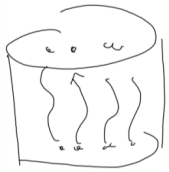
A summary of the open problem sessions of Jan 24, 2019

last update: February 9, 2019

At the end of the BIRS workshop Representation Theory Connections to (q,t)-Combinatorics (19w5131) we reserved two hours for an open problem session. This document serves as a rough written summary of the problems presented by the participants. The video of the two hour periods is available here:
first hour
second hour

Hugh Morton:

For elements that are embedded in a cylinder there are $n!$ elements and you would like to know that they are linearly independent. For HOMFLY PT, there is a scalar product which is non-degenerate that can be used to determine linear independence.



Can we do something like this if we have braids in the torus? That is can we create a bilinear form? What is the HOMFLY skein in this case?

Brendan Pawlowski:

Start with the Stanley symmetric function for a permutation $w \in S_n$ and let $\mathcal{R}(w)$ represent the set of reduced words for w. The Stanley symmetric function is

$$F_w = \sum_{a \in \mathcal{R}(w)} Q_{Asc(a)} \tag{1}$$

$$= \sum_{w=w^1 \cdots w^p} x_1^{\ell(w^1)} \cdots x_p^{\ell(w^p)} \tag{2}$$

where Q_α is the Gessel fundamental quasisymmetric function that are length additive where each w^i has a decreasing reduced word. This thing is not only symmetric, but it is also Schur positive.

Can do the affine version for $w \in \tilde{S}_n$, $\tilde{F}_w = \sum_{w=w^1 \cdot \cdots \cdot w^p} x_1^{\ell(w^1)} \cdots x_p^{\ell(w^p)}$ where each w^i has a cyclically decreasing reduced word. the result is symmetric, but is not Schur positive, but it is affine Schur positive.

This works for any Coxter group. Take any $z \in W$ that is an involution. There is a 0-Hecke product,

$$w \circ s = \begin{cases} ws & \text{if } \ell(ws) > \ell(w) \\ w & \text{else} \end{cases}.$$

Say $s_{a_1}, s_{a_2}, \ldots, s_{a_\ell}$ is an involution word for z if $z = s_{a_\ell} \circ \cdots \circ s_{a_2} \circ s_{a_1} \circ s_{a_2} \circ \cdots \circ s_{a_\ell}$
E.g. $(13) = s_1 s_2 s_1 = s_2 s_1 s_2$ so (s_1, s_2) and (s_2, s_1) are involution words.

Define $\hat{\mathcal{R}}(z)$ is the set of involution words for z.

Then define by putting a hat on first equation $\hat{F}_w = \sum_{a \in \hat{\mathcal{R}}(w)} Q_{Asc(a)} = \sum_{w \in A(z)} F_w$ it is a sum of Stanley symmetric functions so it is symmetric and Schur positive. It is also Schur P-positive.

The open problem is to do the same thing for the affine setting.

Now for $z \in \tilde{S}_n$ there is a definition for an affine $\tilde{\hat{F}}_z = \sum_{w \in A(z)} \tilde{F}_w$ that is known to be affine Schur positive, but what else can be said about it?

A q analogue of a probability

François Bergeron

Problem 1:

A theorem of R.Patrias and V.Reiner, together with F.B., roughly states that: The probability that a monomial positive symmetric function is actually Schur positive is

$$\frac{1}{\prod_{\mu\vdash n}\sum_{\lambda\vdash n} K_{\lambda\mu}} .$$

This is a very precise formula which gives an idea why Schur positivity is a very rare phenomenon. Similarly, the probability that a Schur positive symmetric function is e-positive (or h-positive), is

$$\frac{1}{\prod_{\lambda\vdash n}\sum_{\mu\vdash n} K_{\lambda\mu}} .$$

Observe here that we exchange the role of λ and μ.

Question: (this is slightly vague, but the aim is to try to make this more precise) "Consider the q-analogue of these formulae obtained by replacing the Kostka numbers by the q-Kostka polynomials (or even (q,t)), can we give a natural interpretation of this as a probability of some sort?"

