{k=1}^{\infty} t(k) f(k) = \sum_{k=1}^{\infty} s(k) g(k) \tag{2.1}$$

The proof requires Markov's theorem on rearrangement of series, which can be found in section 142 of [Knopp], which we quote here using notation that will make its application to Theorem 2.1 apparent:

Theorem 2.2 (Markov): Given a semi-infinite matrix $M = \{m_{ij}\}$, the sum of whose row sums converges, for the sum of the row sums to be equal to the sum of the column sums, it is necessary and sufficient that (1) the *n*-remainders of the sums of the rows, $r_{in} = \sum_{j=n}^{\infty} m_{ij}$, converge for all *i* and *n*; (2) for every fixed *n* the sum of the *n*-remainders of all rows, $R_n = \sum_{i=1}^{\infty} r_{in}$ converges; and (3) $\lim_{n\to\infty} R_n = 0$.

We can now use Markov's theorem to prove the PST:

Proof of Theorem 2.1: Consider the semi-infinite matrix *M* whose elements are given by

$$m_{ij} = \begin{cases} s(i)f(j), & j \ge i \\ 0 & \text{otherwise} \end{cases}$$

The sum of the elements of row *i* is

$$\sum_{j=1}^{\infty} m_{ij} = s(i)\sum_{j=i}^{\infty} f(j) = s(i)g(i) \text{ (by the definition of } g\text{).}$$

The sum of the elements of column *j* is

$$\sum_{i=1}^{\infty} m_{ij} = f(j)\sum_{i=1}^{j} s(i) = t(j)f(j) \text{ (by the definition of } t\text{)}$$

These sums will exist and be equal if the conditions of Markoff's theorem are satisfied. First, the sum of the row sums must converge. This is the condition that $\sum_{k=1}^{\infty} s(k)g(k)$ converge. Next, all of the remainders r_{in} must converge. But $r_{in} = \sum_{j=n}^{\infty} m_{1j}$, which is bounded above by $\sum_{j=n}^{\infty} s(i)f(j) = s(i)\sum_{j=n}^{\infty} f(j) = s(i)g(n)$. (The remainder is bounded by, but not necessarily equal to this quantity, because $m_{ij} = 0$ for $j < i$.) Since each of the $s(i)$ is finite, r_{in} converges if each $g(n)$ converges, which it does by the premise of the Theorem. The next condition is that $R_n = \sum_{i=1}^{\infty} r_{in}$ must converge. But $R_n = \sum_{i=1}^{n} s(i)g(n) = t(n)g(n)$ because in column *n* of matrix M only the first *n* elements are non-zero. Therefore all R_n are finite and their limit is zero iff the limit of $t(n)g(n)$ is zero. QED.

Observe that the relationship between *f* and *g* is independent of *s* and *t*. As we shall see later, once convenient pairs of functions (*s*, *t*) and (*f*, *g*) are known, it is often possible to transform sums of the form (1.2) by inspection into simpler ones.

3. Property Enumerators

Let *P* be a subset (not necessarily finite) of the natural numbers $N = \{1,2,3,...\}$. In a slight abuse of notation, we will say that a natural number *m* has property *P* (using *P* both for the name of a set and the name of a property) iff $m \in P$. Corresponding to every $P \subseteq N$, there is an enumeration function $e_P(k)$, which we will call a property enumerator, whose value is the number of natural numbers $\leq k$ having property *P*. Alternatively,

$$e_P(k) = \left| P \cap \{1,...,k\} \right| = \sum_{\substack{i \leq k \\ i \in P}} 1 \tag{3.1}$$

All property enumerators are monotone non-decreasing and trivially satisfy $e_P(k) \leq k$ and $e_P(k+1) \leq e_P(k)+1$.

Property enumerators are very familiar, particularly in number theory. For example, the prime number function $\pi(n)$, which counts the number of primes $\leq n$, is a property enumerator. Some property enumerators are not commonly written but are easy to derive. The number of perfect squares $\leq n$ is given by

$$s(n) = \lfloor \sqrt{n} \rfloor \tag{3.2}$$

In general, if the set *P* taken in increasing order is the range of some function $F_P(k)$ for successive values of $k \geq 1$, then $e_P(k) = \lfloor F^{-1}{}_P(k) \rfloor$.

