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The Golden Ratio family and the Binet equation

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Abstract: The Golden Ratio can be considered as the first member of a family which can generate a set of generalized Fibonacci sequences. Here we consider some related problems in terms of the Binet form of these sequences, $\{F_n(a)\}$, where the sequence of ordinary Fibonacci numbers can be expressed as $\{F_n(5)\}$ in this notation. A generalized Binet equation can predict all the elements of the Golden Ratio family of sequences. Identities analogous to those of the ordinary Fibonacci sequence are developed as extensions of work by Filipponi, Monzingo and Whitford in *The Fibonacci Quarterly*, by Horadam and Subba Rao in the *Bulletin of the Calcutta Mathematical Society*, within the framework of Sloane's *Online Encyclopedia of Interger Sequences*.

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1 Introduction

It can be shown that the Golden Ratio [10] may be considered as the first member of a family which can generate a set of generalized Fibonacci sequences [8]. Here we relate the ideas there to the work of Filipponi [2], Monzingo [12] and Whitford [24] to consider some related problems with their common thread being the Binet form of these sequences, $\{F_n(a)\}$, where the sequence of ordinary Fibonacci numbers can be expressed as $\{F_n(5)\}$ in this notation. Thus, for instance

$$\frac{F_n(a)}{F_{n-1}(a)} \to \varphi_a \tag{1.1}$$

in which

$$\varphi_a = \frac{1+\sqrt{a}}{2} \tag{1.2}$$

and the generalized Binet formula in this notation is

$$F_n(a) = \frac{\left(\frac{1+\sqrt{a}}{2}\right)^n - \left(\frac{1-\sqrt{a}}{2}\right)^n}{\sqrt{a}}. \tag{1.3}$$

which is well-known for the Fibonacci numbers as

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}. \tag{1.4}$$

Hence, elements of the sequences in the family should be similarly predicted. We note in passing that the Binet formula for the Fibonacci numbers is usually attributed to Jacques Philippe Marie Binet (1786–1856), but it was previously known to such famous mathematicians as Abraham de Moivre (1667–1754), Daniel Bernoulli (1700–1782), and Leonhard Euler (1707–1783): “like many results in Mathematics, it is often not the original discoverer who gets the glory of having their name attached to the result, but someone later!” [6].

2 Differences of squares

When n in Equation (1.3) is a power of 2 we can start to develop identities analogous to those of the Fibonacci sequence. For example,

$$x^{2n} - y^{2n} = (x^n - y^n)(x^n + y^n) \tag{2.1}$$

can become

$$F_{2n}(a) = F_n(a)L_n(a) \tag{2.2}$$

in which $L_n(a)$ is the corresponding generalized Lucas sequence. Both types of sequence satisfy the second order recurrence relation

$$u_n(a) = u_{n-1}(a) + r_1 u_{n-2}(a),\ n > 2, \tag{2.3}$$

where r_1 is in Class $\bar{1}_4 \in Z_4$ (a modular ring) [9] (Table 1). We shall use this then with $r_1 = [(a-1)/4]$ as an integer in the recurrence relations which follow

Row $r_i \downarrow$	**Class** $i \rightarrow$	$\bar{0}_4$	$\bar{1}_4$	$\bar{2}_4$	$\bar{3}_4$	**Comments**
0		0	1	2	3	$N = 4r_i + i$
1		4	5	6	7	even $\bar{0}_4$, $\bar{2}_4$
2		8	9	10	11	$(N^n, N^{2n}) \in \bar{0}_4$
3		12	13	14	15	odd $\bar{1}_4, \bar{3}_4$; $N^{2n} \in \bar{1}_4$

Table 1. Classes and rows for Z_4

We can continue the process in (2.1) to get

$$\begin{aligned} x^{2n} - y^{2n} &= \left(x^n + y^n\right)\left(x^n - y^n\right) \\ &= \left(x^n + y^n\right)\left(x^{\frac{n}{2}} + y^{\frac{n}{z}}\right)\left(x^{\frac{n}{2}} - y^{\frac{n}{2}}\right) \end{aligned} \tag{2.4}$$

and so on. For instance, when n = 4, this can be reduced to

$$x^8 - y^8 = \left(x^4 + y^4\right)\left(x^2 + y^2\right)\left(x + y\right)\left(x - y\right)$$

with $x + y = 1$ and $x - y = \sqrt{a}$, and when n = 8, this can be reduced to

$$x^{16} - y^{16} = \left(x^8 + y^8\right)\left(x^4 + y^4\right)\left(x^2 + y^2\right)\left(x + y\right)\left(x - y\right)$$

