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Derangements and asymptotics of the Laplace transforms of large powers of a polynomial

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Abstract. We use a probabilistic approach to produce sharp asymptotic estimates as $n \to \infty$ for the Laplace transform of P^n, where P is a fixed complex polynomial. As a consequence we obtain a new elementary proof of a result of Askey-Gillis-Ismail-Offer-Rashed, [1, 3] in the combinatorial theory of derangements.

Contents

1. Statement of the main results

The generalized derangement problem in combinatorics can be formulated as follows. Suppose X is a finite set and $\sim$ is an equivalence relation on X. For each $x \in X$ we denote by $\hat{x}$ the equivalence class of x. $\hat{X}_\sim$ will denote the set of equivalence classes. The counting function of $\sim$ is the function

$$\nu = \nu_\sim : \hat{X} \longrightarrow \mathbb{Z}, \quad \nu(\hat{x}) = |\hat{x}| = \text{ the cardinality of } \hat{x}.$$

A $\sim$-*derangement* of x is a permutation $\varphi : X \longrightarrow X$ such that

$$x \notin \hat{x}, \quad \forall x \in X.$$

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We denote by $\mathcal{N}(X,\sim)$ the number of $\sim$-derangements. The ratio

$$p(X,\sim) = \frac{\mathcal{N}(X,\sim)}{|X|!}$$

is the probability that a randomly chosen permutation of X is a derangement.

In [2] S. Even and J. Gillis have described a beautiful relationship between these numbers and the Laguerre polynomials

$$L_n(x) = e^x \frac{d^n}{dx^n}\Big(e^{-x}x^n\Big) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-x)^k}{k!}, \quad n = 0, 1, \dots .$$

For example

$$L_0(x) = 1, \quad L_1(x) = 1 - x, \quad L_2(x) = \frac{1}{2!}(x^2 - 4x + 2).$$

We set

$$L_\sim := \prod_{c \in \hat{X}} (-1)^{\nu(c)} \nu(c)!\, L_{\nu(c)}(x).$$

Observe that the leading coefficient of $L_\sim$ is 1. We have the following result.

Theorem 1.1 (Even-Gillis)**.**

$$\mathcal{N}(X,\sim) = \int_0^\infty e^{-x} L_\sim(x)\,dx. \tag{1.1}$$

For several very elegant short proofs we refer to [1, 4].

Given $(X,\sim)$ as above and n a positive integer we define $(X_n,\sim_n)$ to be the disjoint union of n-copies of X

$$X_n = \bigcup_{k=1}^{n} X \times \{k\}$$

equipped with the equivalence relation

$$(x,j) \sim_n (y,k) \Longleftrightarrow j = k, \quad x \sim y.$$

We deduce

$$p(X_n,\sim_n) = \frac{1}{(n|X|)!} \int_0^\infty e^{-x} \big(L_\sim(x)\big)^n dx. \tag{1.2}$$

For example, consider the "marriage relation"

$$(C,\sim), \quad C = \{\pm 1\}, \quad -1 \sim 1.$$

In this case $\hat{C}$ consists of a single element and the counting function is the number $\nu = 2$. Then $(C_n,\sim_n)$ can be interpreted as a group of n married couples. If we set

$$\delta_n := p(C_n,\sim_n)$$

then we can give the following amusing interpretation for δ_n.

Couples mixing problem. *At a party attended by n couples, the guests were asked to put their names in a hat and then to select at random one name from that pile. Then the probability that nobody will select his/her name or his/her spouse's name is equal to δ_n.*

Using (1.2) we deduce

$$\delta_n = \frac{1}{(2n)!}\int_0^\infty e^{-x}\big(x^2-4x+2\big)^n dx. \tag{1.3}$$

We can ask about the asymptotic behavior of the probabilities $p(X_n, \sim_n)$ as $n \to \infty$. In [1, 3], Askey-Gillis-Ismail-Offer-Rashed describe the first terms of an asymptotic expansion in powers of n^{-1}. To formulate their result let us introduce the "momenta"

$$\nu_r = \sum_{c\in \hat{X}} \nu(c)^r.$$

Theorem 1.2 (Askey-Gillis-Ismail-Offer-Rashed)**.**

(1.4)

$$p(X_n, \sim_n) = \exp\left(-\frac{\nu_2}{\nu_1}\right)\left(1-\frac{\nu_1(2\nu_3-\nu_2)-\nu_2^2}{2\nu_1^3}n^{-1}+O(n^{-2})\right) \quad \text{as } n\to\infty.$$

For example we deduce from the above that

$$\delta_n = e^{-2}\left(1-\frac{1}{2}n^{-1}+O(n^{-2})\right), \quad n\to\infty. \tag{1.5}$$

The proof in [3] of the asymptotic expansion (1.4) is based on the saddle point technique applied to the integrals in the RHS of (1.2) and special properties of the Laguerre polynomials. The proof in [1] is elementary but yields a result less precise than (1.4).

