

Some integer sequences related to Pisot sequences*

by

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Dedicated to Ch. Pisot on the occasion of his retirement

1. Introduction. In his 1938 paper [6], Pisot initiated the study of the sequences of integers, now called Pisot sequences, defined by the condition

$$(1) \quad -1/2 < a_{n+1} - a_n^2/a_{n-1} \leq 1/2.$$

Given $0 < a_0 \leq a_1$, (1) determines a sequence $E(a_0, a_1)$. These sequences are modelled on sequences of real numbers of the type $u_n = \lambda \theta^n$ in the sense that for such a sequence $u_{n+1} = u_n^2/u_{n-1}$. Pisot showed in fact that if $a_1 > a_0 + (2a_0)^{1/2}$ then there are $\theta > 1$ and $\lambda > 0$ determined by $\lim (a_{n+1}/a_n) = \theta$ and $\lim (a_{n+1}^2/a_{n+1}) = \lambda$ such that $a_n = \lambda \theta^n + \varepsilon_n$, where ε_n is a bounded sequence with $\limsup |\varepsilon_n| \leq (\theta - 1)^{-2}/2$. The countable set of θ so produced is called E .

Although his motivation was the study of the distribution modulo 1 of sequences of the type $\lambda \theta^n$, Pisot also considered the question of whether the numbers in E are algebraic. In this regard, he showed that $E(2, a_1)$ and $E(3, a_1)$ satisfy linear recurrence relations and that $\theta(2, a_1)$ and $\theta(3, a_1)$ are Pisot numbers. Cantor [4] has recently given an extensive generalization of this result.

In general, if $E(a_0, a_1)$ satisfies a linear recurrence relation, then θ is either a Pisot or Salem number (or $\theta = 1$), as shown by Flor [5]. Cantor [3] proposed using Pisot sequences to search for Salem numbers, but cast doubt on the possibility that all Pisot sequences satisfy linear recurrence relations by his observation that $E(4, 13)$ satisfies no such relation of degree ≤ 100 . The question was settled in [1], by showing that the set of θ in E corresponding to non-recurrent sequences is dense in $[\tau, \infty)$, where $\tau = (5^{1/2} + 1)/2$.

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It is possible that the techniques of [1] could yield positive results about the Salem numbers. For example, if an interval (a, b) , with $1 \leq a < b$, could be found such that $a < \theta(a_0, a_1) < b$ implies $E(a_0, a_1)$ is non-recurrent, then it would follow that $T \cap (a, b) = \emptyset$ and hence that $\inf T > 1$, by a familiar argument. One difficulty with this approach is that the methods of [1] work best when $\theta > \tau$, whereas it is more hopeful to deal with intervals near 1. This is the motivation for the sequences we will introduce here. Now the methods of [1] will apply in a natural way to the entire interval $\theta > 1$.

We model our sequences on $u_n = \lambda\theta^n + \mu\theta^{-n}$. Noting that

$$(u_{n+2} + u_n)/u_{n+1} = \theta + \theta^{-1} = (u_{n+1} + u_{n-1})/u_n,$$

we define the F -sequence $F(a_0, a_1, a_2)$ to be the sequence of integers satisfying

$$(2) \quad -1/2 < a_{n+2} + a_n - (a_{n+1}/a_n)(a_{n+1} + a_{n-1}) \leq 1/2, \quad n \geq 1.$$

Thus a_{n+2} is uniquely determined unless $a_n = 0$. If this should happen for $n \geq 2$, a natural definition is to take $a_{n+2} = -a_{n-2}$.

It is important to realize that we may take $a_0 = 0$, so for example

$$F(0, 5, 11) = 0, 5, 11, 19, 30, 46, 70, 106, \dots$$

We call an F -sequence with $a_0 = 0$ a *special* F -sequence. (This is an appropriate point to mention that Cantor [4] uses the term F -sequence for a quite different concept.)

In Section 2, we will show, under fairly general assumptions, that an F -sequence satisfies $a_n = \lambda\theta^n + \varepsilon_n$ for some $\lambda > 0$, $\theta > 1$, where ε_n is a bounded sequence. Note that the motivating parameter μ is not identifiable since an arbitrary term of the form $\mu\theta^{-n}$ can be added without affecting the asymptotic behaviour of ε_n . The advantage of (2) over (1), for our purposes, is the appearance of an extra factor $(E - \theta^{-1})$ in the "characteristic inequality" (23) for ε_n .