Also associated with a property *P* is its "indicator function" $I_P(k)$, whose value is 1 iff $k \in P$, 0 otherwise. The sequence of values of $I_P(k)$ can be though of as a semi-infinite bit-vector indicating membership in *P*. It follows immediately from (3.1) that the partial sum function of $I_P(k)$ is just $e_P(k)$, the property enumerator of *P*:

$$e_P(n) = \sum_{i=1}^{n} I_p(i) \tag{3.3}$$

Therefore, the PST for property enumerators (PSTE) can be written in the form:

$$\sum_{k=1}^{\infty} e_P(k)\, f(k) \;=\; \sum_{k=1}^{\infty} I_P(k) g(k) \;=\; \sum_{p\,\in P} g(p)\,. \tag{3.4}$$

The PST does not restrict the sequence $t(k)$ to be monotonic nondecreasing and it need not be integer-valued.

Now for some specific examples. Let $f(k) = a^{-k}$, $a > 1$. Then

$$g(p) \;=\; \sum_{k=p}^{\infty} \frac{1}{a^k} \;=\; \frac{1}{(a-1)a^{p-1}}$$

Note that the function *g* is computed once for any *f* and is independent of property *P*.

Now consider the property "is a perfect square", whose enumerator *s* is given in (3.2). By the PSTE (3.4), we have

$$\sum_{k=1}^{\infty} \frac{\lfloor \sqrt{k} \rfloor}{a^k} \;=\; \frac{1}{a-1} \sum_{\substack{p\ a\ perfect \\ square}} \frac{1}{a^{p-1}}\ .$$

But the sum on the right-hand side can be written simply as $\dfrac{1}{a-1}\sum_{k=1}^{\infty} a^{1-k^2}$. Therefore, we have immediately that

$$\sum_{k=1}^{\infty} \frac{\lfloor \sqrt{k} \rfloor}{a^k} \;=\; \frac{a}{a-1} \sum_{k=1}^{\infty} \frac{1}{a^{k^2}}\ .$$

Likewise, if $\pi(n)$ is the enumerator of primes $\leq n$, then

$$\sum_{k=1}^{\infty} \frac{\pi(k)}{a^k} \;=\; \frac{a}{a-1} \sum_{p\ prime} \frac{1}{a^p}\ .$$

Now let $f(k) = \zeta(k) - 1$. Then

$$g(p) \;=\; \sum_{k=p}^{\infty} \big(\zeta(k) - 1\big) \;=\; \sum_{k=p}^{\infty} \sum_{n=2}^{\infty} \frac{1}{n^k} \;=\; \sum_{n=2}^{\infty} \frac{1}{(n-1)n^{p-1}} \;=\; p - 1 - \sum_{j=2}^{p-1} \zeta(k)$$

$$= 1 - \sum_{k=2}^{p-1}\left((\zeta(k) - 1\right)$$

where a sum is understood to be zero if the lower limit of summation exceeds the upper limit.

By the PSTE,

$$\sum_{k=2}^{\infty}\pi(k)\left(\zeta(k) - 1\right) = \sum_{p\ prime}\left(1 - \sum_{k=2}^{p-1}\left(\zeta(k) - 1\right)\right).$$

4. Enumeration transforms

Because $g(p)$ can be computed from $f(k)$ without reference to any particular property *P*, it makes sense to regard *g* as a transform of *f*, which we will call the enumeration transform and write $f \Rightarrow g$. We have already seen that

$$\frac{1}{a^k} \Rightarrow \frac{1}{(a-1)a^{p-1}}$$

We now consider various other functions and their enumeration transforms.

Let $f(k) = k^{-s}$. Then

$g(p) = \sum_{k=p}^{\infty} k^{-s} = \zeta(s,p) = \frac{(-1)^{s+1}}{s!}\psi^{(s-1)}(p)$, where $\psi^{(n)}(z)$ is the polygamma function.

