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A CONJECTURE ON UNIT FRACTIONS INVOLVING PRIMES

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ABSTRACT. We present a conjecture on unit fractions involving primes, and provide numerical data supporting the conjecture.

Unit fractions have the form $1/n$ with $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$. A sum of finitely many distinct unit fractions is called a *Egyptian fraction* as it was first studied by the ancient Egyptians around 1650 B.C. As

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)},$$

any positive rational number $r = m/n$ with $m, n \in \mathbb{Z}^+$ is an Egyptian fraction. (This easy fact was first proved by Fibonacci in 1202 and it implies that the series $\sum_{n=1}^{\infty} 1/n$ diverges.) For example,

$$1 = \frac{1}{2} + \frac{1}{2} = \frac{1}{2} + \left(\frac{1}{2+1} + \frac{1}{2 \times 3} \right) = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}.$$

See also Graham [Gr] and Guy [Gu, pp. 252–262] for known problems and results on Egyptian fractions.

Euclid proved that there are infinitely many primes. In 1737, Euler showed further that $\sum_p 1/p$ diverges, where p runs over all the primes. Equivalently, $\sum_p 1/(p-1)$ and $\sum_p 1/(p+1)$ diverge. By Dirichlet's theorem, for any $d = \pm 1$ and $n \in \mathbb{Z}^+$ there are infinitely many primes p with $p \equiv d \pmod{n}$. Motivated by this, we formulate the following conjecture.

Conjecture. (i) (Sept. 9, 2015) *For any positive rational number r , there is a finite set P_r^- of primes such that*

$$\sum_{p \in P_r^-} \frac{1}{p-1} = r. \tag{1}$$

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(ii) (Sept. 10, 2015) *For any positive rational number r , there is a finite set P_r^+ of primes such that*

$$\sum_{p \in P_r^+} \frac{1}{p+1} = r. \quad (2)$$

The author made the conjecture public by adding comments (cf. [S1]) on the sequence A000040 of primes in OEIS. He also sent a message (cf. [S2]) to Number Theory Mailing List to report part (i) of the conjecture. The author would like to offer 500 US dollars as the first complete solution to the conjecture.

Recall that a positive integer n is called a *practical number* if each $m = 1, \dots, n$ can be written as the sum of some distinct (positive) divisors of n . 1 is the only odd practical numbers, and all powers of two are practical numbers. The distribution of practical numbers is quite similar to that of prime numbers. For $x > 0$ let $P(x)$ denote the number of practical numbers not exceeding x . Similar to the Prime Number Theorem, we have

$$P(x) \sim c \frac{x}{\log x} \quad \text{for some constant } c > 0,$$

which was conjectured by M. Margenstern [M] in 1991 and proved by A. Weingartner [W] in 2014. In view of the above conjecture on unit fractions involving primes, on Sept. 12, 2015 the author conjectured that any positive rational number r can be written as $\sum_{j=1}^k 1/q_j$, where $q_1, \dots, q_k$ are distinct practical numbers. (See the author's comments (cf. [S3]) added to the sequence A005153 of practical numbers in OEIS.) For example,

$$\frac{10}{11} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{48} + \frac{1}{132} + \frac{1}{176}$$

with 2, 4, 8, 48, 132, 176 all practical numbers.

We have checked the conjecture for all those rational numbers $r \in (0, 1]$ with denominators among $1, \dots, 30$. Below we provide 12 tables containing related data. Note that Tables 8 and 12 were produced by Prof. Qing-Hu Hou at Tianjin Univ. (Nov. 6, 2015) on the author's request.

Table 1: P_r^- and P_r^+ for $r \in (0, 1]$ with denominators among $1, \dots, 8$