It should be mentioned that Pisot [6] studied sequences of integers modelled on quite general functions of r parameters. In his case, the parameters are all determinable from the asymptotic behaviour of a_n , so our F -sequences do not quite fit into his framework.

In Section 4, we discuss the relation between F -sequences and the Pisot and Salem numbers. In Section 5, we apply the methods of [1] to deduce a criterion for an F -sequence to satisfy a T -recurrence; some examples are discussed there. Finally, in Section 6, we list a few other related sequences.

2. Geometric F -sequences

DEFINITION. We say that an F -sequence is *geometric* if $a_n > 0$ eventually, and $\liminf(a_{n+1}/a_n) > 1$.

The following three lemmas generalize results of [6] and [4]:

LEMMA 1. Let a_n be a geometric F -sequence. Then $\lim(a_{n+1}/a_n) = \theta > 1$, $\lim(a_{n+1}^n/a_{n+1}) = \lambda > 0$, and $\varepsilon_n = a_n - \lambda\theta^n$ is a bounded sequence.

Proof. Write $\theta_n = a_{n+1}/a_n$ and $\lambda_n = a_{n+1}^n/a_{n+1} = a_n\theta_n^{-n} = a_{n+1}\theta_n^{-n-1}$. We begin by showing $b_n = a_{n+1} - a_n^2/a_{n-1}$ is a bounded sequence. Given $1 < \alpha < \liminf \theta_n$, we have $\theta_n \geq \alpha$ for $n \geq m$, say. By (2), $|b_{n+1} - \theta_{n+1}^{-1}b_n| \leq 1/2$ and hence $|b_{n+1}| \leq \alpha^{-1}|b_n| + 1/2$ for $n > m$. Iterating this inequality, we have

$$(3) \quad |b_{n+1}| \leq \alpha^{m-n}|b_{m+1}| + \alpha/(2(\alpha-1)) \\ \leq |b_{m+1}| + \alpha/(2(\alpha-1)) = c, \quad \text{for } n \geq m.$$

But $b_{n+1} = a_{n+1}(\theta_{n+1} - \theta_n)$, hence (3) shows that

$$(4) \quad |\theta_{n+1} - \theta_n| \leq c/a_{n+1} \leq c\alpha^{m-n-1}/a_m \quad \text{for } n \geq m.$$

This shows that θ_n converges to θ , say, where $\theta > 1$ and

$$(5) \quad |\theta - \theta_n| \leq c\alpha^{-n}(a-1)^{-1} \quad \text{for } n \geq m.$$

Now write

$$(6) \quad |\lambda_{n+1} - \lambda_n| = a_{n+1}|\theta_{n+1}^{-n-1} - \theta_n^{-n-1}| \leq (n+1)\alpha^{-n-2}a_{n+1}|\theta_{n+1} - \theta_n| \\ = (n+1)\alpha^{-n-2}|b_{n+1}| \leq c(n+1)\alpha^{-n-2},$$

so λ_n converges to λ , say, and it will be obvious later that $\lambda > 0$.

We next show that $a_n\theta^{-n} \rightarrow \lambda$. Multiply (5) by $a_n\theta^{-n-1}$ to get

$$(7) \quad |a_n\theta^{-n} - a_{n+1}\theta^{-n-1}| \leq c\theta^{-n-1}(a-1)^{-1},$$

which shows $a_n\theta^{-n}$ converges to l , say. But $l = \lambda$, since

$$(8) \quad |a_n\theta^{-n} - a_n\theta_n^{-n}| \leq na_n\alpha^{-n-1}|\theta - \theta_n| \leq cn\alpha^{-n-1}(a-1)^{-1},$$

by (5). Thus (7) implies

$$(9) \quad |a_n\theta^{-n} - \lambda| \leq c\theta^{-n}(\theta-1)^{-1}(a-1)^{-1},$$

that is

$$(10) \quad |\varepsilon_n| \leq c(\theta-1)^{-1}(a-1)^{-1}.$$

Since $a_n \rightarrow \infty$, and ε_n is bounded, it is clear that $\lambda > 0$.

LEMMA 2. Suppose $a_1 > 0$ and $a_2 > 2a_1$, then the special F -sequence $F(0, a_1, a_2)$ is geometric. Furthermore, the set of limits θ from such sequences is dense in $[1, \infty)$.