In particular,

$$\frac{1}{k^2} \Rightarrow \psi^{(1)}(p)$$

$$\frac{1}{k^a} \Rightarrow \zeta(a,p) = \frac{(-1)^a}{(a-1)!}\psi^{(a-1)}(p)$$

$$\frac{1}{(k+b)^a} \Rightarrow \zeta(a,p+b) = \frac{(-1)^a}{(a-1)!}\psi^{(a-1)}(p+b)$$

$$\frac{1}{k^2+k} \Rightarrow \frac{1}{p}$$

$$\frac{1}{k^2-k} \Rightarrow \frac{1}{p-1}$$

$$\frac{1}{k^2+k-2} \Rightarrow \frac{3p^2-1}{3p(p+1)(p-1)}$$

$$\frac{1}{k^2-1} \Rightarrow \frac{2p-1}{2p(p-1)}$$

$$\frac{1}{k(k+2)} \Rightarrow \frac{2p+1}{2p(p+1)}$$

$$\frac{1}{k(k+3)} \Rightarrow \frac{3p^2+6p+2}{3p(p+1)(p+2)}$$

$$\frac{1}{(k+1)(k+2)} \Rightarrow \frac{1}{p+1}$$

$$\frac{1}{(k-1)(k+2)} \Rightarrow \frac{3p^2-1}{3p(p^2-1)}$$

$$\frac{1}{(k-2)(k+1)} \Rightarrow \frac{3p^2-6p+2}{3(p^3-3p^2+2p)}$$

$$\frac{1}{k(k+1)(k+2)} \Rightarrow \frac{1}{2(p^2+p)}$$

$$\frac{2}{k^3-k} \Rightarrow \frac{1}{p^2-p}$$

$$\frac{1}{(k+1)(k+2)(k+3)} \Rightarrow \frac{1}{2(p^2+3p+2)}$$

$$\frac{1}{(k+1)(k+2)(k+3)(k+4)} \Rightarrow \frac{1}{3(p^3+6p^2+11p+6)}$$

In general,

$\prod_{i=1}^{n}\frac{1}{(k+i)}=\frac{1}{(n-1)}\left(\sum_{j=1}^{n}(-1)^{j}S_1(n,j)p^{j-1}\right)^{-1}$, where the $S_1(n,j)$ are Stirling numbers of the first kind.

$$\prod_{i=0}^{n}\frac{1}{(k+i)}=\frac{1}{n}\left(\sum_{j=1}^{n}(-1)^{j}S_1(n,j)p^{j-1}\right)^{-1}$$

$$\frac{(-1)^k}{k^2-1} \Rightarrow \frac{(-1)^p}{2p(p-1)}$$

$$\frac{(-1)^k}{k(k+2)} \Rightarrow \frac{(-1)^p}{2p(p+1)}$$

$$\frac{1}{4k^2-1} \Rightarrow \frac{1}{2(2p-1)}$$

$$\frac{1}{k^3-k} \Rightarrow \frac{1}{2p(p-1)}$$

$$\frac{1}{k^3-k^2} \Rightarrow \frac{1}{p-1}-\psi^{(1)}(p)$$

$$\frac{1}{k^4-k^2} \Rightarrow \frac{2p-1}{2p(p-1)}-\psi^{(1)}(p)$$

$$\frac{1}{a^k} \Rightarrow \frac{a^{1-p}}{a-1}$$

$$\frac{(-1)^k}{a^k} \Rightarrow (-1)^p\frac{a^{1-p}}{a+1}$$

$$\frac{k}{a^k} \Rightarrow \frac{1-p+ap}{(a-1)^2a^{p-1}}$$

$$\frac{k^2}{a^k} \Rightarrow \frac{(a-1)^2p+2(a-1)p+a+1}{(a-1)^3a^{p-1}}$$

$$(-1)^k \frac{k}{a^k} \Rightarrow (-1)^p \frac{ap+p-1}{(a+1)^2 a^{p-1}}$$

$$\frac{1}{k!} \Rightarrow e\left(1-\frac{\Gamma(p,1)}{\Gamma(p)}\right),$$ where $\Gamma(p,q)$ is the incomplete gamma function.

$$\frac{(-1)^k}{k!} \Rightarrow \frac{1}{e}\left(1-\frac{\Gamma(p,-1)}{\Gamma(p)}\right)$$

$$\frac{\sin k}{a^k} \Rightarrow \frac{e^i\left(\sin(1-p)+a\sin p\right)}{a^{p-1}(e^i-a)(ae^i-1)}$$

$$\frac{\cos k}{a^k} \Rightarrow \frac{e^i\left(\sin(1-p)-a\cos p\right)}{a^{p-1}(e^i-a)(ae^i-1)}$$

$$\frac{\sinh k}{a^k} \Rightarrow \frac{1}{2a^{p-1}e^p}\left(\frac{e^{2p}}{a-e}-\frac{e}{ae-1}\right)$$

$$\frac{\cosh k}{a^k} \Rightarrow \frac{1}{2a^{p-1}e^p}\left(\frac{e^{2p}}{a-e}+\frac{e}{ae-1}\right)$$