r	P_r^-	P_r^+
1	$\{2\}, \{3, 5, 7, 13\}$	$\{2, 3, 5, 7, 11, 23\}$
1/2	$\{3\}$	$\{2, 5\}$
1/3	$\{5, 11\}$	$\{2\}$
2/3	$\{3, 7\}$	$\{2, 3, 11\}$
1/4	$\{5\}$	$\{3\}$
3/4	$\{3, 5\}$	$\{2, 3, 5\}$
1/5	$\{7, 31\}$	$\{5, 29\}$
2/5	$\{5, 11, 29, 71\}$	$\{2, 17, 89\}$
3/5	$\{3, 11\}$	$\{2, 3, 59\}$
4/5	$\{3, 5, 29, 71\}$	$\{2, 3, 5, 19\}$
1/6	$\{7\}$	$\{5\}$
5/6	$\{3, 5, 13\}$	$\{2, 3, 5, 11\}$
1/7	$\{13, 29, 43\}$	$\{7, 71, 251\}$
2/7	$\{5, 29\}$	$\{3, 31, 223\}$
3/7	$\{5, 13, 17, 43, 113\}$	$\{2, 11, 83\}$
4/7	$\{3, 17, 113\}$	$\{2, 5, 13\}$
5/7	$\{3, 7, 31, 71\}$	$\{2, 3, 7, 167\}$
6/7	$\{3, 5, 13, 43\}$	$\{2, 3, 5, 11, 41\}$
1/8	$\{11, 41\}$	$\{7\}$
3/8	$\{5, 11, 41\}$	$\{2, 23\}$
5/8	$\{3, 11, 41\}$	$\{2, 3, 23\}$
7/8	$\{3, 5, 11, 41\}$	$\{2, 3, 5, 7\}$

Table 2: P_r^- and P_r^+ for $r \in (0, 1)$ with denominators among $9, \dots, 12$

r	P_r^-	P_r^+
1/9	{13, 37}	{11, 41, 251}
2/9	{7, 19}	{5, 17}
4/9	{5, 7, 37}	{2, 11, 41, 251}
5/9	{3, 19}	{2, 5, 17}
7/9	{3, 5, 37}	{2, 3, 5, 41, 251}
1/10	{11}	{11, 59}
3/10	{5, 29, 71}	{3, 19}
7/10	{3, 7, 31}	{2, 3, 11, 29}
9/10	{3, 5, 11, 29, 71}	{2, 3, 5, 7, 47, 239}
1/11	{23, 67, 73, 89, 199}	{11, 131}
2/11	{7, 67}	{7, 43, 47, 109, 239}
3/11	{7, 19, 23, 199}	{3, 43}
4/11	{5, 13, 73, 89, 199}	{2, 43, 131}
5/11	{5, 13, 19, 23, 67, 199}	{2, 11, 47, 109, 239, 263}
6/11	{3, 23}	{2, 5, 23, 263}
7/11	{3, 11, 41, 89}	{2, 3, 23, 131, 263}
8/11	{3, 7, 19, 199}	{2, 3, 7, 71, 197}
9/11	{3, 5, 37, 67, 73, 89}	{2, 3, 5, 17, 131, 197}
10/11	{3, 5, 13, 19, 67, 199}	{2, 3, 5, 7, 43, 131, 263}
1/12	{13}	{11}
5/12	{5, 7}	{2, 11}
7/12	{3, 13}	{2, 3}
11/12	{3, 5, 7}	{2, 3, 5, 7, 23}

Table 3: P_r^- and P_r^+ for $r \in (0, 1)$ with denominators among 13, 14, 15

r	P_r^-	P_r^+
1/13	{29, 53, 71, 131}	{23, 71, 83, 181, 251}
2/13	{11, 31, 79, 131}	{11, 23, 79, 103, 239, 389}
3/13	{7, 29, 71, 131, 157}	{7, 13, 59, 103, 181, 389}
4/13	{5, 29, 71, 131}	{3, 29, 79, 179, 239, 467}
5/13	{5, 11, 41, 157, 313}	{2, 31, 103, 223, 251, 503}
6/13	{5, 7, 37, 73, 313}	{2, 11, 29, 151, 311, 569}
7/13	{3, 53, 79, 157}	{2, 5, 41, 167, 181, 311}
8/13	{3, 13, 53, 79}	{2, 3, 53, 179, 233, 269}
9/13	{3, 7, 53, 157}	{2, 3, 11, 71, 103, 467}
10/13	{3, 5, 53}	{2, 3, 5, 71, 311, 467}
11/13	{3, 5, 13, 79}	{2, 3, 5, 11, 103, 311}
12/13	{3, 5, 7, 157}	{2, 3, 5, 7, 23, 233, 467}
1/14	{17, 113}	{13}
3/14	{7, 31, 71}	{5, 31, 83, 223}
5/14	{5, 13, 43}	{2, 41}
9/14	{3, 13, 29, 43}	{2, 3, 19, 179, 251}
11/14	{3, 5, 29}	{2, 3, 5, 31, 223}
13/14	{3, 5, 7, 109, 379}	{2, 3, 5, 7, 23, 83}
1/15	{17, 241}	{19, 59}
2/15	{11, 31}	{11, 19}
4/15	{5, 61}	{3, 59}
7/15	{5, 7, 29, 71}	{2, 11, 19}
8/15	{3, 31}	{2, 5, 29}
11/15	{3, 7, 17, 241}	{2, 3, 7, 47, 239}
13/15	{3, 5, 11, 61}	{2, 3, 5, 11, 29}
14/15	{3, 5, 7, 61}	{2, 3, 5, 7, 17, 359}