Proof. We shall prove by induction that for all $n \geq 1$,

$$(11) \quad a_{n+1} - 2a_n + a_{n-1} \geq n \quad \text{and} \quad a_{n+1}/a_n \geq (n+1)/n.$$

For $n = 1$, this has been assumed. From (2), assuming (11), we have

$$(12) \quad a_{n+2} + a_n \geq (a_{n+1}/a_n)(a_{n+1} + a_{n-1}) - 1/2 \\ \geq (a_{n+1}/a_n)(2a_n + n) - 1/2 \geq 2a_{n+1} + n + 1 - 1/2,$$

which proves $a_{n+2} - 2a_{n+1} + a_n \geq n+1$, since the left member of (12) is an integer. This in turn implies that

$$a_{n+2}/a_{n+1} \geq 2 - a_n/a_{n+1} + (n+1)/a_{n+1} > 2 - n/(n+1) = (n+2)/(n+1),$$

so the induction is complete.

Summing (11) twice gives

$$(13) \quad a_n \geq \binom{n+1}{3} + na_1 > \binom{n+1}{3}.$$

By (2), we have

$$(14) \quad |(a_{n+2} + a_n)/a_{n+1} - (a_{n+1} + a_{n-1})/a_n| \leq (2a_{n+1})^{-1}.$$

Summing (14) and using (13) shows that $(a_{n+1} + a_{n-1})/a_n \rightarrow \varrho$, with

$$(15) \quad |\varrho - (a_{n+1} + a_{n-1})/a_n| < 3/(2n(n+1)).$$

By (11), we have

$$(16) \quad (a_{n+1} + a_{n-1})/a_n - 2 \geq n/a_n,$$

so that $\varrho \geq 2$. But, if $\varrho = 2$, then (15) and (16) imply that

$$(17) \quad a_n \geq 2n^2(n+1)/3 > 4\binom{n+1}{3}.$$

This improvement of (13) can be used as above giving a corresponding improvement of (15), and by induction $a_n > 4^k \binom{n+1}{3}$ for all k , which is not possible.

Since $\lim((a_{n+1} + a_{n-1})/a_n) > 2$ and $a_{n-1}/a_n < 1$, it follows that $\liminf(a_{n+1}/a_n) > 1$ so a_n is geometric.

To see that the set of θ is dense in $[1, \infty)$, it suffices to prove

$$(18) \quad |(\theta + \theta^{-1}) - a_2/a_1| < (\log a_1)/(4a_1) + 3/(2(a_1 - a_1^{1/2})).$$

To do this, we have from (13) that $a_n > \max\left(na_1, \binom{n+1}{3}\right)$. Hence if $m = [a_1^{1/2}]$, (14) implies

$$(19) \quad |\varrho - a_2/a_1| < \sum_{n=2}^m (2na_1)^{-1} + \sum_{n=m+1}^{\infty} 3/((n+1)n(n-1)) < \log m/(2a_1) + 3/(2m(m+1)),$$

which implies (18).

LEMMA 3. Suppose that $a_2 - 2a_1 + a_0 > (2a_1)^{1/2}$ and $a_1 \geq a_0$. Then $F(a_0, a_1, a_2)$ is geometric. Furthermore $d_n = a_{n+1} - 2a_n + a_{n-1} > (2a_n)^{1/2}$ for all n , and

$$(20) \quad |(\theta + \theta^{-1}) - (a_{n+1} + a_{n-1})/a_n| \leq 1/(d_n + (d_n^2 - 2a_n)^{1/2}).$$

Proof. By the assumptions, there is a $c > 0$ satisfying

$$(21) \quad d_n = a_{n+1} - 2a_n + a_{n-1} \geq a_n/(2c) + c,$$

for $n = 1$; namely $c = a_1/(d_1 + (d_1^2 - 2a_1)^{1/2})$. We shall prove by induction that (21) and $a_n \geq a_{n-1}$ hold for all n . For, by (2)

$$(22) \quad d_{n+1} \geq (a_{n+1}/a_n)d_n - 1/2 \geq (a_{n+1}/a_n)(a_n/(2c) + c) - 1/2 \geq a_{n+1}/(2c) + (a_{n+1}/a_n)c - 1/2.$$

However, $a_n \geq a_{n-1}$ so that (21) implies $a_{n+1} \geq (1 + (2c)^{-1})a_n + c$, and this combines with (22) to show (21) for d_{n+1} . It also shows that $a_{n+1} \geq a_n$ so the induction is complete.