See reference: `arXiv:1810.11038`
The probability of positivity in symmetric and quasisymmetric functions
Authors: Rebecca Patrias, Stephanie van Willigenburg

Positivity of matrices of symmetric functions

Lauren Williams

Is there an analogy between matrices being totally positive and Schur positivity?

A square matrix is said to be totally positive if each sub-matrix has a positive determinant.

Given a matrix $M = (m_{ij})$ entries are symmetric polynomials. M is totally Schur positive if each square sub-matrix has Schur positive determinant.

An example of this are the Jacobi-Trudi matrices, but are there others?

The paper by Fomin-Zelevinsky "Total positivity: tests and parametrizations"
`https://arxiv.org/abs/math/9912128`
is a notable reference for this question. They address various questions there (how to parametrize the set of all totally positive matrices, and how to find "total positivity tests" – collections of minors whose positivity implies the positivity of ALL minors), and so I wonder if the same questions might have nice answers in this Schur-positivity setting.

Some conjectures in Science Fiction

Marino Romero

Denote the cells of a partition by their coordinates $(i,j) \in \mu$ as follows: for the partition $(3,1)$ we have

$(1,0)$		
$(0,0)$	$(0,1)$	$(0,2)$

Define the alternant

$$\Delta_\mu[X_n;Y_n] = \det ||x_r^j y_r^i||_{\substack{(i,j)\in\mu \\ r=1,\dots,n}}.$$

For instance,

$$\Delta_{(3,1)} = \det \begin{bmatrix} 1 & 1 & 1 & 1 \\ y_1 & y_2 & y_3 & y_4 \\ y_1^2 & y_2^2 & y_3^2 & y_4^2 \\ x_1 & x_2 & x_3 & x_4 \end{bmatrix}.$$

And define M_μ as the linear span of all the derivatives of Δ_μ:

$$M_\mu = \{P(\partial_{X_n}, \partial_{Y_n})\Delta_\mu : P \in \mathbb{Q}[X_n, Y_n]\}.$$

The $n!$-Conjecture states that the dimension of M_μ is $n!$, and as a consequence, the modified Macdonald polynomials are the bigraded Frobenius image of an S_n module:

$$\mathrm{Frob}_{qt}(M_\mu) = \widetilde{H}_\mu[X;q,t].$$

This is now a theorem due to Mark Haiman. The conjectured Frobenius characteristic of Diagonal Harmonics led François Bergeron to the operator ∇, which was introduced by François Bergeron and Adriano Garsia in Science Fiction:
F. Bergeron and A. M. Garsia. "Science Fiction and Macdonald's Polynomials". In: arXiv:9809128 (1998).
This operator is defined by its diagonal action on the modified Macdonald basis:

$$\nabla \widetilde{\mathrm{H}}_\mu = T_\mu \widetilde{\mathrm{H}}_\mu \quad \text{with} \quad T_\mu \prod_{(i,j)\in\mu} q^i t^j.$$

We will write $\alpha \to \mu$ if the partition μ can be attained from α by adding a single cell. For example, $(2,1) \to (3,1)$. One approach to the $n!$ conjecture from Adriano Garsia was to prove the $n!/2$-Conjecture. Let $\alpha, \beta \to \mu$, then

Conjecture 1.

$$\dim(M_\alpha \bigwedge M_\beta) = n!/2$$

Science Fiction conjectures the following formula for the Frobenius image of the intersection:

Conjecture 2.

$$\mathrm{Frob}_{qt}\left(M_\alpha \bigwedge M_\beta\right) = \frac{T_\beta \tilde{H}_\alpha - T_\alpha \tilde{H}_\beta}{T_\beta - T_\alpha} = \left(\frac{1}{1 - T_\alpha/T_\beta}\right) \widetilde{\mathrm{H}}_\alpha + \left(\frac{1}{1 - T_\beta/T_\alpha}\right) \widetilde{\mathrm{H}}_\beta$$

.