Table 4: P_r^- and P_r^+ for $r \in (0, 1)$ with denominators among 16 and 17

r	P_r^-	P_r^+
1/16	{17}	{23, 47}
3/16	{7, 61, 241}	{5, 47}
5/16	{5, 17}	{3, 19, 79}
7/16	{5, 7, 61, 241}	{2, 11, 47}
9/16	{3, 17}	{2, 5, 19, 79}
11/16	{3, 7, 61, 241}	{2, 3, 11, 47}
13/16	{3, 5, 17}	{2, 3, 5, 19, 79}
15/16	{3, 5, 7, 61, 241}	{2, 3, 5, 7, 19, 79}
1/17	{19, 307}	{17, 467, 883} (Qing-Hu Hou)
2/17	{13, 73, 103, 137, 307}	{11, 101, 107, 179, 269, 271, 431}
3/17	{7, 103}	{7, 31, 167, 223, 239, 271, 509}
4/17	{7, 19, 103, 307}	{5, 23, 79, 179, 239, 359, 509}
5/17	{5, 37, 73, 409}	{3, 67, 101, 109, 239, 271, 373}
6/17	{5, 13, 73, 307, 409}	{3, 13, 83, 101, 239, 271, 509}
7/17	{5, 13, 19, 103, 137, 307, 409}	{2, 17, 103, 197, 263, 373, 571}
8/17	{5, 7, 31, 103, 211, 281, 409}	{3, 7, 19, 31, 107, 431, 647, 1699, 2591, 4049} (Qing-Hu Hou)
9/17	{3, 113, 137, 211, 239, 241}	{2, 5, 101, 109, 239, 271, 373}
10/17	{3, 17, 127, 137, 239, 307, 337}	{2, 5, 17, 89, 101, 109, 373}
11/17	{3, 13, 29, 43, 239}	{2, 3, 29, 59, 109, 373, 509}
12/17	{3, 7, 43, 127, 239, 307}	{2, 3, 11, 53, 67, 271, 431}
13/17	{3, 7, 13, 239, 241, 337, 421, 1021}	{2, 3, 7, 29, 67, 151, 569}
14/17	{3, 5, 29, 43, 103, 239}	{2, 3, 5, 19, 101, 109, 373, 509}
15/17	{3, 5, 11, 41, 137}	$P_{2/17}^+ \cup P_{13/17}^+$
16/17	{3, 5, 7, 73, 137, 307}	$P_{4/17}^+ \cup P_{12/17}^+$