or

$$F_{16}(a) = L_8(a)L_4(a)L_2(a)\,,$$

which can be readily confirmed when a = 5. More generally,

$$\frac{x^n - y^n}{x - y} = \left(x^{n-1} + y^{n-1}\right) + \frac{xy}{x - y}\left(x^{n-2} - y^{n-2}\right) \tag{2.5}$$

can be expressed as

$$F_n(a) = L_{n-1}(a) + \left(\frac{1-a}{4}\right)F_{n-2}(a), \tag{2.6}$$

which, when a = 5 and n = 7, $F_7(5) = 13,$ and $L_6(5) - F_5(5) = 18 - 3.$

Equation (2.5) can be factorised further

$$\frac{x^n - y^n}{x - y} = \left(x^{n-1} + y^{n-1}\right) + \left(\frac{1-a}{4}\right)\left(x^{n-3} - y^{n-3}\right) + \left(\frac{1-a}{4}\right)^2\left(x^{n-5} - y^{n-5}\right) + \left(\frac{1-a}{4}\right)^3$$

This in turn can be re-written as

$$F_n(a) = L_{n-1}(a) + \left(\frac{1-a}{4}\right)L_{n-3}(a) + \left(\frac{1-a}{4}\right)^2 L_{n-5}(a) + \left(\frac{1-a}{4}\right)^3; \tag{2.7}$$

for instance,

$$F_7(5) = L_6(5) - L_4(5) + L_2(5) - 1.$$

3 Extensions of Whitford's results

Direct calculations of $\left(\frac{1+\sqrt{a}}{2}\right)^n$ and $\left(\frac{1-\sqrt{a}}{2}\right)^n$ as in the Binet equation (1.3) and from (2.6) yield the patterns set out in Table 2. Each n yields an infinity of 'golden ratios':

n	$L_n(a)$	$L_n(5)$	$L_n(13)$	$L_n(17)$
2	$\frac{1}{2^1}(a+1)$	3	7	9
4	$\frac{1}{2^3}(a^2+6a+1)$	7	31	49
6	$\frac{1}{2^5}(a^3+15a^2+15a+1)$	18	154	297
8	$\frac{1}{2^7}(a^4+28a^3+70a^2+28a+1)$	47	799	1889
n	$F_n(a)$	$F_n(5)$	$F_n(13)$	$F_n(17)$
3	$\frac{1}{2^2}(a+3)$	2	4	5
5	$\frac{1}{2^4}(a^2+10a+5)$	5	19	29
7	$\frac{1}{2^6}(a^3+21a^2+35a+7)$	13	97	181
9	$\frac{1}{2^8}(a^4+36a^3+126a^2+84a+9)$	34	508	1165

Table 2. Various Golden Ratio Sequences

That is, for example, as in (2.7):

$$\frac{1}{2^8}\left(a^4+36a^3++126a^2+84a+9\right)$$

so that

$$u_9(5)=34=F_9(5),\ u_9(13)=508=F_9(13),\ \ u_9(17)=1165=F_9(17),$$

in which the sequences satisfy the second order recurrence relation (2.3) in the form

$$u_n(a)=u_{n-1}(a)+\left(\frac{a-1}{4}\right)u_{n-2}(a),\ n>2, \tag{3.1}$$

with unity as the initial terms as in Whitford [24]. Some of the properties of particular forms of these sequences have been developed in [8]. Thus Simson's identity becomes

$$F_n(a)F_{n+2}(a)-F_{n+1}^2(a)=(-1)^{n+1}\left(\frac{a-1}{4}\right)^n \tag{3.2}$$

which had previously been proved by Lucas [11]. The work of Filipponi and Monzingo extended that of Whitford. In turn we can extend it further by considering the recurrence relation

$$v_n(a)=v_{n-1}(a)+4(a-n)v_{n-2}(a),\ n>2, \tag{3.3}$$

which with unit initial conditions again generates the sequence

$$\{v_n(a)\} \equiv \{1,1,4a-11,8a-27,16a^2-120a+209,48a^2-412a+830,...\},$$

$$\{v_n(0)\} \equiv \{1,1,-11,-27,193,...\},$$

$$\{v_n(4)\} \equiv \{1,1,5,5,-15,...\},$$

$$\{v_n(5)\} \equiv \{1,1,9,13,9,-30,...\},$$

which invite a separate study particularly in relation to negative signs and intersections [16]. Instead we shall briefly consider Whitford's table of sequences which we have slightly extended (Table 3).

