Pattern avoidance in rook monoids

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Definition

Let $n \in \mathbb{N}$. The *rook monoid* $\mathcal{R}_n$ is the set of all $n \times n$ $\{0,1\}$-matrices such that each row and each column contains at most one 1.

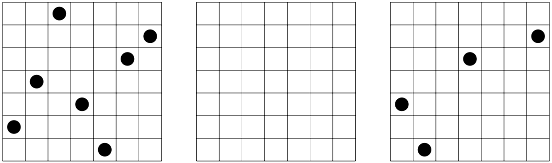
Example members of $\mathcal{R}_7$:

$$\begin{bmatrix} 0&0&\mathbf{1}&0&0&0&0\\ 0&0&0&0&0&0&\mathbf{1}\\ 0&0&0&0&0&\mathbf{1}&0\\ 0&\mathbf{1}&0&0&0&0&0\\ 0&0&0&\mathbf{1}&0&0&0\\ \mathbf{1}&0&0&0&0&0&0\\ 0&0&0&0&\mathbf{1}&0&0 \end{bmatrix} \quad \begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{bmatrix} \quad \begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&\mathbf{1}\\ 0&0&0&\mathbf{1}&0&0&0\\ 0&0&0&0&0&0&0\\ \mathbf{1}&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&\mathbf{1}&0&0&0&0&0 \end{bmatrix}$$

Notice: $n \times n$ permutation matrices are a submonoid of $\mathcal{R}_n$.

Rook Placements

- Have an $n \times n$ grid.
- Place k rooks ($0 \leq k \leq n$) in non-attacking position. (No more than one rook in each row, no more than one rook in each column).



Rook Polynomials

$R_n(x) = \sum_{k=0}^{n} r_{n,k}x^k$ where $r_{n,k}$ is the number of placements of k rooks on an $n \times n$ board.

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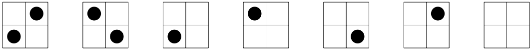
$R_1(x) = x + 1$

$R_2(x) = 2x^2 + 4x + 1$



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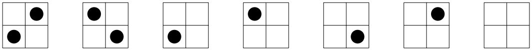

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In general $r_{n,k} = \binom{n}{k}^2 k!$.

A new enumeration problem

Known: How many ways can we place k rooks on an $n \times n$ grid?

- $r_{n,k} = \binom{n}{k}^2 k!$
- $\displaystyle\sum_{n=0}^{\infty} R_n(1)\frac{x^n}{n!} = \frac{e^{\left(\frac{x}{1-x}\right)}}{1-x}$

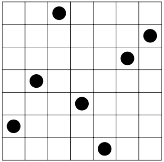
  Sequence: $2, 7, 34, 209, 1546, 13327, \ldots$ (OEIS A002720)

New question: How many ways can we place k rooks on an $n \times n$ grid so they *avoid* a given smaller rook placement pattern?

Rook Strings

Given an $n \times n$ rook placement, associate a string $r_1 \cdots r_n$ such that:

- If there is a rook in column i, row j, then $r_i = j$.
- If column i is empty, then $r_i = 0$.



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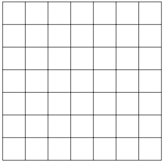

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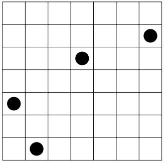

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Rook string avoidance

Definition

Given a rook pattern $q \in \mathcal{R}_m$ and any element $r \in \mathcal{R}_n$, r *contains* q if there exist $1 \leq i_1 < \cdots < i_m \leq n$ such that:

- $q_j = 0$ if any only if $r_{i_j} = 0$
- The nonzero members of $r_{i_1} \cdots r_{i_n}$ are order-isomorphic to the non-zero enties of q.

Otherwise r *avoids* q.

Example: $3402 \in \mathcal{R}_4$

- contains 0, 1, 01, 10, 12, 21, 201.
- avoids 102.

Notation

- $\mathcal{R}_n(q) = \{r \in \mathcal{R}_n \mid r \text{ avoids } q\}$
- $\mathcal{R}_{n,k}(q) = \{r \in \mathcal{R}_n \mid r \text{ avoids } q, r \text{ has } k \text{ nonzero entries}\}$
- $r_n(q) = |\mathcal{R}_n(q)|$
- $r_{n,k}(q) = |\mathcal{R}_{n,k}(q)|$

For example:

- $\mathcal{R}_2(01) = \{00, 10, 20, 12, 21\}$
- $\mathcal{R}_{2,0}(01) = \{00\}$
- $\mathcal{R}_{2,1}(01) = \{10, 20\}$
- $\mathcal{R}_{2,2}(01) = \{12, 21\}$
- $r_2(01) = 5,\ r_{2,0}(01) = 1,\ r_{2,1}(01) = 2,\ r_{2,2}(01) = 2$

The pattern $0\cdots 0$

r avoids $\underbrace{0\cdots 0}_{j}$

$\iff$ r has at most $j-1$ 0s.