Table 5: P_r^- and P_r^+ for $r \in (0, 1)$ with denominators among 18 and 19

r	P_r^-	P_r^+
1/18	{19}	{17}
5/18	{5, 37}	{3, 53, 107}
7/18	{5, 13, 19}	{2, 17}
11/18	{3, 13, 37}	{2, 3, 43, 197}
13/18	{3, 7, 19}	{2, 3, 7, 71}
17/18	{3, 5, 7, 37}	{2, 3, 5, 7, 17, 71}
1/19	{37, 137, 191, 229, 331, 397, 761, 1021}	{37, 107, 227, 239, 311, 359, 701, 911}
2/19	{13, 101, 151, 191}	{13, 59, 223, 251, 269, 359, 863, 911}
3/19	{11, 29, 127, 229, 271, 379, 457, 761}	{7, 71, 151, 239, 311, 379, 683, 1039}
4/19	{7, 53, 131, 157, 211, 281, 457}	{5, 41, 139, 223, 311, 379, 607, 1039}
5/19	{7, 17, 61, 191, 229, 241, 457, 761}	{5, 13, 79, 239, 311, 389, 727, 797}
6/19	{5, 29, 71, 191, 211, 281, 457}	{3, 23, 83, 239, 311, 379, 797, 1039}
7/19	{5, 13, 61, 101, 241, 401, 571}	{2, 53, 227, 269, 307, 359, 659, 1063}
8/19	{5, 11, 17, 229, 241}	{2, 13, 227, 263, 307, 379, 769, 1063}
9/19	{5, 7, 31, 67, 229, 419, 571}	{2, 11, 23, 167, 251, 359, 683, 839}
10/19	{3, 101, 151, 191, 229}	{2, 5, 71, 239, 311, 379, 683, 1039}
11/19	{3, 29, 37, 127, 281, 457, 571}	{2, 5, 17, 71, 227, 379, 719, 911}
12/19	{3, 11, 61, 211, 229, 281, 457}	{2, 3, 29, 151, 307, 379, 659, 1063}
13/19	{3, 11, 23, 37, 181, 331, 419}	{2, 3, 11, 127, 227, 383, 607, 911}
14/19	{3, 7, 31, 41, 191, 229, 457}	{2, 3, 7, 59, 151, 359, 719, 911}
15/19	{3, 5, 67, 101, 151, 191, 419}	{2, 3, 5, 37, 127, 383, 607, 911}
16/19	{3, 5, 17, 61, 191, 241, 457, 761}	{2, 3, 5, 13, 71, 227, 683, 1063}
17/19	{3, 5, 11, 31, 211, 281, 571, 761}	{2, 3, 5, 7, 71, 379, 569, 683}
18/19	{3, 5, 7, 61, 151, 229, 601, 761}	{2, 3, 5, 7, 17, 71, 503, 1063}

Table 6: P_r^- and P_r^+ for $r \in (0, 1)$ with denominators among 20 and 21

r	P_r^-	P_r^+
1/20	{29, 71}	{19}
3/20	{11, 29, 71}	{7, 71, 89}
7/20	{5, 11}	{2, 59}
9/20	{5, 7, 31}	{2, 11, 29}
11/20	{3, 29, 71}	{2, 5, 19}
13/20	{3, 11, 29, 71}	{2, 3, 17, 89}
17/20	{3, 5, 11}	{2, 3, 5, 11, 59}
19/20	{3, 5, 7, 31}	{2, 3, 5, 7, 23, 29}
1/21	{31, 71}	{47, 107, 167, 179, 269, 431}
2/21	{17, 43, 113}	{17, 43, 167, 197, 251, 503}
4/21	{7, 43}	{7, 23, 83, 167, 251, 503}
5/21	{7, 17, 113}	{5, 19, 103, 179, 233, 503}
8/21	{5, 11, 61, 71}	{2, 31, 167, 223, 251, 503}
10/21	{5, 7, 29, 43}	{2, 7, 131, 197, 307, 503}
11/21	{3, 43}	{2, 5, 83, 167, 251, 503}
13/21	{3, 13, 29}	{2, 3, 41, 167, 251, 503}
16/21	{3, 7, 17, 43, 113}	{2, 3, 5, 167, 251, 503}
17/21	{3, 5, 29, 43}	{2, 3, 5, 19, 139, 419}
19/21	{3, 5, 13, 17, 113}	{2, 3, 5, 7, 41, 167}
20/21	{3, 5, 7, 29}	{2, 3, 5, 7, 13, 167}

Table 7: P_r^- and P_r^+ for $r \in (0, 1)$ with denominators among 22 and 24

r	P_r^-	P_r^+
1/22	{23}	{53, 107, 149, 199, 263, 449}
3/22	{11, 41, 89}	{11, 29, 109, 179, 197}
5/22	{7, 23, 67}	{5, 23, 71, 197}
7/22	{5, 37, 67, 73, 89}	{3, 23, 71, 131, 197}
9/22	{5, 13, 19, 67, 199}	{2, 17, 89, 109}
13/22	{3, 17, 61, 199, 241, 397}	{2, 5, 11, 131}
15/22	{3, 7, 67}	{2, 3, 13, 59, 139, 307}
17/22	{3, 7, 19, 23, 199}	{2, 3, 5, 43}
19/22	{3, 5, 11, 181, 199, 331}	{2, 3, 5, 11, 43, 131}
21/22	{3, 5, 7, 37, 127, 463}	{2, 3, 5, 7, 13, 167, 461}
1/24	{37, 73}	{23}
5/24	{7, 37, 73}	{5, 41, 139, 223, 239, 479}
7/24	{5, 37, 73}	{3, 41, 139, 223, 239, 479}
11/24	{5, 7, 37, 73}	{2, 11, 31, 223, 263, 461}
13/24	{3, 37, 73}	{2, 5, 31, 223, 263, 461}
17/24	{3, 7, 37, 73}	{2, 3, 11, 29, 167, 419}
19/24	{3, 5, 37, 73}	{2, 3, 5, 29, 167, 419}
23/24	{3, 5, 7, 37, 73}	{2, 3, 5, 7, 13, 83}