By Lemma 1, $a_{n+1}/a_n \rightarrow \theta > 1$. We obtain (20) by using (14) and $a_k \geq (1 + (2c)^{-1})^{k-n} a_n$ for $k \geq n$, where now we can use $c = a_n/(d_n + (d_n^2 - 2a_n)^{1/2})$.

3. The characteristic inequality

THEOREM 1. (a) Suppose $a_n = \lambda\theta^n + \varepsilon_n$ is a geometric F -sequence. Then

$$(23) \quad \limsup |(E - \theta)^2 (E - \theta^{-1}) \varepsilon_n| \leq 1/2,$$

where $E\varepsilon_n = \varepsilon_{n+1}$ denotes the (backward) shift operator.

(b) Conversely, if $a_n = \lambda\theta^n + \varepsilon_n$ with $\lambda > 0$ and $\theta > 1$ and ε_n bounded, and if

$$(24) \quad \limsup |(E - \theta)^2 (E - \theta^{-1}) \varepsilon_n| < 1/2,$$

then a_n satisfies (2) for all sufficiently large n .

(c) The inequality (23) implies

$$(25) \quad \limsup |\varepsilon_n| \leq \theta/(2(\theta - 1)^2),$$

and (24) is implied by

$$(26) \quad \limsup |\varepsilon_n| < \theta/(2(\theta + 1)^3).$$

Proof. Write $D_n(a) = a_n(a_{n+2} + a_n) - a_{n+1}(a_{n+1} + a_{n-1})$. Then (2) implies $|D_n(a)| \leq a_n/2$. But

$$D_n(a) = \lambda\theta^n\{(E - \theta)^2 (E - \theta^{-1}) \varepsilon_{n-1}\} + D_n(\varepsilon),$$

which immediately implies (a) and (b).

To prove (c), we write $(E - \theta)^2 (E - \theta^{-1}) \varepsilon_n = \zeta_n$, so that $\limsup |\zeta_n| \leq 1/2$. Since E is an operator of norm 1 on bounded sequences, we can immediately conclude that $E - \theta = -\theta(1 - \theta^{-1}E)$ is invertible and

$$(E - \theta)(E - \theta^{-1}) \varepsilon_n = -(\theta^{-1} + \theta^{-2}E + \dots)\zeta_n,$$

so that

$$\limsup |(E - \theta)(E - \theta^{-1}) \varepsilon_n| \leq (\theta - 1)^{-1} \limsup |\zeta_n| \leq (\theta - 1)^{-1}/2.$$

A repetition of this gives $\limsup |(E - \theta^{-1})\varepsilon_n| \leq 1/(2(\theta - 1)^2)$. The operator $E - \theta^{-1}$ is not invertible but has a one-dimensional null space. Thus if $\varepsilon_0 = \mu$ is prescribed, we may solve $\varepsilon_{n+1} - \theta^{-1}\varepsilon_n = \eta_n$, say by successive substitution to obtain

$$\varepsilon_{n+1} = \eta_n + \theta^{-1}\eta_{n-1} + \dots + \theta^{-n}\eta_0 + \mu\theta^{-n-1},$$

from which $\limsup |\varepsilon_n| \leq \theta/(\theta - 1) \limsup |\eta_n|$ follows.

The proof that (26) implies (24) is straightforward.

Remarks. The inequality for E -sequences corresponding to (23) is

$$(27) \quad \limsup |(E - \theta)^2 \varepsilon_n| \leq 1/2$$

which implies Pisot's result that $\limsup |\varepsilon_n| \leq 1/(2(\theta - 1)^2)$, as above.

4. Linear recurrence relations satisfied by F -sequences. This section contains generalizations of some of Flor's [5] results to F -sequences. The proof of Theorem 2 is familiar but is included for completeness. In Theorem 3, we use Dirichlet's theorem rather than Minkowski's lemma to obtain a somewhat more constructive result.

THEOREM 2. Suppose $a_n = \lambda\theta^n + \varepsilon_n$ is a geometric F -sequence which satisfies a linear recurrence relation. Then θ is a Pisot or Salem number.