In this paper we will investigate the large n asymptotics of Laplace transforms

$$\mathcal{F}_n(\mathcal{Q}, z) = \frac{z^{dn+1}}{(dn)!}\int_0^\infty e^{-zt}\mathcal{Q}(t)^n dt, \quad \mathfrak{Re}\, z > 0, \tag{1.6}$$

where $\mathcal{Q}(t)$ is a degree d complex polynomial with leading coefficient 1. If we denote by $\mathcal{L}[f(t), z]$ the Laplace transform of $f(t)$

$$\mathcal{L}[f(t), z] = \int_0^\infty e^{-zt} f(t)dt$$

then

$$\mathcal{F}_n(\mathcal{Q}, z) = \frac{\mathcal{L}[\mathcal{Q}(t)^n, z]}{\mathcal{L}[t^{dn}, z]}.$$

The estimate (1.4) will follow from our results by setting

$$z = 1, \quad \mathcal{Q} = L_\sim.$$

To formulate the main result we first write $\mathcal{Q}$ as a product

$$\mathcal{Q}(t) = \prod_{i=1}^d (t + r_i).$$

We set

$$\vec{r} = (r_1, \dots, r_d) \in \mathbb{C}^d, \quad \mu_s = \mu_s(\vec{r}) = \frac{1}{d}\sum_{i=1}^d r_i^s.$$

Theorem 1.3 (Existence theorem)**.** *For every* $\mathfrak{Re}\, z > 0$ *we have an asymptotic expansion as* $n \to \infty$

$$\mathcal{F}_n(\mathcal{Q}, z) = \sum_{k=0}^{\infty} A_k(z) n^{-k}. \tag{1.7}$$

Above, the term $A_k(z)$ *is a holomorphic function on* $\mathbb{C}$ *whose coefficients are universal elements in the ring of polynomials* $\mathbb{C}(d)[\mu_1, \mu_2, \dots, \mu_k]$*, where* $\mathbb{C}(d)$ *denotes the field of rational functions in the variable* $d = \deg \mathcal{Q}$*.*

The proof of this theorem is given in the second section of this paper and it is probabilistic in flavor. In the third section we compute the terms A_k in some cases. For example we have

$$A_0(z) = e^{\mu_1 z}, \quad A_1(z) = \frac{1}{2d} e^{\mu_1 z}(\mu_1^2 - \mu_2) z^2, \tag{1.8}$$

and we can refine (1.5) to

$$\delta_n = e^{-2}\left(1 - \frac{1}{2} n^{-1} - \frac{23}{96} n^{-2} + O(n^{-3})\right), \quad n \to \infty. \tag{1.9}$$

These computations will lead to a proof of the following result.

Theorem 1.4 (Structure theorem)**.** *For any* k *and any degree* d *we have*

$$A_k(z) = e^{\mu_1 z} B_k(z),$$

where $B_k \in \mathbb{C}(d)[\mu_1, \dots, \mu_k][z]$ *is a universal polynomial in* z *with coefficients in* $\mathbb{C}(d)[\mu_1, \dots, \mu_k]$*.*

The formulæ (1.8) have an immediate curious consequence which was mentioned as an open question in [3].

Corollary 1.5. *Suppose* $P(t) = t^d + at^{d-1} + \cdots$ *is a degree* d *polynomial with real coefficients. Then*

$$\int_0^{\infty} e^{-t} P(t)^n dt > 0, \quad \forall n \gg 0.$$

Notations. A d*-dimensional (multi)index* will be a vector $\vec{\alpha} \in \mathbb{Z}^d_{\geq 0}$. For every vector $\vec{x} \in \mathbb{C}^d$ and any d-dimensional index $\vec{\alpha}$ we define

$$\vec{x}^{\,\vec{\alpha}} = x_1^{\alpha_1} \dots x_d^{\alpha_d}, \quad |\vec{\alpha}| = \alpha_1 + \cdots + \alpha_d, \quad S(\vec{x}) = x_1 + \cdots + x_d.$$

If $n = |\vec{\alpha}|$ then we define the multinomial coefficient

$$\binom{n}{\vec{\alpha}} := \frac{n!}{\prod_{i=1}^d \alpha_i!}.$$

These numbers appear in the *multinomial formula*

$$S(\vec{x})^n = \sum_{|\vec{\alpha}|=n} \binom{n}{\vec{\alpha}} \vec{x}^{\,\vec{\alpha}}.$$

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2. Proof of the existence theorem

The key to our approach is the following elementary result.