Table 8 (Qing-Hu Hou): P_r^- and P_r^+ for $r \in (0, 1)$ with denominator 23

r	P_r^-	P_r^+
1/23	{29, 139, 1933}	{23, 643, 3863}
2/23	{13, 277}	{11, 367, 1103}
3/23	{11, 47, 139, 691}	{7, 229, 919}
4/23	{7, 139}	{5, 137}
5/23	{11, 13, 31, 1381}	{5, 19, 2069, 4139}
6/23	{5, 139, 277}	{3, 137, 367, 1103}
7/23	{5, 31, 61, 277, 1381}	{3, 19, 229}
8/23	{5, 13, 79, 599}	{2, 139, 229, 410, 1609}
9/23	{5, 11, 47, 61, 461, 1381}	{2, 17, 643, 1609, 4139}
10/23	{5, 11, 13, 691}	{2, 19, 29, 59, 827, 4139}
11/23	{5, 11, 13, 31, 139, 277, 1381}	{2, 11, 19, 137, 229}
12/23	{3, 47}	{2, 5, 47, 1103}
13/23	{3, 29, 47, 139, 1933}	{2, 5, 17, 181, 467, 1091, 1103, 4783}
14/23	{3, 11, 139, 691}	{2, 5, 17, 29, 89, 139, 643}
15/23	{3, 11, 31, 61, 461}	{2, 5, 19, 29, 41, 59, 83, 137, 139, 643, 1931}
16/23	{3, 7, 47, 139}	{2, 3, 11, 47, 137, 1103}
17/23	{3, 7, 23, 67, 139, 277, 1013}	{2, 3, 11, 19, 59, 229, 827, 4139}
18/23	{3, 5, 47, 139, 277}	{2, 3, 5, 31, 1103, 2207}
19/23	{3, 7, 11, 31, 47, 277, 1381}	{2, 3, 5, 13, 229, 5519, 7727}
20/23	{3, 5, 11, 61, 461, 1381}	{2, 3, 5, 11, 29, 367, 5519}
21/23	{3, 5, 11, 31, 61, 139, 277, 461}	{2, 3, 5, 11, 13, 137, 1103, 7727}
22/23	{3, 5, 11, 13, 47, 691}	{2, 3, 5, 7, 17, 71, 137, 229, 2069}

Table 9: P_r^- and P_r^+ for $r \in (0, 1)$ with denominator 25

r	P_r^-	P_r^+
1/25	{31, 151}	{59, 149, 167, 223, 239, 479}
2/25	{29, 61, 71, 151, 241, 401}	{17, 139, 167, 199, 251, 419}
3/25	{13, 71, 151, 211, 241, 337, 421, 701}	{11, 53, 149, 179, 269, 449}
4/25	{11, 29, 101, 181, 271, 379, 421}	{7, 53, 167, 251, 269, 349}
6/25	{7, 29, 43, 211, 281, 337, 401}	{7, 17, 149, 197, 263, 439}
7/25	{5, 109, 151, 211, 241, 379, 401}	{3, 71, 197, 199, 263, 439}
8/25	{5, 23, 113, 241, 337, 401, 421, 463, 701}	{3, 19, 139, 149, 251, 449}
9/25	{5, 13, 101, 127, 271, 379, 421}	{2, 103, 167, 233, 251, 349}
11/25	{5, 7, 109, 211, 241, 379, 401}	{2, 11, 103, 149, 233, 359}
12/25	{5, 7, 29, 71, 151, 241, 401}	{2, 7, 103, 179, 233, 449}
13/25	{5, 7, 13, 109, 379, 401, 433, 541, 701}	{2, 7, 103, 179, 233, 359}
14/25	{3, 29, 71, 101}	{2, 5, 23, 89, 199, 449}
16/25	{3, 11, 31, 151}	{2, 3, 23, 139, 199, 349}
17/25	{3, 11, 17, 151, 157, 401, 521}	{2, 3, 11, 139, 251, 449}
18/25	{3, 7, 29, 131, 401, 433, 541, 547, 701}	{2, 3, 11, 23, 149, 199}
19/25	{3, 5, 101}	{2, 3, 7, 23, 139, 349}
21/25	{3, 5, 13, 151}	{2, 3, 5, 13, 83, 149}
22/25	{3, 5, 11, 61, 151, 241, 401}	{2, 3, 5, 11, 23, 199}
23/25	{3, 5, 11, 19, 181, 379, 401, 433, 701}	{2, 3, 5, 7, 29, 149, 199}
24/25	{3, 5, 7, 41, 101, 241, 433, 541}	{2, 3, 5, 7, 17, 41, 349, 359}