There are similar formulae for the intersection of k predecessors of the partition μ. And there is the $n!/k$-Conjecture. These can be stated as follows:

Conjecture 3. *For* $\alpha^1, \ldots, \alpha^k \to \mu$,

$$\dim\left(M_{\alpha^1} \bigwedge \cdots \bigwedge M_{\alpha^k}\right) = n!/k$$

and

$$\mathrm{Frob}_{qt}\left(M_{\alpha^1} \bigwedge \cdots \bigwedge M_{\alpha^k}\right) = \sum_{i=1}^{k} \left(\prod_{j \neq i} \frac{1}{1 - T_{\alpha^i}/T_{\alpha^j}}\right) \widetilde{\mathrm{H}}_{\alpha^i}\,.$$

An interesting operator to consider here is the "flip" with respect to α, which is the nonsingular transformation sending $f(X_n, Y_n) \in M_\alpha$ to

$$\mathrm{flip}_\alpha(f) = f(\partial_{X_n}, \partial_{Y_n}) \Delta_\alpha(X_n, Y_n) \quad \in M_\alpha.$$

This map explains the identity

$$T_\mu \omega \widetilde{\mathrm{H}}_\mu[X; 1/q, 1/t] = \widetilde{\mathrm{H}}_\mu[X; q, t],$$

but in Science Fiction, this map acts on the subspaces defined by intersecting and excluding some of the subspace from $M_{\alpha^1}, \ldots, M_{\alpha^k}$. For example, in the case of two predecessors $\alpha, \beta \to \mu$, we should have

$$M_\alpha = \left(M_\alpha \bigwedge M_\beta\right) \bigoplus \mathrm{flip}_\alpha\left(M_\alpha \bigwedge M_\beta\right).$$

See the paper for more formulas and conjectures.

Rational Dyck paths v. Simultaneous (M, N)-cores

Mikhail Mazin

Take m and n relatively prime. There is a bijection between lattice paths which lie below the diagonal in an $n \times m$ rectangle and partitions which are simultaneously m and n cores.

In the relatively prime case there are q, t- countings of both sides which agree. That is, on the left hand side you have a q, t counting of paths below the diagonal in an $m \times n$ rectangle and this is equal to a right hand side of a q, t counting of the partitions which are simultaneously m and n cores.

Consider again the m and n relatively prime and let $(M, N) = (dm, dn)$. The q, t-counting can be naturally generalized both for lattice paths that stay below the diagonal in an $M \times N$ rectangle and for the simultaneous M, N-core partitions. However, they are not equal anymore. In fact, the set of simultaneous M, N-core partitions is infinite in the non relatively prime case, and the resulting generating function is not a polynomial, but rather a power series (a rational function with denominator $(1 - q)^{d-1}$).

Both counts are related to some deep and interesting mathematics. The paths under the diagonal appear in the compositional Shuffle theorem by Erik Carlsson and Anton Mellit, and simultaneous cores correspond to invariants of torus links.

The open question is what is the precise relation between these two q, t-countings in the non relatively prime case. One can also upgrade this question by replacing lattice paths by rational parking functions, and the simultaneous cores by the M-stable affine permutations in $\widetilde{S}_n$. Similarly, in the relatively prime case, these set are in bijection, and produce the same q, t polynomials, but in the non relatively prime case the set of M-stable permutations became infinite, and the exact relation between the two generating functions is unknown.

A reference:

E. Gorsky, M. Mazin, M. Vazirani, Rational Dyck Paths in the Non Relatively Prime Case, Electron. J. Combin. 24 (2017), no. 3, Paper P3.61.

We define equivalence classes on the set of simultaneous cores, which are in bijection with the lattice paths under the diagonal (in the non relatively prime case). The dinv statistic is constant on equivalence classes, and the area statistic changes in a predictable way. However, the equivalence classes come in different sizes, so the relation between the generating functions remains unclear.

Matt Hogancamp mentioned that he and Anton Mellit are going to soon upload a preprint about their recursive computation of the torus link invariants, which is closely related to the generating functions for the simultaneous M, N-cores.

Question for Banff Conference 19w5131

Peter Samuelson

The elliptic Hall algebra $E_{q,t}$ is an algebra over $\mathbb{C}(q,t)$ which was defined by Burban and Schiffmann as a "universal Hall algebra of elliptic curves over finite fields." It is generated by elements $u_{m,n}$ with $m,n \in \mathbb{Z}$, modulo certain commutator relations. In particular, $u_{m,n}$ and $u_{m',n'}$ commute if (m,n) and (m',n') are collinear in $\mathbb{Z}^2$. The $u_{m,n}$ are PBW generators, of $E_{q,t}$, in the sense that picking a total order on $\mathbb{Z}^2$ gives a basis of $E_{q,t}$ indexed by ordered words in the $u_{m,n}$.