Lemma 2.1. *If $P(x) = p_m t^m + \cdots + p_1 t + p_0$ is a degree m with complex coefficients then for every $\mathfrak{Re}\, z > 1$ we have*

$$\frac{\mathcal{L}[P(t), z]}{\mathcal{L}[t^m, z]} = \frac{z^{m+1}}{m!}\int_0^\infty e^{-zt}P(t)dt = \sum_{a+b=m} \frac{p_a}{\binom{m}{a}}\frac{z^b}{b!}. \tag{2.1}$$

Proof.

$$\begin{aligned}\frac{z^{m+1}}{m!}\int_0^\infty e^{-zt}P(t)dt &= \frac{z^{m+1}}{m!}\sum_{a=0}^m p_a \int_0^\infty e^{-zt}t^a dt \\ &= \frac{z^{m+1}}{m!}\sum_{a=0}^m p_a \frac{a!}{z^{a+1}} = \sum_{a+b=m}\frac{p_a}{\binom{m}{a}}\frac{z^b}{b!}.\end{aligned}$$

□

Denote by $\mathcal{Q}(n,a)$ the coefficient of t^a in $\mathcal{Q}(t)^n$. From (2.1) we deduce

$$\mathcal{F}_n(\mathcal{Q}, z) = \sum_{a+b=dn}\frac{\mathcal{Q}(n,a)}{\binom{dn}{a}}\frac{z^b}{b!}. \tag{2.2}$$

Using the equality

$$\mathcal{Q}^n = \prod_{i=1}^d \underbrace{\left(\sum_{j+k=n}^n \binom{n}{i} t^j r_i^k\right)}_{(t+r_i)^n}$$

we deduce that if $a + b = dn$ then

$$\mathcal{Q}(n,a) = \sum_{|\vec{\alpha}|=b}\left(\prod_{i=1}^d \binom{n}{\alpha_j}\right)\vec{r}^{\alpha}. \tag{2.3}$$

For $|\vec{\alpha}| = b$ we set

$$B(n,\vec{\alpha}) := \prod_{i=1}^d \binom{n}{\alpha_j}, \;\; P_{n,b}(\vec{\alpha}) := \frac{B(n,\vec{\alpha})}{\binom{dn}{b}}, \;\; \rho_b(\vec{\alpha}) = \vec{r}^{\vec{\alpha}},$$

so that

$$\mathcal{F}_n(\mathcal{Q}, z) = \sum_{a+b=dn}\left(\sum_{|\vec{\alpha}|=b} P_{n,b}(\vec{\alpha})\rho_b(\vec{\alpha})\right)\cdot\frac{z^b}{b!}. \tag{2.4}$$

Observe that we have

$$P_{n,b}(\vec{\alpha}) = \frac{\prod_{i=1}^d (1-\frac{1}{n})\cdots(1-\frac{\alpha_i - 1}{n})}{\prod_{k=1}^{b-1}(1-\frac{k}{dn})}\cdot\underbrace{\frac{1}{d^b}\binom{b}{\vec{\alpha}}}_{:=P_b(\vec{\alpha})}. \tag{2.5}$$

The coefficients $P_b(\vec{\alpha})$ define the multinomial probability distribution P_b on the set of multiindices

$$\Lambda_b = \Big\{ \vec{\alpha} \in \mathbb{Z}^b_{\geq 0};\ \ |\vec{\alpha}| = b \Big\}.$$

For every random variable ζ on Λ_b we denote by $E_b(\zeta)$ its expectation with respect to the probability distribution P_b. For each n we have a random variable $\zeta_{n,b}$ on Λ_b defined by

$$\zeta_{n,b}(\vec{\alpha}) = \frac{\prod_{i=1}^d (1-\frac{1}{n})\cdots(1-\frac{\alpha_i-1}{n})}{\prod_{k=1}^{b-1}(1-\frac{k}{dn})}\rho_b(\vec{\alpha}).$$

Form (2.4) and (2.5) we deduce

$$\mathcal{F}_n(\mathcal{Q}, z) = \sum_{a+b=dn} E_b(\zeta_{n,b})\frac{z^b}{b!}. \tag{2.6}$$

To find the asymptotic expansion for $\mathcal{F}_n$ we will find asymptotic expansions in powers of n^{-1} for the expectations $E_b(\zeta_{n,b})$ and them add them up using (2.6).