a	$\frac{a-1}{4}$	$F_1(a)$	$F_2(a)$	$F_3(a)$	$F_4(a)$	$F_5(a)$	$F_6(a)$	$F_7(a)$	$F_8(a)$	$F_9(a)$	$F_{10}(a)$
1	**0**	1	1	1	1	1	1	1	1	1	1
5	**1**	1	1	2	3	5	8	13	21	34	55
9	**2**	1	1	3	5	11	21	43	85	171	341
13	**3**	1	1	4	7	19	40	97	217	508	1159
17	**4**	1	1	5	9	29	65	181	441	1165	2929
21	**5**	1	1	6	11	41	96	301	781	2286	6191
25	**6**	1	1	7	13	55	133	463	1261	4039	11605

Table 3. Whitford's table of Generalized Fibonacci numbers – extended

We note that Table 3 is Sloane's A083856 with many individual rows and columns also listed there, as is the sequence of forward diagonals {1, 2, 3, 5, 9, 17, 34, 71, ...} [A110113] and the first backward diagonals {1, 3, 7, 29, 99, 463, ...} [A171180].

We choose now to consider finite difference operators, $\Delta_{i.j}$, [14] acting on the sequences within these rows and columns (i, j) for various values of a and n (the position of an element within each sequence $\{F_n(a)\}$). We define sequence row difference operators $\Delta_{a,j}F_j(a)$, sequence column difference operators $\Delta_{i,k}F_k(a)$, and sequence vector difference operators $\Delta_{a,k}F_k(a)$ [20], respectively by

$$\Delta_{a,j}F_j(a) = F_j(a) - F_j(a-1) \tag{3.4}$$

and

$$\Delta_{i,k}F_k(i) = F_k(i) - F_{k-1}(i) \tag{3.5}$$

and

$$\Delta_{a,k}F_k(a) = F_k(a) - F_{k-1}(a-1) \tag{3.6}$$

We can then apply these operators to the rows, columns and forward and backward diagonals of the elements of Table 3 to find a variety of inter-related sequences, some expected, some unexpected, as we see in Tables 4, 5 and 6.

$\Delta_{i,k}F_k(a)$ →	$F_2(a)$	$F_3(a)$	$F_4(a)$	$F_5(a)$	$F_6(a)$	$F_7(a)$	$F_8(a)$	$F_9(a)$	$F_{10}(a)$
***a* = 1**	0	0	0	0	0	0	0	0	0
5	0	1	1	2	3	5	8	13	21
9	0	2	2	6	10	22	42	86	170
13	0	3	3	12	21	57	120	291	651
17	0	4	4	20	36	116	260	724	1764
21	0	5	5	30	55	205	480	1505	3905
25	0	6	6	42	78	330	798	2778	7566

Table 4. First column differences, $\Delta_{i,k}F_k(a)$, from Table 3

$\Delta^2_{i,k}F_k(a)$ →	$F_3(a)$	$F_4(a)$	$F_5(a)$	$F_6(a)$	$F_7(a)$	$F_8(a)$	$F_9(a)$	$F_{10}(a)$
***a* = 1**	0	0	0	0	0	0	0	0
5	1	0	1	1	2	3	5	8
9	2	0	4	4	12	20	44	84
13	3	0	9	9	36	63	171	360
17	4	0	16	16	80	144	464	1040
21	5	0	25	25	150	275	1025	2400
25	6	0	36	36	252	468	1980	4788

Table 5. Second column differences, $\Delta_{i,k}(\Delta_{i,k}F_k(a))$, from Table 4

If we take the second five rows associated with the last six columns of Table 5 we get the patterns displayed in Table 6 in which the obvious common factor (the squares) in the fourth column of Table 5 has been taken out of each of the rows.

a	$\frac{a-1}{4}$	Common factor	$F_3(a)$	$F_4(a)$	$F_5(a)$	$F_6(a)$	$F_7(a)$	$F_8(a)$	**a**

4 Concluding comments

Further related investigations can include

- establishing a general form for $F_n(a)$ as a polynomial in a as in Table 2;
- determining intersections of the generalized sequences in Table 4 [3, 16, 22, 23];
- exploring identities analogous to those of the ordinary Fibonacci and Lucas numbers [4, 6];
- using simple factorials to calculate generalized Pascal–Fibonacci numbers and Pascal-type triangles [1, 5, 7, 13];
- extending the ideas to third order recursive sequences [4, 16, 17, 18, 19];
- finding the missing patterns inherent in Table 4 where some of the sequences do not appear in the *OEIS* [21]);
- developing generating functions and polynomials for these generalizations [14, 15].