There is a central extension $\hat{E}_{q,t}$ which is generated by $u_{m,n}$ and additional central elements $\kappa_{m,n}$ with $m,n \in \mathbb{Z}$. In this extension, generators on a line in $\mathbb{Z}^2$ generate a Heisenberg algebra instead of a commutative algebra, with relations of the form $[u_x, u_{-x}] = c_x \kappa_x$, where $x \in \mathbb{Z}^2$ and $c_x \in \mathbb{C}(q,t)$. For later, we remark that if $E^+_{q,t}$ and $\hat{E}^+_{q,t}$ are defined to be the subalgebras generated by $\{u_{m,n} \mid m > 0 \text{ or } (m = 0 \text{ and } n \geq 0\}$, then $E^+_{q,t}$ is isomorphic to $\hat{E}^+_{q,t}$.

Theorem: [Schiffmann-Vasserot, also independently proved by others in a different language] The central extension $\hat{E}_{q,t}$ acts on Sym.

These operators are very interesting combinatorially, and there is a long story which we can't tell here, but instead recommend the article by Bergeron below, and references therein.

Theorem: [Morton-Samuelson] $E_{q,t=q}$ is isomorphic the Homflypt skein algebra $Sk_q(T^2)$.

The skein algebra acts naturally on the skein module of any 3-manifold with torus boundary. In particular, $Sk_q(T^2)$ acts on $Sk_q(annulus \times [0,1]) \simeq Sym \otimes Sym$. The "positive subalgebra" $Sk_q(T^2)^+$ acts on the subspace $Sym \otimes 1$, and so does $E^+_{q,t}$ by the remark above the previous theorem.

Theorem [Morton, Samuelson] The action of $Sk^+_q(T^2)$ on Sym is the $t = q$ specialization of the $\mathcal{E}^+_{q,t=q}$ action on Sym.

Question: Does the action of $\mathcal{E}_{q,t=q}$ on $Sym \otimes Sym$ extend to generic t?.

The "vertical subalgebra" (generated by the $u_{0,n}$) acts on Sym (essentially) by Macdonald operators, which are diagonalized by Macdonald polynomials. The module $Sym \otimes Sym$ has a basis $s_{\lambda,\mu}$ of "double Schur functions," and this diagonalizes the "vertical subalgebra" of the skein algebra. A positive answer to this question would (presumably) lead to "double" Macdonald polynomials, which would be indexed by pairs of Young diagrams, so a followup question is "how much of Macdonald theory carries through to double Macdonald polynomials"?

Some references:

"The Homflypt skein algebra of the torus and the elliptic Hall algebra" [Morton, Samuelson]
"On the Hall algebra of an elliptic curve" [Burban, Schiffmann]
"The elliptic Hall algebra and the K-theory of the Hilbert scheme of A^2" [Schiffmann, Vasserot]
"Open Questions for Operators Related to Rectangular Catalan Combinatorics" [by F. Bergeron]

A crystal structure for 3-column Macdonald polynomials?

A major open problem is to find a combinatorial interpretation of the coefficients in the Schur expansion of the modified Macdonald polynomial $\widetilde{H}_\mu(x;q,t)$. Haglund, Haiman, and Loehr gave a combinatorial formula for the monomial expansion [HHL05]:

$$\widetilde{H}_\mu(x;q,t) = \sum_{\sigma\,:\,\mu\to\mathbb{Z}_{>0}} q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} x^{\mathrm{wt}(\sigma)}.$$

One appealing approach to finding the Schur expansion would be to define a crystal structure on the set of fillings of μ that preserves the statistics maj and inv; the Schur expansion would then be given by the weights of the highest elements in the crystal structure.