For every nonnegative integer α we define a polynomial

$$W_\alpha(x) = \begin{cases} 1 & \text{if } \alpha = 0,1 \\ \prod_{j=1}^{\alpha-1}(1-jx) & \text{if } \alpha > 1. \end{cases}$$

For a d-dimensional multiindex $\vec{\alpha}$ we set

$$W_{\vec{\alpha}}(x) = \prod_{i=1}^d W_{\alpha_i}(x).$$

We can now rewrite (2.5) as

$$P_{n,b}(\vec{\alpha}) = P_b(\vec{\alpha})\frac{W_{\vec{\alpha}}(\frac{1}{n})}{W_b(\frac{1}{dn})}.$$

We set

$$R_b(\vec{\alpha}, x) = W_{\vec{\alpha}}(x),\ \ K_b(\vec{\alpha}, x) = \frac{1}{W_b(\frac{x}{d})}R_b(\vec{\alpha}, x)\rho_b(\alpha).$$

We regard the correspondences

$$\vec{\alpha} \mapsto R_b(\vec{\alpha}, x),\ K_b(\vec{\alpha}, x)$$

as random variables $R_b(x)$ and $K_b(x)$ on Λ_b valued in the field of rational functions. We deduce

$$\zeta_{n,b} = K_b(n^{-1}).$$

Observe

$$E_b(x) = E_b(K_b(x)) = \frac{1}{W_b(x)}E_b(R_b(x)).$$

From the fundamental theorem of symmetric polynomials we deduce that the expectations $E_b(R_b(x))$ are *universal* polynomials

$$E_b(R_b(x)) \in \mathbb{C}[\mu_1, \dots, \mu_b][x],\ \ \deg_x E_b(R_b(x)) \leq b-d,$$

whose coefficients have degree b in the variables μ_i, $\deg \mu_i = i$. We deduce that $E_b(x)$ has a Taylor series expansion

$$E_b(x) = \sum_{m \geq 0} E_b(m) x^m$$

such that $E_b(m) \in \mathbb{C}(d)[\mu_1, \dots, \mu_b]$. The rational function $x \to K_b(\vec{\alpha}, x)$ has a Taylor expansion at $x = 0$ convergent for $|x| < \frac{d}{b-1}$ so the above series converges for $|x| < \frac{d}{b-1}$. We would like to estimate the size of the coefficients $E_b(m)$. The tricky part is that the radius of convergence of $E_b(x)$ goes to zero as $b \to \infty$.

Lemma 2.2. *Set*

$$R = \max_{1 \leq i \leq d} |r_i|.$$

There exists a constant C which depends only on R and d such that for every $b \geq 0$ and every $1 \leq \lambda_b < \frac{b}{b-1}$ we have the inequality

$$|E_b(m)| \leq \left(\frac{b}{\lambda_b d} \right)^m C^b \frac{b^{b-1}}{(b-2)! \left(1 - \lambda_b \frac{b-1}{b}\right)}. \tag{2.7}$$

Proof. Note first that

$$|\rho_b(\vec{\alpha})| \leq R^b, \ \ \forall |\vec{\alpha}| = b.$$

For $b = 0, 1$ we deduce form the definition of the polynomials W_α that $E_b(x) = 1$. Fix m and $b > 1$. Using the Cauchy residue formula we deduce

$$E_b(m) = \frac{1}{2\pi\sqrt{-1}} \int_{|x| = \hbar} \frac{1}{x^{m+1}} E_b(x) dx, \quad \hbar = \lambda_b \cdot \frac{d}{b}.$$

Hence

$$|E_b(m)| \leq \frac{1}{\hbar^m} \sup_{|x| = \hbar} |E_b(x)| \leq \frac{b^m R^b}{(\lambda_b d)^m \min_{|x|=\hbar} |W_b(x/d)|} \cdot \max_{|x| = \hbar} E_b(R_b(x)).$$

Next observe that

$$W_b(x/d) = (b-1)! \prod_{k=1}^{b-1} \left(\frac{1}{k} - x/d \right), \quad \hbar/d < 1/k, \forall k \leq b-1,$$

from which we conclude

$$\begin{aligned} \min_{|x|=\hbar} |W_b(x)| = W_b(\hbar) &= \prod_{k=1}^{b-1} \left(1 - \frac{k\lambda_b}{b} \right) = \frac{1}{b^{b-1}} \prod_{k=1}^{b-1} (b - k\lambda_b) \\ &\geq \frac{(b-2)!(1 - \lambda_b \frac{b-1}{b})}{b^{b-1}}. \end{aligned}$$