Table 10: P_r^- and P_r^+ for $r \in (0, 1)$ with denominators among 26 and 28

r	P_r^-	P_r^+
1/26	{71, 127, 211, 271, 313, 379, 521}	{67, 139, 181, 263, 311, 461, 509}
3/26	{13, 71, 157, 241, 337, 421, 547}	{11, 67, 197, 233, 263, 439, 509}
5/26	{7, 109, 181, 271, 337, 433, 547}	{5, 109, 181, 263, 307, 439, 571}
7/26	{7, 13, 131, 229, 313, 457, 571}	{5, 11, 139, 251, 311, 359, 467}
9/26	{5, 17, 53, 137, 337, 443, 547}	{3, 13, 83, 233, 311, 359, 389}
11/26	{5, 11, 19, 157, 181, 271, 541}	{2, 13, 167, 233, 263, 461, 467}
15/26	{3, 19, 103, 239, 307, 443, 547}	{2, 5, 17, 79, 239, 389, 467}
17/26	{3, 11, 29, 113, 241, 313, 547}	{2, 3, 17, 181, 233, 311, 503}
19/26	{3, 7, 23, 131, 157, 421, 463}	{2, 3, 7, 67, 311, 389, 509}
21/26	{3, 5, 29, 79, 281, 313, 421}	{2, 3, 5, 23, 89, 359, 467}
23/26	{3, 5, 11, 61, 79, 313, 521}	{2, 3, 5, 7, 233, 311, 467}
25/26	{3, 5, 7, 37, 73, 313}	{2, 3, 5, 7, 11, 311}
1/28	{29}	{41, 83}
3/28	{13, 43}	{11, 41}
5/28	{13, 17, 43, 113}	{5, 83}
9/28	{5, 17, 113}	{3, 13}
11/28	{5, 13, 29, 43}	{2, 19, 179, 251}
13/28	{5, 7, 31, 71}	{2, 7, 167}
15/28	{3, 29}	{2, 5, 131, 223}
17/28	{3, 13, 43}	{2, 3, 41}
19/28	{3, 13, 17, 43, 113}	{2, 3, 11, 83}
23/28	{3, 5, 17, 113}	{2, 3, 5, 13}
25/28	{3, 5, 13, 29, 43}	{2, 3, 5, 7, 71, 251}
27/28	{3, 5, 7, 31, 71}	{2, 3, 5, 7, 11, 167}

Table 11: P_r^- and P_r^+ for $r \in (0, 1)$ with denominators among 27 and 30

r	P_r^-	P_r^+
1/27	{37, 109}	{43, 107, 197}
2/27	{29, 67, 127, 211, 271, 337, 433, 661}	{17, 53}
4/27	{11, 37, 157, 211, 271, 281, 521}	{7, 71, 107}
5/27	{11, 17, 109, 211, 241, 379, 541}	{5, 53}
7/27	{7, 17, 61, 211, 241, 379, 541}	{3, 107}
8/27	{5, 37, 181, 211, 271, 379, 541}	{3, 41, 53, 251}
10/27	{5, 11, 157, 211, 271, 281, 521}	{2, 41, 107, 251}
11/27	{5, 11, 29, 101, 151, 379, 421}	{2, 17, 53}
13/27	{5, 7, 23, 127, 181, 271, 463}	{2, 7, 71, 107}
14/27	{5, 7, 13, 109, 211, 379, 541}	{2, 5, 53}
16/27	{3, 17, 61, 211, 241, 379, 541}	{2, 3, 107}
17/27	{3, 11, 61, 211, 271, 379, 541}	{2, 3, 29, 107, 269}
19/27	{3, 7, 43, 241, 271, 337, 421}	{2, 3, 11, 29, 269}
20/27	{3, 7, 19, 109, 211, 379, 541}	{2, 3, 7, 53, 71}
22/27	{3, 5, 23, 127, 181, 271, 463}	{2, 3, 5, 17, 107}
23/27	{3, 5, 13, 109, 211, 379, 541}	{2, 3, 5, 11, 53}
25/27	{3, 5, 11, 17, 113, 379, 541}	{2, 3, 5, 7, 23, 107}
26/27	{3, 5, 7, 29, 181, 379, 421}	{2, 3, 5, 7, 17, 53, 71}
1/30	{31}	{29}
7/30	{7, 29, 61, 71}	{5, 17, 89}
11/30	{5, 11, 61}	{2, 29}
13/30	{5, 7, 61}	{2, 11, 59}
17/30	{3, 29, 61, 71}	{2, 5, 17, 89}
19/30	{3, 11, 31}	{2, 3, 19}
23/30	{3, 5, 61}	{2, 3, 5, 59}
29/30	{3, 5, 7, 29, 71}	{2, 3, 5, 7, 19, 23}