Applying some of the above transforms, we have virtually by inspection that

$$\sum_{k=2}^{\infty}\frac{\lfloor \lg k\rfloor}{k^2+k} = \sum_{\substack{p\ a\ positive\\ power\ of\ 2}}\frac{1}{p} = \sum_{k=1}^{\infty}\frac{1}{2^k} = 1$$

$$\sum_{k=2}^{\infty}\frac{\lfloor \lg k\rfloor}{a^k} = \frac{1}{a-1}\sum_{\substack{p\ a\ positive\\ power\ of\ 2}}\frac{1}{a^{p-1}} = \frac{a}{a-1}\sum_{k=1}^{\infty}\frac{1}{a^{2^k}}$$

$$\sum_{k=2}^{\infty}\frac{\lfloor \sqrt{k}\rfloor}{k^2+k} = \sum_{p\ a\ square}\frac{1}{p} = \sum_{k=1}^{\infty}\frac{1}{k^2} = \zeta(2)$$

$$2\sum_{k=2}^{\infty}\frac{\pi(k)}{k^3-k} = \sum_{p\ prime}\frac{1}{p(p-1)}$$

$$\sum_{k=2}^{\infty} \frac{(2k+1)\pi(k)}{k^2(k+1)^2} = \sum_{p\ prime} \frac{1}{p^2}$$

$$\sum_{k=2}^{\infty} \frac{k\pi(k)}{(k+1)!} = \sum_{p\ prime} \frac{1}{p!}$$

5. The Trivial Property

Consider the trivial property T "is a natural number". Its enumerator is $e_T(k) = k$. We then have

$$\sum_{k=1}^{\infty} k\, f(k) = \sum_{p=1}^{\infty} g(p)$$

Taking $f(k) = 2^{-k}$, for example, we have that

$$\sum_{k=1}^{\infty} \frac{k}{a^k} = \sum_{k=1}^{\infty} \frac{1}{(a-1)a^{k-1}} = \frac{a}{(a-1)^2} .$$

$$\sum_{k=1}^{\infty} \frac{k}{k!} = e = e\sum_{k=1}^{\infty}\left(1 - \frac{\Gamma(k,1)}{\Gamma(k)}\right), \text{ which implies that } \sum_{k=1}^{\infty}\left(1 - \frac{\Gamma(k,1)}{\Gamma(k)}\right) = 1 .$$

In general,

$$\sum_{k=p}^{\infty} \frac{a^k}{k!} = e^a\left(1 - \frac{\Gamma(k,p)}{\Gamma(k)}\right), \text{ and since } \sum_{k=1}^{\infty} \frac{ka^k}{k!} = ae^a \text{, this implies that}$$

$$\sum_{k=1}^{\infty}\left(1 - \frac{\Gamma(k,a)}{\Gamma(k)}\right) = a .$$

The number of even natural numbers $\le k$ is $\lfloor k/2 \rfloor$. Therefore,

$$\sum_{k=1}^{\infty}\left\lfloor \frac{k}{2} \right\rfloor a^{-k} = \sum_{p\ even}^{\infty} \frac{1}{(a-1)a^{p-1}} = \frac{a}{(a-1)}\sum_{k=1}^{\infty}\frac{1}{2^{2k}} = \frac{a}{3(a-1)}$$

$$\sum_{k=1}^{\infty}\left\lceil \frac{k}{2} \right\rceil a^{-k} = \sum_{p\ odd}^{\infty} \frac{1}{(a-1)a^{p-1}} = \frac{a}{(a-1)}\sum_{k=1}^{\infty}\frac{1}{2^{2k-1}} = \frac{4a}{3(a-1)}$$