$\iff$ r has at least $n-j+1$ nonzero entries.

$$r_{n,k}(\underbrace{0\cdots 0}_{j}) = \begin{cases} r_{n,k} = \binom{n}{k}^2 k! & k \geq n-j+1 \\ 0 & k < n-j+1 \end{cases}$$

The pattern $0 \cdots 0$

$$r_n(\underbrace{0 \cdots 0}_{j}) = \sum_{k=n-j+1}^{n} \binom{n}{k}^2 k!$$

In particular:

$r_n(0) = \sum_{k=n}^{n} \binom{n}{k}^2 k! = n!$

$r_n(00) = \sum_{k=n-1}^{n} \binom{n}{k}^2 k! = (n+1)!$

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In general for fixed j

$$\sum_{n=0}^{\infty} r_n(\underbrace{0\cdots0}_{j})\frac{x^n}{n!} = \sum_{i=1}^{j} \frac{x^{i-1}}{(i-1)!(1-x)^i}$$

Permutation patterns

Consider $\rho \in \mathcal{S}_j$.

Then $r_{n,k}(\rho) = \binom{n}{k}^2 s_k(\rho)$ and $r_n(\rho) = \sum_{k=0}^{n} \binom{n}{k}^2 s_k(\rho)$

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We have:

$$r_n(1) = \sum_{k=0}^{n} \binom{n}{k}^2 s_k(1) = \binom{n}{0} s_0(1) = 1$$

$$r_n(12) = r_n(21) = \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \text{ (OEIS A000984)}$$

For $\rho \in \mathcal{S}_3$,

$$r_n(\rho) = \sum_{k=0}^{n} \binom{n}{k}^2 C_k \text{ where } C_k = \frac{\binom{2k}{k}}{(k+1)} \text{ (OEIS A086618)}$$

Small patterns

Rook patterns of length 3 or less include:

- 0,1
- 00, 01, 10, 12, 21
- 000, 001, 010, 100, 012, 102, 120, 021, 201, 210, 123, 132, 213, 231, 312, 321

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$r_n(p) = r_n(q)$ if rook placement p can be obtained from q by the action of the dihedral group on the $n \times n$ square (then reducing non-zero entries).

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$r_n(001) = r_n(010) = r_n(100)$.

The pattern 01

n\ k	0	1	2	3	4	5	6	total
1	1	1						2
2	1	2	2					5
3	1	3	6	6				16
4	1	4	12	24	24			65
5	1	5	20	60	120	120		326
6	1	6	30	120	360	720	720	1957

$$r_{n,k}(01) = \binom{n}{k} k! = \frac{n!}{(n-k)!}$$

$$\sum_{n=0}^{\infty} r_n(01) \frac{x^n}{n!} = \frac{e^x}{1-x} \text{ (OEIS A000522)}$$

The pattern 001

n\ k	0	1	2	3	4	5	6	total
1	1	1						2
2	1	4	2					7
3	1	6	18	6				31
4	1	8	36	96	24			165
5	1	10	60	240	600	120		1031
6	1	12	90	480	1800	4320	720	7423

$$r_{n,k}(001) = \begin{cases} \binom{n}{k}^2 k! & k \geq n-1 \\ \binom{n}{k}(k+1)! & k \leq n-2 \end{cases}$$

$$\sum_{n=0}^{\infty} r_n(001)\frac{x^n}{n!} = \frac{e^x - x}{(1-x)^2} \text{ (OEIS A193657)}$$

The pattern 012

n\ k	0	1	2	3	4	5	6	total
1	1	1						2
2	1	4	2					7
3	1	9	15	6				31
4	1	16	54	64	24			159
5	1	25	140	310	325	120		921
6	1	36	300	1040	1935	1956	720	5988

$$r_{n,k}(012) = \begin{cases} n! & k = n \\ \displaystyle\sum_{j=1}^{k+1} \binom{n-j}{n-k-1}\binom{n}{k}\binom{k}{j-1}(j-1)! & k \leq n-1 \end{cases}$$

The pattern 102

n\ k	0	1	2	3	4	5	6	total
1	1	1						2
2	1	4	2					7
3	1	9	15	6				31
4	1	16	54	64	24			159
5	1	25	140	310	320	120		916
6	1	36	300	1040	1890	1872	720	5859

$$r_{n,k}(102) = \begin{cases} n! & k = n \\ \sum_P \binom{n}{k}(\Delta P)! & k \leq n-1 \end{cases}$$

where the sum is over sets $P = \{p_1, \ldots, p_{n-k}\} \subset \{1, \ldots, n\}$ where $1 \leq p_1 < p_2 < \cdots < p_{n-k} \leq n$.