In the case where μ is a single column, inv is always zero, and the usual crystal structure on words (see, e.g., [Shi05]) preserves maj (which is the usual major index on words in this case). The highest weights are Yamanouchi words, which leads to the formula

$$\widetilde{H}_{(1^n)}(x;q,t) = \sum_{w \text{ Yamanouchi}} t^{\mathrm{maj}(w)} s_{\mathrm{wt}(w)}(x).$$

In the case where μ has two columns, Haglund–Haiman–Loehr defined a suitable crystal structure on the fillings [HHL05, §9] (in fact, this crystal structure is a translation of a construction of Carré–Leclerc and van Leeuwen on domino tableaux [CL95, vL00]). The highest weight elements are again the fillings whose (row) reading word is Yamanouchi, but the crystal operators are a bit more complicated than the usual crystal operators on words.

In the case where μ has three columns, Blasiak [Bla16] recently proved a conjectured rule of Haglund [Hag04] for the Schur expansion. The proof uses Fomin and Blasiak's machinery of noncommutative Schur functions and switchboards [BF17]. **We propose the problem of finding a** maj**- and** inv**-preserving crystal structure on fillings of partitions with three columns.** The Yamanouchi fillings no longer have the correct statistics to form the set of highest weight elements (for example, when $\mu = (3,3)$), so one will have to find the correct replacement for the Yamanouchi condition. We hope that the Haglund–Blasiak rule may serve as a guide. Perhaps 3-ribbon tableaux could be useful for intuition, as in the 2-column/domino case (see [HHL05, §3] for the connection between modified Macdonald polynomials and ribbon-tableaux generating functions).

We note that Kaliszewski and Morse [KM17] have given a maj-preserving crystal structure on the fillings of an arbitrary shape, thereby (re)proving a formula for the Schur expansion of the Macdonald polynomials at $q = 1$. The Hall–Littlewood case ($q = 0$) can also be understood in terms of crystals: one may use Haglund–Haiman–Loehr's bijection between inversion-less fillings and row-strict fillings, which turns maj into the cocharge of the RSK recording tableau [HHL05, §7], and then use the usual crystal structure on row-strict fillings (i.e., tensor products of single row crystals), which preserves the recording tableau (see, e.g., [Shi05]).

References

[BF17] J. Blasiak and S. Fomin. Noncommutative Schur functions, switchboards, and Schur positivity. *Selecta Math. (N.S.)*, 23(1):727–766, 2017.

[Bla16] J. Blasiak. Haglund's conjecture on 3-column Macdonald polynomials. *Math. Z.*, 283(1-2):601–628, 2016.

[CL95] C. Carré and B. Leclerc. Splitting the square of a Schur function into its symmetric and antisymmetric parts. *J. Algebraic Combin.*, 4(3):201–231, 1995.

[Hag04] J. Haglund. A combinatorial model for the Macdonald polynomials. *Proc. Natl. Acad. Sci. USA*, 101(46):16127–16131, 2004.

[HHL05] J. Haglund, M. Haiman, and N. Loehr. A combinatorial formula for Macdonald polynomials. *J. Amer. Math. Soc.*, 18(3):735–761, 2005.

[KM17] R. Kaliszewski and J. Morse. Colorful combinatorics and Macdonald polynomials. `arXiv:1710.00801`, 2017.

[Shi05] M. Shimozono. Crystals for dummies. `https://www.aimath.org/WWN/kostka/crysdumb.pdf`, 2005.

[vL00] M. van Leeuwen. Some bijective correspondences involving domino tableaux. *Electron. J. Combin.*, 7:Research Paper 35, 25, 2000.

An elliptic Hall algebra identity

François Bergeron

For any operator $F_{m,n}$, on symmetric functions coming from the elliptic Hall algebra. Define the operation $\hat{F}_{m,n} := F_{m,n}|_{t=1}$. There is an identity that is simple to state, but begs for a proof:

$$\hat{F}_{m,n} \cdot f = f \cdot (\hat{F}_{m,n} \cdot 1).$$

In other words, $\hat{F}_{m,n}$ is simply a multiplication operator. This is certainly not the case (in general) for $F_{m,n}$.