Conversely, if θ is a Pisot or Salem number then there is an F -sequence with $\theta = \lim(a_{n+1}/a_n)$.

Proof. If a_n has generating function $f(z) = A(z)/Q(z)$, with $Q(0) = 1$, then Fatou's lemma implies that Q has integer coefficients. Thus if $P(z) = z^k Q(z^{-1})$, $k = \deg Q$, then the roots of P are algebraic integers. Clearly f has a simple pole at $z = \theta^{-1}$, since $a_n = \lambda\theta^n + \varepsilon_n$. Since ε_n is bounded the remaining poles are outside or on the unit circle with those on the circle being simple. Thus P has exactly one root θ outside the unit circle which thus must be in S or T . The roots inside the circle must be conjugates of θ hence are also simple roots. The roots on the circle are either roots of unity or conjugates of θ , if θ is in T .

Conversely if θ is in S or T , then one can find λ in $Q(\theta)$ so that $\|\lambda\theta^n\| < \theta(\theta+1)^{-3}/2$ for all $n \geq 0$, and hence by Theorem 1, $a_n = \text{Tr}(\lambda\theta^n)$ eventually satisfies (2).

THEOREM 3. Let θ be a Salem number or a reciprocal quadratic Pisot number. Then there is a special F -sequence $F(0, a_1, a_2)$ with $\lim(a_{n+1}/a_n) = \theta$.

Proof. Let $T(z)$, of degree $2m$, be the minimal polynomial of θ . Then $T(z)$ is reciprocal, and we may write $T(z) = z^m R(\zeta)$, with $\zeta = z + z^{-1}$. We will construct a polynomial $B(\zeta)$ of degree $m-1$ so that if $A(z) = z^{m-1} B(\zeta)$, then $zA(z)/T(z)$ will be the generating function of a special F -sequence, i.e.

$$(28) \quad zA(z)/T(z) = a_1 z + a_2 z^2 + \dots$$

Let $\theta, \theta^{-1}, \alpha_2, \bar{\alpha}_2, \dots, \bar{\alpha}_m$ be the zeros of $T(z)$ so that the real numbers $\varrho = \theta + \theta^{-1}, \varrho_2 = \alpha_2 + \bar{\alpha}_2, \dots, \varrho_m = \alpha_m + \bar{\alpha}_m$ are the zeros of $R(\zeta)$. The relation (28) implies

$$(29) \quad a_n = \lambda\theta^n + \mu\theta^{-n} + \sum_{k=2}^m (\beta_k \alpha_k^n + \bar{\beta}_k \bar{\alpha}_k^n),$$

where

$$(30) \quad \lambda = -\theta A(\theta^{-1})/T'(\theta^{-1}) = (\theta - \theta^{-1})^{-1} B(\varrho)/R'(\varrho) = -\mu,$$

while

$$(31) \quad \beta_k = (\alpha_k - \bar{\alpha}_k)^{-1} B(\varrho_k)/R'(\varrho_k) = -\bar{\beta}_k,$$

so β_k is pure imaginary.

Writing (29) as $a_n = \lambda\theta^n - \lambda\theta^{-n} + \delta_n$, we see from (2) that the condition to be satisfied is

$$(32) \quad |(E - \theta)^2 (E - \theta^{-1}) \delta_{n-1} - \theta^{-2n} (E - \theta) (E - \theta^{-1})^2 \delta_{n-1} + D_n(\delta) \lambda^{-1} \theta^{-n}| < 2^{-1} (1 - \theta^{-2n} + \lambda^{-1} \theta^{-n} \delta_n), \quad \text{for } n \geq 1,$$

using the notation of Theorem 1. This can clearly be satisfied by making $\beta_2, \dots, \beta_m$ sufficiently small (thus forcing λ to be large). To be more explicit, let

$$R_\varrho(\zeta) = R(\zeta)/(\zeta - \varrho) = \zeta^{m-1} + c_1 \zeta^{m-2} + \dots + c_{m-1},$$

where $c_1, \dots, c_{m-1}$ are certain real numbers. By Dirichlet's theorem, there are arbitrarily large integers a_1 and $n_1, \dots, n_{m-1}$ so that

$$|a_1 c_k - n_k| < a_1^{-1/(m-1)} \quad \text{for } k = 1, 2, \dots, m-1.$$

Let

$$B(\zeta) = a_1 \zeta^{m-1} + n_1 \zeta^{m-2} + \dots + n_{m-1},$$

and then determine $\lambda, \dots, \beta_m$ by (30) and (31). Then $R_\varrho(\varrho_k) = 0$ so

$$(33) \quad |B(\varrho_k)/R'(\varrho_k)| = |(B(\varrho_k) - a_1 R_\varrho(\varrho_k))/R'(\varrho_k)| < a_1^{-1/(m-1)} 2^m / R'(\varrho_k).$$

By taking a_1 sufficiently large, $\beta_2, \dots, \beta_m$ may be made as small as desired. Note that $\lambda \sim a_1/(\theta - \theta^{-1})$.