The number of natural numbers $\le k$ that are powers of two is $1+\lfloor \lg k \rfloor$. Therefore,

$$\sum_{k=1}^{\infty}(1+\lfloor \lg k \rfloor)a^{-k} = \sum_{\substack{p\,a\,power\\ of\,2}}^{\infty} \frac{1}{(a-1)a^{p-1}} = \frac{a}{(a-1)}\sum_{k=0}^{\infty}\frac{1}{2^{2^k}}$$

The number of natural numbers $\le k$ that are positive powers of two is $\lfloor \lg k \rfloor$. Therefore,

$$\sum_{k=2}^{\infty}\frac{\lfloor \lg k \rfloor}{k^2-1} = \sum_{\substack{p\ a\ positive\\ power\ of\ 2}}^{\infty} \frac{2p-1}{2p(p-1)} = \sum_{k=1}^{\infty}\frac{2^{k+1}-1}{2^{k+1}(2^k-1)}$$

The number of natural numbers $\le k$ that are squares exceeding 1 is $\lfloor \sqrt{k} \rfloor - 1$. Therefore,

$$\sum_{k=2}^{\infty}\frac{\lfloor \sqrt{k} \rfloor - 1}{k^2-1} = \sum_{p\ a\ square > 1}^{\infty} \frac{2p-1}{2p(p-1)} = \sum_{k=2}^{\infty}\frac{2k^2-1}{2k^2(k^2-1)} = \frac{\pi^2}{12} - \frac{1}{8},$$

where the last value on the right is obtained by traditional summation methods. Since

$$\sum_{k=2}^{\infty}\frac{1}{k^2-1} = \frac{3}{4}$$

it follows that

$$\sum_{k=2}^{\infty}\frac{\lfloor \sqrt{k} \rfloor}{k^2-1} = \frac{\pi^2}{12} + \frac{5}{8},$$

6. The Inverse Transform

Given $g(p)$, when can we find a function *f* such that

$$g(p) = \sum_{k=p}^{\infty} f(k)$$

for all $p \in N$? If such an *f* exists and is unique, we will refer to *f* as the inverse enumeration transform of *g*, written $g \Leftarrow f$. Consider two consecutive values $g(q)$ and $g(q+1)$ of *g*. Then

$$g(q) - g(q+1) = \sum_{k=q}^{\infty} f(k) - \sum_{k=q+1}^{\infty} f(k) = f(q) \tag{6.1}$$

So *f* is simply obtained from *g* by means of the difference operator.

Following is a table of some inverse transforms:

$g(p)$	$\Leftarrow$	$f(k)$	
$\frac{1}{p}$		$\frac{1}{k(k+1)}$	(6.2.1)
$\frac{1}{p-1}$		$\frac{1}{k(k-1)}$	(6.2.2)
$\frac{1}{p^2}$		$\frac{2k+1}{k^2(k+1)^2}$	(6.2.3)
$\frac{1}{p^2-1}$		$\frac{2k+1}{k(k^2-1)(k+2)}$	(6.2.4)
$\frac{1}{ap+b}$		$\frac{a}{(ak+b)(ak+a+b)}$	(6.2.5)
$\frac{1}{ap^2+b}$		$\frac{a(2k+1)}{(ak^2+b)(ak^2+2ak+a+b)}$	(6.2.6)
$\frac{1}{p^3}$		$\frac{3k^2+3k+1}{k^3(k+1)^3}$	(6.2.7)
p^{-a}		$\frac{(k+1)^a-k^a}{k^a(k+1)^a}$	(6.2.8)
a^{-p}		$\frac{a-1}{a^{k+1}}$	(6.2.9)
$\frac{1}{a^p p^b}$		$\frac{ak+a-1}{a^{k+1}(k+1)^b}$	(6.2.10)

$\dfrac{1}{p!}$	$\dfrac{k}{(k+1)!}$	(6.2.11)
$\dfrac{1}{p \cdot p!}$	$\dfrac{k^2+k+1}{k(k+1)(k+1)!}$	(6.2.12)
$\dfrac{a^p}{p!}$	$\dfrac{a^k(k+1+a)}{(k+1)!}$	(6.2.13)
$\dfrac{1}{a^p p!}$	$\dfrac{ak+a+1}{a^{k+1}(k+1)!}$	(6.2.14)
$\dfrac{1}{(2p)!}$	$\dfrac{4k^2+6k+1}{(2k+2)!}$	(6.2.15)
$\dfrac{1}{p!p!}$	$\dfrac{k^2+2k}{(k+1)!(k+1)!}$	(6.2.16)
$\dfrac{1}{F_p}$	$\dfrac{F_{k-1}}{F_k F_{k+1}}$, where F_k is the k^{th} Fibonacci number	(6.2.17)