References

[1] Cook, C. K., & Shannon, A.G. (2006) Generalized Fibonacci and Lucas Sequences with Pascal-type Arrays. *Notes on Number Theory & Discrete Mathematics*. 12(4): 1–9.

[2] Filipponi, P. (1991) A Note on a Class of Lucas Sequences. *The Fibonacci Quarterly*. 29(3), 256–263.

[3] Horadam, A. F. (1966) Generalizations of Two Theorems of K. Subba Rao. *Bulletin of the Calcutta Mathematical Society*. 58, 23–29.

[4] Jarden, D. (1966) *Recurring Sequences*. Jerusalem: Riveon Lematematika.

[5] Kapur, J. N. (1976) Generalized Pascal's Triangles and Generalized Fibonacci Numbers. *Pure and Applied Mathematical Sciences*. 3(1–2), 93–100.

[6] Knott, R. (2015) Fibonacci Numbers and the Golden Section. Available online: `http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fib.html`.

[7] Leyendekkers, J. V., Shannon, A. G. (2013) The Pascal–Fibonacci Numbers. *Notes on Number Theory and Discrete Mathematics*. 19(3), 5–11.

[8] Leyendekkers, J. V., Shannon, A. G. (2015) The Golden Ratio Family and Generalized Fibonacci Numbers. *Journal of Advances in Mathematics*. 10(1), 3130–3137.

[9] Leyendekkers, J. V., Shannon, A. G., Rybak, J. M. (2007) *Pattern Recognition: Modular Rings and Integer Structure*. North Sydney: Raffles KvB Monograph No.9.

[10] Livio, Mario. 2002. The Golden Ratio. New York: Golden Books.

[11] Lucas, E. (1969) *The Theory of Simply Periodic Numerical Functions*. (Edited by D.A. Lind; translated by S. Kravitz.) San Jose: The Fibonacci Association.

[12] Monzingo, M. G. (1980) An Observation Concerning 'Binet's Formula Generalized'. In V.E. Hoggatt Jr, M. Bicknell-Johnson (eds). *A Collection of Manuscripts related to the Fibonacci Sequence*. Santa Clara: The Fibonacci Association, 93–94.

[13] Peelle, H. A. (1975) Euclid, Fibonacci and Pascal – Recursed! *International Journal of Mathematical Education in Science and Technology*. 6(4), 395–405.

[14] Rota, G.-C., Doubilet, P., Greene, C., Kahaner, D., Odlyzko, A., Stanley, R. (1975) *Finite Operator Calculus*. London: Academic Press, Ch.2.

[15] Shannon, A. G. (1975) Fibonacci analogs of the classical polynomials. *Mathematics Magazine*. 48(3), 3–7.

[16] Shannon, A. G. (1983) Intersections of second order linear recursive sequences. *The Fibonacci Quarterly*. 21(1), 6–12.

[17] Shannon, A. G., Anderson, P. G., Horadam, A.F. (2006) Properties of Cordonnier, Perrin and Van der Laan Numbers. *International Journal of Mathematical Education in Science & Technology*. 37 (7), 825–831.

[18] Shannon, A. G., Wong, C.K. (2010) Some Properties of Generalized Third Order Pell Numbers. *Congressus Numerantium*. 201, 345–354.

[19] Shannon, A. G., Cook, C. K., Hillman, R. A. (2015) An Extension of Some Results due to Jarden. *Journal of Advances in Mathematics*. 9(8), 2937–2945.

[20] Shannon, A. G., Turner, J. C., Atanassov, K.T. (1991) A Generalized Tableau Associated with Colored Convolution Trees. *Discrete Mathematics*. 92, 329–340.

[21] Sloane, N.J.A., & Plouffe, S. (1995) *The Encyclopedia of Integer Sequences*. San Diego, CA: Academic Press; + Sloane, N.J.A. 1964+. `https://oeis.org/`.

[22] Subba Rao, K. (1954) Some Properties of Fibonacci Numbers – I. *Bulletin of the Calcutta Mathematical Society*. 46, 253–257.

[23] Subba Rao, K. (1959) Some Properties of Fibonacci Numbers – II. *Mathematics Student*. 27, 19–23.

[24] Whitford, A.K. (1977) Binet's Formula Generalized. *The Fibonacci Quarterly*. 15(1), 21, 24, 29.