EXAMPLE. The F -sequence $F(0, 5, 11)$ can be shown to be of the type discussed in Theorem 3, with $A(z) = 5 + 6z + 8z^2 + 6z^3 + 5z^4$, and $T(z) = 1 - z - z^3 - z^5 + z^6$, so $a_{n+1}/a_n \rightarrow \theta = 1.5061356795$, a Salem number. In this case $B(\zeta) = 5\zeta^2 + 6\zeta - 2$, while $5R_\varrho(\zeta) \simeq 5\zeta^2 + 5.85\zeta - 2.30$. Using (31), one verifies that $|\beta_2| = .053260$ and $|\beta_3| = .009382$. To check (32), one needs

$$(E - \theta)^2 (E - \theta^{-1}) \delta_n = 2 \text{Re} \left\{ \sum_{k=2}^m \beta_k (\alpha_k - \theta) (\varrho_k - \varrho) \alpha_k^{n-1} \right\},$$

so that

$$\gamma = \sup_n |(E - \theta)^2 (E - \theta^{-1}) \delta_n| = 2 \sum_{k=2}^m |\beta_k| |\alpha_k - \theta| (\varrho - \varrho_k).$$

We can estimate

$$\sup_n |(E - \theta)(E - \theta^{-1})^2 \delta_n| \leq \theta(\theta + 1)(\theta - 1)^{-1} \gamma.$$

In this case, $\gamma = .492015$ so that (32) is readily established for $n \geq 9$, and (2) can be verified directly for $n < 9$.

Remark. We have not attempted a complete investigation of non-geometric F -sequences. For symmetry, if a_n is allowed to be negative, the rule (2) should be modified to "round down" to the nearest integer if $a_{n+2} < 0$. The proof of Lemma 2 shows that if a_n exhibits a rate of growth greater than n^2 , then a_n is geometric. This is sharp since $a_n = \binom{n+k}{2}$ is an F -sequence for $k \geq 5$, since $a_{n+1} - 2a_n + a_{n-1} = 1$.

For special F -sequences, it may be true that a superlinear rate of growth implies a_n is geometric. By Lemma 2, $F(0, a, 2a + b)$ is geometric if $b > 0$, and clearly $F(0, a, 2a)$ is linear. But, for example, $F(0, 20, 39)$ is periodic (with period 26), while $F(0, 21, 41)$ tends to $-\infty$ linearly, after a single positive excursion.

5. A criterion for pure T -recurrence. The following result is proved using the techniques of [1]. The only basic change is that we now have (23) rather than (27), so we avoid the estimate

$$\limsup |(E - \theta)^2 (E - \theta^{-1}) \varepsilon_n| \leq (1 + \theta^{-1}) \limsup |(E - \theta)^2 \varepsilon_n|.$$

As in [1], we say that a_n satisfies a *pure recurrence* if its generating function A/Q has $\deg A < \deg Q$. We say it is *T -recurrent* if $Q(z) = T(z)K(z)$, where T is the minimal polynomial of a Salem number and K is cyclotomic with simple roots.

Using the estimates obtainable from Lemma 1, and the techniques of Theorem 1 of [1], it is possible to effectively deal with T -recurrences which are not pure, and thus show that there are F -sequences satisfying no linear recurrence whatsoever. However, because of Theorem 3, the following result is sufficient for the application suggested in the introduction. In fact, the convergence results of Section 2 are not really needed for this purpose, except to suggest that the method will be fairly efficient.

As in [1], we note that if $F(a_0, a_1, a_2)$ satisfies a pure T -recurrence then

$$(34) \quad a_n = \lambda \theta^n + \mu \theta^{-n} + \delta_n,$$

where δ_n is a linear combination of powers of numbers of modulus 1, and hence is almost periodic in n ([1], p. 92). The equation (34) may be used to define a_n for $n < 0$, and since the leading coefficient of Q is ± 1 , it follows that a_n is an integer for all n .