By (6.2.3) we have

$$\sum_{k=1}^{\infty}\left\lfloor \frac{k}{2} \right\rfloor \frac{(2k+1)}{k^2(k+1)^2} = \sum_{k=1}^{\infty}\frac{1}{(2k)^2} = \frac{\pi^2}{24}$$

$$\sum_{k=1}^{\infty}\frac{\lfloor \lg k \rfloor (2k+1)}{k^2(k+1)^2} = \sum_{k=1}^{\infty}\frac{1}{(2^k)^2} = \sum_{k=1}^{\infty}\frac{1}{4^k} = \frac{1}{3}$$

$$\sum_{k=1}^{\infty}\frac{\lfloor \sqrt{k} \rfloor (2k+1)}{k^2(k+1)^2} = \sum_{k=1}^{\infty}\frac{1}{\left(k^2\right)^2} = \zeta(4) = \frac{\pi^4}{90}$$

$$\sum_{k=1}^{\infty}\frac{\pi(k)\ (2k+1)}{k^2(k+1)^2} = \sum_{p\ prime}^{\infty}\frac{1}{p^2}$$

From (6.2.8) we have

$$\sum_{k=1}^{\infty}\frac{\pi(k)\left((k+1)^a-k^a\right)}{k^a(k+1)^a} = \sum_{p\ prime}^{\infty}\frac{1}{p^a} = \sum_{k=1}^{\infty}\frac{\mu(k)}{k}\log\zeta(ak), \quad a>1$$

where the equality on the right is from [Titchmarsh], Eq. 1.6.1.

From (6.2.9) we have the following:

$$\sum_{k=1}^{\infty}\frac{\lfloor\sqrt{k}\rfloor(a-1)}{a^{k+1}} = \sum_{i=1}^{\infty}\frac{1}{a^{k^2}}\ ,\ a>1$$

$$\sum_{k=1}^{\infty}\frac{\pi(k)\ (a-1)}{a^{k+1}} = \sum_{p\ prime}\frac{1}{a^p}\ ,\ a>1$$

From (6.2.11) we have:

$$\sum_{k=1}^{\infty}\left\lfloor\frac{k}{2}\right\rfloor\frac{k}{(k+1)!} = \sum_{k=1}^{\infty}\frac{1}{(2k)!} = \cosh 1 - 1$$

$$\sum_{k=1}^{\infty}\left\lceil\frac{k}{2}\right\rceil\frac{k}{(k+1)!} = \sum_{k=1}^{\infty}\frac{1}{(2k-1)!} = \sinh 1$$

$$\sum_{k=1}^{\infty}\frac{\lfloor\sqrt{k}\rfloor k}{(k+1)!} = \sum_{k=1}^{\infty}\frac{1}{(k^2)!}$$

$$\sum_{k=1}^{\infty}\frac{\lfloor\lg k\rfloor k}{(k+1)!} = \sum_{k=1}^{\infty}\frac{1}{(2^k)!}$$

$$\sum_{k=1}^{\infty}\frac{\pi(k)k}{(k+1)!} = \sum_{p\ prime}\frac{1}{p!}$$

We observe a close connection between the act of transforming $f \Rightarrow g$ and transforming $s(k)$ into $t(n)$. The values $g(k)$ are the partial sums of the tails of the sequence produced by successive values of the $f(k)$. The $t(n)$ are the partial sums of the heads of the sequence produced by successive values of the $s(k)$. The $f(k)$ are finite differences of the $g(k)$, while the $s(k)$ are finite differences of the $t(n)$. In many cases we can easily produce $t(n)$ from $s(k)$:

$s(k)$	$t(n)$
1	n
$(-1)^{k+1}$	$\frac{1}{2}-\frac{(-1)^n}{2}$
k	$\frac{n(n+1)}{2}$
$(-1)^k k$	$(-1)^n(2n+1)-\frac{1}{4}$
$\frac{1}{k}$	H_n
$\frac{1}{k(k+1)}$	$\frac{n}{n+1}$
$\frac{1}{k(k+2)}$	$\frac{n(3n+5)}{4(n+1)(n+2)}$
$\frac{1}{k(k+1)(k+2)}$	$\frac{n(n+3)}{4(n+1)(n+2)}$
$\frac{(-1)^{k+!}}{k(k+2)}$	$\frac{n^2+3n+2-2(-1)^n}{4(n+1)(n+2)}$
$\frac{k}{(k+1)(k+2)}$	$H_n+\frac{2}{n+2}-2$
2^{-n}	$1-2^{-n}$
a^{-k}	$\frac{a^n-1}{(a-1)a^n}$
$\frac{k}{a^k}$	$\frac{a}{(a-1)^2}-\frac{an-n+a}{a^n(a-1)^2}$

$\frac{(-1)^{k+1}}{a^k}$	$\frac{1}{a+1}+\frac{(-1)^{n+1}}{a^n}$
$\frac{(-1)^k k}{2^k}$	$\frac{(-1)^n(3n+2)}{2^n}-\frac{1}{3}$
$\frac{k^2}{2^k}$	$6-\frac{n^2+4n+6}{2^n}$
$\log k$	$\log\Gamma(n)$
$\frac{1}{k!}$	$\frac{e\Gamma(n+1,1)}{\Gamma(n+1)}-1$

7. Harmonic Summation

We have already observed that if $s(k) = k^{-1}$ then $t(n) = H_n$, the n^th harmonic number. From the PST it follows that:

$$\sum_{k=1}^{\infty} H_k f(k) = \sum_{k=1}^{\infty} \frac{g(k)}{k} \tag{7.1}$$

and for generalized harmonic numbers $H_n^{(m)} = \sum_{k=1}^{n} \frac{1}{k^m}$ we have:

$$\sum_{k=1}^{\infty} H_k^{(m)} f(k) = \sum_{k=1}^{\infty} \frac{g(k)}{k^m} \tag{7.2}$$

Theorem 7.1 $\sum_{k=1}^{\infty} \frac{H_k^{(m)}}{a^k} = \frac{a}{a-1}\sum_{k=1}^{\infty}\frac{1}{a^k k^m}$.

Proof: This follows directly from the PST by taking $s(k) = k^{-m}$.

Theorem 7.2. $\sum_{m=1}^{\infty}\left(\left(\sum_{k=1}^{\infty}\frac{H_k^{(m)}}{a^k}\right) - \frac{1}{a-1}\right) = \sum_{k=1}^{\infty}\frac{H_{k-1}}{a^k} = \frac{1}{a-1}\log\frac{a+1}{a}$

8. Partial Sums of the Arithmetic Functions

Consider the Moebius function $\mu(k)$ and let $M(n) = \sum_{k=1}^{n} \mu(k)$. Then by the PST we have:

$$\sum_{k=1}^{\infty} M(k) f(k) = \sum_{k=1}^{\infty} \mu(k) g(k) \qquad (8.1)$$

It follows from the fact that

$$\sum_{k=1}^{\infty} \frac{\mu(k)}{k} = 0 \qquad \text{[Titchmarsh, p. 63]} \qquad (8.2)$$

that we immediately have (taking $g(k) = 1/k$)

$$\sum_{k=1}^{\infty} \frac{M(k)}{k(k+1)} = 0. \qquad (8.3)$$

Likewise, since

$$\sum_{k=1}^{\infty} \frac{\mu(k)}{k^s} = \frac{1}{\zeta(s)} \qquad \text{for s > 1 [HW Thm. 287]} \qquad (8.4)$$

we know that (taking $s = 2$)

$$\sum_{k=1}^{\infty} \frac{M(k)(2k+1)}{k^2(k+1)^2} = \frac{1}{\zeta(2)}. \qquad (8.5)$$

Let $q(n)$ be the number of quadratfrei natural numbers $\le n$. Then $q(n) = \sum_{k=1}^{n} |\mu(k)|$. From

$$\sum_{k=1}^{\infty} \frac{|\mu(k)|}{k^s} = \frac{\zeta(s)}{\zeta(2s)} \qquad \text{[Titchmarsh 1.2.7]} \qquad (8.6)$$

we have

$$\sum_{k=1}^{\infty} \frac{q(k)(2k+1)}{k^2(k+1)^2} = \sum_{k=1}^{\infty} \frac{|\mu(k)|}{k^2} = \frac{\zeta(2)}{\zeta(4)} = \frac{15}{\pi^2}. \qquad (8.7)$$