$(\Delta P)! :=$

$(p_1 - 1)!(p_2 - p_1 - 1)! \cdots (p_{n-k} - p_{n-k-1} - 1)!(n - p_{n-k})!$

Length 4 and beyond

- Have *enumeration scheme* algorithm programmed in Maple
  - Input: set of rook patterns
  - Output: encoding for system of recurrences enumerating rook placements avoiding those patterns
  - Recurrence determined completely algorithmically
  - Once a scheme is found, can compute $r_n(p)$ and $r_{n,k}(p)$ for n as large as 30 or 40.
- Using scheme data, have determined closed form for $\sum_{n=0}^{\infty} r_n(0\cdots0)\frac{x^n}{n!}$ and $\sum_{n=0}^{\infty} r_n(0\cdots01)\frac{x^n}{n!}$.

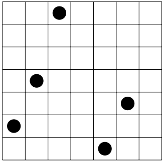
Alternate rook pattern definition

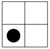
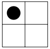
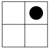
Definition

Rook placement R (on an $n \times n$ board) contains rook placement r (on a $m \times m$ board) if there exist m rows and m columns of R such that

- If R is restricted to those m columns, the empty columns equal the empty columns of r.
- If R is restricted to those m rows, the empty rows equal the empty rows of r.
- R restricted to those m rows and m columns is equal to r.

Example:



contains  and  but avoids 

2d enumeration data

Notation

$r_n^*(p)$ is the number of $n \times n$ rook placements avoiding pattern p in the 2-dimensional sense.

Note: $r_n^*(p) = r_n(p)$ if p has all 0s or p has no 0s.

$r_n^*(p)$ for small 2-dimensional rook patterns

p \ n	1	2	3	4	5	6	OEIS
01	2	6	23	108	605	3956	A093345
001	2	7	33	191	1299	10119	new
012	2	7	31	159	921	5988	new
102	2	7	31	159	916	5859	new

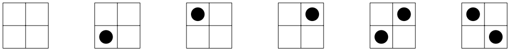
rc-invariant avoidance

- $r_n(321) = r_n^*(321) = \sum_{k=0}^{n} \binom{n}{k}^2 C_k$ (OEIS A086618)
  - Is equal to the number of permutations of length $2n$ which avoid the pattern 4321 and are invariant under the reverse-complement map (Egge, 2010).
  - Have bijective proof.

Signed pattern avoidance

$r_n^*\left(\begin{array}{|c|c|}\hline & \\ \hline & \bullet \\ \hline \end{array}\right)$ is the number of $\{12, \overline{2}1\}$-avoiding signed permutations (studied by Mansour and West in 2002).

Example: $r_2^*\left(\begin{array}{|c|c|}\hline & \\ \hline & \bullet \\ \hline \end{array}\right) = 6$



The six $\{12, \overline{2}1\}$-avoiding signed permutations are:

$$\overline{1}2, \quad 1\overline{2}, \quad \overline{12}, \quad 21, \quad 2\overline{1}, \quad \overline{21}$$

Another B_n sighting

- $r_n(000) = r_n^*(000) = \frac{(n+2)!}{4} + \frac{n!}{2}$ (OEIS A006595)
  - OEIS: this is number of A-reducible ($\overline{12}$ and $1\overline{32}$ avoiding) elements of B_n (Stembridge, 1997).
  - Have bijective proof.

Summary

- Rook monoids provide a natural generalization of permutations.
- The enumeration of rook placements is well-known, but pattern-avoiding rook placements provide a plethora of new enumeration questions.
- Rook placements avoiding one-dimensional patterns can be enumerated via automated enumeration schemes.
- Less is known about two-dimensional avoidance.
- Connections exist to special cases of other pattern-avoidance problems.

Thank You!

References

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- T. Mansour and J. West, Avoiding 2-letter signed patterns, Séminaire Lotharingien de Combinatoire 49 (2002), Article B49a.
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