THEOREM 4. Suppose that $F(a_0, a_1, a_2)$ satisfies a pure T -recurrence corresponding to the Salem number θ . Then θ satisfies

$$(35) \quad |\theta + \theta^{-1} - (a_{n+1} + a_{n-1})/a_n| \leq 1/(2|a_n|(\theta - 1)) \quad \text{for all } n.$$

Furthermore, if $\Delta_\theta(x, y, z) = \{x\theta(2 + \theta^2) - y(1 + 2\theta^2) + z\theta\}/\theta^2$, then for all n , positive or negative,

$$(36) \quad |a_n - \Delta_\theta(a_{n+1}, a_{n+2}, a_{n+3})| \leq 1/(2\theta).$$

Proof. Since $\varepsilon_n = \mu\theta^{-n} + \delta_n$, we have $(E - \theta^{-1})\varepsilon_n = (E - \theta^{-1})\delta_n$, so (23) implies that $\limsup |(E - \theta)^2 (E - \theta^{-1}) \delta_n| \leq 1/2$, and since δ_n is almost periodic we have

$$(37) \quad |(E - \theta)^2 (E - \theta^{-1}) \delta_n| \leq 1/2 \quad \text{for all } n.$$

This implies, as in Theorem 1, that

$$(38) \quad |(E - \theta)(E - \theta^{-1}) \delta_n| \leq 1/(2(\theta - 1)) \quad \text{for all } n.$$

But $(E - \theta)(E - \theta^{-1})\delta_n = (E - \theta)(E - \theta^{-1})a_n$, so (38) implies (35). Similarly, (37) implies (36) since $(E - \theta)^2 (E - \theta^{-1})a_n = \theta(\Delta_\theta(a_{n+1}, a_{n+2}, a_{n+3}) - a_n)$.

APPLICATION. To test $F(a_0, a_1, a_2)$ for pure T -recurrence, one generates $a_1, \dots, a_N$ for large N , then uses (35) to estimate θ . For (35) to be effective, one needs a crude lower bound $\theta \geq \alpha > 1$, since $\theta = 1$ satisfies (35). This bound may be obtained from the results of Lemma 2 or Lemma 3, or from the estimate $|(E - \theta)^2 (E - \theta^{-1})a_n| \leq 1/2$, regarded as a polynomial in θ . Given $\theta \geq \alpha$, (35) allows θ to be computed with arbitrary (and known) accuracy. One uses this approximation in Δ_θ and then uses (36) to compute $a_{-1}, a_{-2}, \dots, a_{-M}$, which can be done accurately for an M which tends to infinity with N , since $1/(2\theta) < 1/2$. If (36) fails (to within the known accuracy) for any n , then $F(a_0, a_1, a_2)$ does not satisfy a pure T -recurrence.

For example $F(0, 30, 61)$ has $\theta = 1.184214547464$, and is not pure T -recurrent since (36) fails for $n = -8$. In fact, since $\theta < \min S$, it is not hard to show that $F(0, 30, 61)$ satisfies no recurrence whatsoever.

We used a combination of Theorem 4 and the method of Cantor [3] to test a large number of special F -sequences for pure T -recurrence. If a sequence passes the test (36) for a reasonably large interval of n , we use the algorithm of [3] to determine the possible recurrence relation. The ideas discussed for the example $F(0, 5, 11)$ can then be used to test the validity of the recurrence so determined.

For example $F(0, 185, 375)$ is recurrent with $\theta = \sigma_1$, the smallest known Salem number [2]. An interesting example is $F(0, 46, 95)$ which satisfies (36) for $-80 \leq n \leq -1$ and appears to have the generating function $zA(z)/T(z) = B(\zeta)/R(\zeta)$ where $T(z) = z^8 - z^5 - z^4 - z^3 + 1$, $B(\zeta) = 46\zeta^3 + 95\zeta^2 + 12\zeta - 22$, $R(\zeta) = \zeta^4 - 4\zeta^2 - \zeta - 1$, and $A(z) = z^3 B(z + z^{-1})$. Here $\theta = \sigma_{23}$ in the notation of [2]. However, computations show that $\sup |(E - \theta)^2 (E - \theta^{-1}) \varepsilon_n| = .525458$ so that $zA(z)/T(z)$ does not generate an F -sequence. On the other hand, if we write $a_n = \text{Tr}(\lambda \theta^n)$ then it may be shown that $\text{Tr}(\lambda \theta^n)$ and $\text{Tr}(\lambda \theta^{2n})$ are the F -sequences $F(0, 95, 196)$ and $F(0, 196, 404)$. It is perhaps worth recording that $196R_\theta(\zeta) = 196\zeta^3 + 404.05\zeta^2 + 48.96\zeta - 95.08$.