If $d(n)$ is the number of divisors of n and $D(n)$ is its partial sum function, then from

$$\sum_{k=1}^{\infty} \frac{d(k)}{k^s} = \zeta^2(s) \qquad \text{[Titchmarsh 1.2.1]} \tag{8.8}$$

we have

$$\sum_{k=1}^{\infty} \frac{D(k)(2k+1)}{k^2(k+1)^2} = \sum_{k=1}^{\infty} \frac{d(k)}{k^2} = \zeta^2(2) = \frac{\pi^4}{36}. \tag{8.9}$$

9. Integral Relations (future research)

Suppose we are given a function $s(x)$. Then let $t(y)$ be the integral of the "left tail" of s:

$$t(y) = \int_0^y s(x)\,dx \tag{9.1}$$

in which case we will refer to *t* as the left integral of *s* at *y*. Likewise, for any function $f(x)$, denote by $g(y)$ the integral of the "right tail" of *f*: $g(y) = \int_y^{\infty} f(x)\,dx$ (and call *g* as the "right integral" of *f* at *y*).

The goal is to investigate transformations on integrals having the form

$$\int_a^{\infty} t(x) f(x)\,dx \tag{9.2}$$

where $t(x)$ is the left integral of a known function $s(x)$.

The fundamental result is:

Theorem 8.1: Let $s(x)$ be a function with left integral $t(y)$ and $f(x)$ be a function with right integral $g(y)$.

$$\int_a^{\infty} t(x) f(x)\,dx = \int_a^{\infty} s(x) g(x)\,dx \tag{9.3}$$

[It remains to derive the conditions under which the Theorem is true, e.g. that the relevant functions are of bounded variation. The proof is obtained by rewriting the integrals in (2.2) in terms of infinitesimals, applying the Partial Sum Theorem 1.2 and passing to the limit as $\Delta x \to 0$. Furthermore, there is no requirement that the limits of integration be 0 and ∞.]

Example:

If $f(x) = (1+x)^{-2}$ and $s(x) = (1+x^2)^{-1}$. Then $g(x) = \int_0^x f(y)dy = \int_0^x (1+y)^{-2}dy = (1+x)^{-1}$ and $t(x) = \int_0^x s(y)\,dy = \arctan x$. Then we have by (2.2) that $\int_0^\infty \frac{\arctan x}{(1+x)^2} = \int_0^\infty \frac{dx}{(1+x)(1+x^2)}$.

The theorem is particularly useful when one of the integrals cannot be evaluated in closed form through the usual algorithms of indefinite integration.

Consider the special case in which $f(x) = g(x)$. That is, we must have $g(y) = \int_y^\infty f(x)\,dx$, so $f(x) = e^{-x}$. In this case, applying the Theorem, we find that

$$\int_a^\infty t(x)e^{-x}\,dx = \int_a^\infty s(x)e^{-x}\,dx \tag{9.4}$$

As another example, take $s(x) = (1+x^2)^{-1/2}$, then $t(x) = \operatorname{arcsinh} x$ and we immediately have:

$$\int_0^\infty e^{-x}\operatorname{arcsinh} x\,dx = \int e^{-x}(1+x^2)^{-1/2}\,dx$$

The integral on the left cannot be evaluated as an indefinite integral in closed form. However, the integral on the right evaluates to (in terms of Bessel functions) $\frac{\pi}{2}\left(H_o(1) - Y_0(1)\right)$.

If $s(x)$ is a probability density function that is zero for $x < 0$, then $t(x)$ as defined above is its cumulative distribution function. Therefore the author believes that the theorem has application in statistics.

Afterword

This paper was motivated by a computer program written by the author that "noticed" that

$$\sum_{k=1}^\infty \frac{\pi(k)}{2^k} = 2\sum_{p\,prime} \frac{1}{2^p}$$

and it was in an attempt to understand and prove this equality that the results of this paper emerged. For many years the author has made daily use of Mathematica®, a product of Wolfram Research, Inc., without which this work would have been impossible.

References

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