See reference: arXiv:1603.04476
Open Questions for operators related to Rectangular Catalan Combinatorics, Journal of Combinatorics Vol. 8, No. 4 (2017), pp. 673-703.
Author: François Bergeron

Super Harmonics and a representation theoretic model for the Delta conjecture

York: Nantel Bergeron, Laura Colmenarejo, Shu Xiao Li, John Machacek, Robin Sulzgruber, Mike Zabrocki; UCSD: Adriano Garsia, Marino Romero, Don Qui, Nolan Wallach

Define the bi-graded S_n module by the following quotient

$$SCoinv_n := \mathbb{Q}[x_1, x_2, \ldots, x_n; \theta_1, \theta_2, \ldots, \theta_n] / \left\langle \sum_{i=1}^{n} x_i^r, \sum_{i=1}^{n} \theta_i x_i^s : 0 < r \leq n, 0 \leq s < n \right\rangle$$

where $\theta_i \theta_j = -\theta_j \theta_i$ and $\theta_i^2 = 0$, but otherwise the variables commute. We can show that as an S_n module the space of polynomials

$$SHar_n = \left\{ P(X_n; \Theta_n) : \sum_{i=1}^{n} \partial_{x_i}^r P(X_n; \Theta_n) = \sum_{i=1}^{n} \partial_{x_i}^s \partial_{\theta_i} P(X_n; \Theta_n) = 0 \text{ for } 0 < r \leq n, 0 \leq s < n \right\}$$

is isomorphic to $SCoinv_n$ where the partial differential operators satisfy the relations $\partial_{\theta_i} \partial_{\theta_j} = -\partial_{\theta_j} \partial_{\theta_i}$ and $\theta_i \partial_{\theta_j} = -\partial_{\theta_j} \theta_i$.

The first few of the q,t-dimensions for $n = 1, 2, 3, 4$ are 1, $1 + q + t$, $1 + (2\,q + 2\,t) + (2\,q^2 + 3\,qt + t^2) + (q^3 + q^2 t)$, $1 + (3\,q + 3\,t) + (5\,q^2 + 8\,qt + 3\,t^2) + (6\,q^3 + 11\,q^2 t + 6\,qt^2 + t^3) + (5\,q^4 + 9\,q^3 t + 4\,q^2 t^2) + (3\,q^5 + 4\,q^4 t + q^3 t^2) + (q^6 + q^5 t)$.

The first three q,t-Frobenius images are s_1, then

$$(q + t)\, s_{1,1} + s_2, \text{ and}$$

$$\left(q^3 + q^2 t + qt + t^2\right) s_{1,1,1} + \left(q^2 + qt + q + t\right) s_{2,1} + s_3 \ .$$

Conjecture 1. *For a fixed $n \geq 1$,*

$$\dim_{qz} SCoinv_n = \sum_{k=1}^{n} S_q(n,k)[k]_q! z^{n-k}$$

where $S_q(n,k) = S_q(n-1,k-1) + [k]_q S_q(n-1,k)$ and $S_q(n,1) = S_q(n,n) = 1$. The dimension of $SCoinv_n$ is equal to the number of ordered set partitions (see sequence:

`http://oeis.org/A000670`*). Furthermore, the Frobenius image of the symmetric group character is*

$$\mathrm{Frob}_{qz}(SCoinv_n) = \sum_{r=1}^{n}(-z/q)^{n-r}H_{n,r}[X;1/q] = \Delta'_{e_{n-1}[X-\epsilon z]}(e_n)\Big|_{t=0}$$

where $e_{n-1}[X-\epsilon z] = e_{n-1} + ze_{n-2} + z^2e_{n-3}\cdots + z^{n-1}$ *and where* $H_{n,k}[X;q] = \omega E_{n,k}[X;q]$ *is defined by the equation*

$$h_n\left[X\frac{1-u}{1-q}\right] = \sum_{r=1}^{n}\frac{(u;q)_r}{(q;q)_r}H_{n,k}[X;q]$$

and $\Delta_f(\tilde{H}_\mu[X;q,t]) = f[B_\mu]\tilde{H}_\mu[X;q,t]$.

Fix an integer n, and let $\Delta_n(X_n) = \prod_{1\le i<j\le n}(x_i - x_j)$ be the Vandermonde determinant. Define polarize power sums $E_r = \sum_{i=1}^{n}\theta_i\partial_{x_i}^r$. They satisfy the relations $E_rE_s = -E_sE_r$ and $E_r^2 = 0$. We conjecture the following analogue of 'The Operator Conjecture.'