6. Some other sequences. Let $N(x) = [x + 1/2]$ denote the "nearest" integer to x . The following are some interesting integer sequences, their "typical" behaviour, and their characteristic inequalities:

- (a) $D(a_m, m)$: $a_{n+1} = N(a_n^{(n+1)/n})$, $n \geq m$,
 $a_n = \theta^n + \varepsilon_n$,
 $\limsup |(E - \theta) \varepsilon_n| \leq 1/2$.
- (b) $E(a_0, a_1)$: $a_{n+1} = N(a_n^2/a_{n-1})$, $n \geq 2$,
 $a_n = \lambda \theta^n + \varepsilon_n$,
 $\limsup |(E - \theta)^2 \varepsilon_n| \leq 1/2$.
- (c) $D'(a_{m-1}, a_m, m)$: $a_{n+1} = N(a_n^{(n+1)/n} + a_n^{(n-1)/n} - a_{n-1})$, $n \geq m$,
 $a_n = \theta^n + \theta^{-n} + \varepsilon_n$,
 $\limsup |(E - \theta)(E - \theta^{-1}) \varepsilon_n| \leq 1/2$.
- (d) $F(a_0, a_1, a_2)$: $a_{n+1} = N(a_n(a_n + a_{n-2})/a_{n-1} - a_{n-1})$, $n \geq 2$,
 $a_n = \lambda \theta^n + \mu \theta^{-n} + \varepsilon_n$,
 $\limsup |(E - \theta)^2 (E - \theta^{-1}) \varepsilon_n| \leq 1/2$.
- (e) $G(a_0, a_1, a_2, a_3)$:
 $a_{n+2} = N(\{a_{n+1}^2 + 2(a_{n+1}a_{n-1} - a_n^2) - (a_na_{n-2} - a_{n-1}^2)\}/a_n)$, $n \geq 2$,
 $a_n = \lambda \theta^n + \mu \theta^{-n} + \varepsilon_n$,
 $\limsup |(E - \theta)^2 (E - \theta^{-1})^2 \varepsilon_n| \leq 1/2$.

Of these, (a) and (b) were considered by Pisot [6]. In (a), it can be shown that a_n grows geometrically if and only if $a_m^{m+1} > (a_m + 1/2)^m$. The relation between the E -, F - and G -sequences is that in each case a certain finite difference of $a_{n+1}a_{n-1} - a_n^2$ is made as small as possible. An analogue of Theorem 2 applies to all these sequences, but for D and D' sufficiently large Salem numbers are ruled out by the fact that θ^n is dense modulo 1

if θ is a Salem number. An analogue of Theorem 4 can be proved for G -sequences but the analogue of (36) now gives no effective test for $n < 0$, since the final coefficient of the polynomial $(E - \theta)^2 (E - \theta^{-1})^2$ is 1.

A sequence of a somewhat different sort is obtained by analogy with the fact that the F -sequence recursion is

$$(a_{n+2} - 2a_{n+1} + a_n) = N((a_{n+1}/a_n)(a_{n+1} - 2a_n + a_{n-1})).$$

Namely

$$(f) \quad a_{n+2} = 2a_{n+1} + N((a_n/a_{n-1})(a_{n+1} - 2a_n)),$$

$$a_n = \lambda \theta^n + \mu(2 - \theta)^n + \varepsilon_n,$$

$$\limsup |(E - \theta)^2 (E + \theta - 2) \varepsilon_n| \leq 1/2.$$

In contrast to (c)–(e), the second term $\mu(2 - \theta)^n$ has a real meaning if $\theta > 2$. The set of limits θ includes, in addition to S and T , such quadratics as $1 + 5^{1/2}$ which is obtained from the sequence 4, 13, 42, 136, ...

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