Table 12 (Qing-Hu Hou): P_r^- and P_r^+ for $r \in (0, 1)$ with denominator 29

r	P_r^-	P_r^+
1/29	{59, 61, 1741}	{31, 347, 4639, 6959}
2/29	{17, 241, 661, 1277}	{19, 59, 463, 6959}
3/29	{13, 59, 349}	{11, 59, 347, 1913, 19139}
4/29	{11, 31, 233, 4931, 11833}	{11, 19, 347, 811, 2029}
5/29	{7, 211, 1741, 2437}	{5, 173}
6/29	{7, 29, 281, 2437, 2521, 7309}	{5, 29, 271, 509, 1217, 4079, 7307, 17747}
7/29	{7, 19, 59, 523}	{5, 13, 521, 811, 7307}
8/29	{5, 41, 1451, 5801}	{5, 11, 47, 347, 463}, {3, 43, 347, 4871, 17863}
9/29	{5, 19, 233, 2089}	{3, 17, 347, 521}
10/29	{5, 13, 97, 929}	{2, 139, 419, 521, 18269}, {3, 19, 29, 89, 2609}
11/29	{5, 13, 37, 59, 1277, 5743}	{2, 23, 347, 811, 4871}, {3, 7, 347, 811, 4871}
12/29	{5, 13, 19, 79, 131, 349, 1171, 1741, 11311}	{2, 13, 167, 347, 463}
13/29	{5, 13, 19, 37, 59, 73, 2089}	{2, 11, 47, 173, 347, 463}
14/29	{5, 11, 13, 41, 61, 233, 349, 1741}	{2, 11, 17, 173, 347, 521}
15/29	{3, 59}	{2, 11, 19, 29, 89, 173, 2609}, {3, 5, 19, 29, 89, 173, 2609}
16/29	{3, 31, 59, 929, 13921}	{2, 5, 19, 811, 2029}
17/29	{3, 13, 349}	{2, 3, 347}
18/29	{3, 11, 59, 349, 1741}	{2, 3, 43, 131, 173, 811, 13397}
19/29	{3, 11, 23, 199, 349, 991, 1277}	{2, 3, 19, 59, 347, 463, 6959}
20/29	{3, 7, 67, 233, 419, 1103, 4409}	{2, 3, 11, 47, 463}
21/29	{3, 7, 19, 523}	{2, 3, 11, 17, 521}
22/29	{3, 5, 181, 349, 8353, 13921}	{2, 3, 5, 173, 347}
23/29	{3, 5, 41, 59, 1451, 5801}	{2, 3, 5, 23, 811, 4871}
24/29	{3, 5, 19, 59, 233, 2089}	{2, 3, 5, 13, 251, 811, 1217, 7307}
25/29	{3, 5, 13, 43, 349, 547, 7541, 11311}	{2, 3, 5, 11, 47, 173, 463}
26/29	{3, 5, 13, 19, 233, 349, 2089}	{2, 3, 5, 11, 17, 173, 521}
27/29	{3, 5, 13, 19, 37, 73, 2089}	{2, 3, 5, 11, 13, 41, 463, 4871, 9743}
28/29	{3, 5, 11, 13, 41, 241, 349, 6961}	{2, 3, 5, 7, 11, 173, 811, 4871}

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