Conjecture 2.

$$SHar_n \simeq \mathrm{span}\{\partial_x^a E_1^{\epsilon_1}E_2^{\epsilon_1}\cdots E_{n-1}^{\epsilon_1}\Delta_n(X_n)\}$$

with $\epsilon_i \in \{0,1\}$ *and* ∂_x^a *represents the derivatives with respect to the* x_i *variables in all possible ways.*

We can show containment of the linear span in $SHar_n$ and this surprisingly allows us to prove a basis of 2^{n-1} alternants $E_1^{\epsilon_1}E_2^{\epsilon_1}\cdots E_{n-1}^{\epsilon_1}\Delta_n(X_n)$.

The Frobenius image expressed in terms of the Delta operator is quite suggestive and leads us to the following conjecture. Let

$$\mathbb{Q}[X_n, Y_n; \Theta_n] := \mathbb{Q}[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n; \theta_1, \theta_2, \ldots, \theta_n]$$

be the polynomial ring in three sets of variables, the first two are commuting and the third one is anti-commuting (and the variables of different flavors commute). The invariants of this polynomial ring are generated by analogues of the power sums

$$p_{r,s} = \sum_{i=1}^{n}x_i^r y_i^s \text{ and } \tilde{p}_{r',s'} = \sum_{i=1}^{n}\theta_i x_i^{r'}y_i^{s'}$$

for $0 < r+s \le n$ and $0 \le r'+s' < n$. Define the analogue of the diagonal harmonics as

$$SDCoinv_n := \mathbb{Q}[X_n, Y_n; \Theta_n]/\left\langle p_{r,s}, \tilde{p}_{r',s'} | 0 < r+s \le n, 0 \le r'+s' < n\right\rangle \ .$$

Calculating the dimensions of the quotients for $1 \leq n \leq 6$, the graded dimensions are $1, 4, 28, 288, 3936, 67328$ (see sequence: `https://oeis.org/A201595`). Moreover we can calculate the Frobenius image for $1 \leq n \leq 4$ and conjecture the following model for the 'The Delta Conjecture.' The variable q will keep track of the degree in the X_n variables, t for the degree in the Y_n variables and z is the degree in in the Θ_n variables.

Conjecture 3. *For $n \geq 1$,*

$$\mathrm{Frob}_{qtz}(SDCoinv_n) = \Delta'_{e_{n-1}[X-\epsilon z]}(e_n)$$

where $e_{n-1}[X - \epsilon z] = e_{n-1} + ze_{n-2} + z^2 e_{n-3} + \cdots + z^{n-1}$ and $\Delta'_f(\tilde{H}_\mu[X; q, t]) = f[B_\mu - 1]\tilde{H}_\mu[X; q, t]$.

Based on (potentially spotty, but significant enough to believe that it is true) data computed this week, we can make an interesting further conjecture. Let $\mathcal{E}_n[Q; Z] = \sum_\mu a_\mu[Q]s_\mu[Z]$ be the symmetric functions described in François Bergeron's talk on Monday and Tuesday as a symmetric function expression for the multivariate analogue of the diagonal harmonics, then let

$$Coinv_n^{k,k'} := \mathbb{Q}[X_n^{(1)}, \ldots, X_n^{(k)}; \Theta_n^{(1)}, \ldots, \Theta_n^{(k')}] / \langle Sym^+ \rangle$$

be the coinvariant space in k sets of commuting variables and k' sets of anti-commuting variables (such that the anticommuting variables also anticommute among themselves).

Conjecture 4. *Let Q_k represent the alphabet $q_1, q_2, \ldots, q_k$ to keep track of the degrees in the $X_n^{(i)}$ variables and $T_{k'}$ represent the alphabet $t_1, t_2, \ldots, t_{k'}$ as variables which keep track of the degrees in the $\Theta_n^{(i)}$ variables, then*

$$\mathrm{Frob}_{Q_k, T_{k'}}(Coinv_n^{k,k'}) = \mathcal{E}_n[Q_k - \epsilon T_{k'}; Z] \ .$$