To estimate $E_b(R_b(x))$ from above observe that for every $1 \leq k \leq (b-1)$ and $|x| = \hbar$ we have

$$|1 - kx| \leq 1 + k|x| = 1 + \frac{k\lambda_b d}{b} < 1 + d.$$

This shows that for every $|\vec{\alpha}| = b$ and $|x| = \hbar$ we have

$$|R_b(\vec{\alpha}, x)| < (1 + d)^b.$$

The lemma follows by assembling all the facts established above. □

Define the formal power series

$$A_m(z) := \sum_{b\geq 0} E_b(m)\frac{z^b}{b!} \in \mathbb{C}[[z]].$$

The estimate (2.7) shows that this series converges for all z.

For every formal power series $f = \sum_{k\geq 0} a_k T^k$ and every nonnegative integer ℓ we denote by $J_T^\ell(f)$ its ℓ-th jet

$$J_T^\ell(f) = \sum_{k=0}^{\ell} a_k T^k.$$

For $x = n^{-1}$ we have

$$\mathcal{F}_x(z) = \mathcal{F}_n(\mathcal{Q}, z) = \sum_{b\leq d/x} E_b(x)\frac{z^b}{b!} = \sum_{b\leq d/x}\left(\sum_{m\geq 0} E_b(m)x^m\right)\frac{z^b}{b!}$$

$$= \sum_{m\geq 0}\left(\sum_{b\leq d/x} E_b(m)\frac{z^b}{b!}\right)x^m = \sum_{m\geq 0} J_z^{d/x}(A_m(z))x^m.$$

Consider the formal power series in x with coefficients in the ring $\mathbb{C}\{z\}$ of convergent power series in z

$$\mathcal{F}_\infty(z) = \sum_{m\geq 0} A_m(z)x^m \in \mathbb{C}\{z\}[[x]].$$

We will prove that for every $\ell \geq 0$ and every $z \in \mathbb{C}$ we have

$$|\mathcal{F}_n(z) - J_x^\ell \mathcal{F}_\infty(z)| = O(n^{-\ell-1}), \text{ as } n \to \infty. \tag{2.8}$$

To prove this it is convenient to introduce the "rectangles"

$$D_{u,v} = \Big\{(b,m) \in (\mathbb{Z}_{\geq 0})^2;\ \ b \leq u,\ \ m \leq v\Big\}.$$

In this notation we have ($x = n^{-1}$)

$$\mathcal{F}_n(z) = \sum_{(b,m)\in D_{n,\infty}} E_b(m)x^m\frac{z^b}{b!}, \quad J_x^\ell\mathcal{F}_\infty(z) = \sum_{(b,m)\in D_{\infty,\ell}} E_b(m)x^m\frac{z^b}{b!}.$$

Then

$$\mathcal{F}_n(z) - J_x^\ell\mathcal{F}_\infty(z) = \underbrace{\sum_{b\leq dn}\left(\sum_{m>\ell} E_b(m)x^m\right)\frac{z^b}{b!}}_{S_1(n)} + \underbrace{\sum_{m\leq \ell}\left(\sum_{b>dn} E_b(m)\frac{z^b}{b!}\right)x^m}_{S_2(n)}.$$

We estimate each sum separately. Using (2.7) with a $\lambda_b > 1$ to be specified later we deduce

$$\sum_{m>\ell} |E_b(m)x^m| \leq \frac{C^b b^{b-1}}{(b-2)!(1-\lambda_b\frac{b-1}{b})}\sum_{m>\ell}\left(\frac{bx}{\lambda_b d}\right)^m.$$

The inequality $b \leq dn$ can be translated into $\frac{bx}{d} \leq 1$ so that the above series is convergent for $b \leq dn$ whenever $\lambda_b > 1$ so that

$$\sum_{m>\ell} |E_b(m)x^m| \leq \frac{C^b b^{b-1}}{(b-2)!(1-\lambda_b \frac{b-1}{b})} \left(\frac{bx}{\lambda_b d}\right)^{\ell+1} \frac{1}{1-\frac{bx}{\lambda_b d}}.$$

When $b \leq dn$ we have

$$1 - \frac{bx}{\lambda_b d} > 1 - \frac{1}{\lambda_b}.$$

If we choose

$$\lambda_b = \left(\frac{b}{b-1}\right)^{1/2}$$

we deduce

$$1 - \lambda_b \frac{b-1}{b} = 1 - \left(\frac{b-1}{b}\right)^{1/2} \Longrightarrow \frac{1}{1-\lambda_b \frac{b-1}{b}} < b$$

and, since $\frac{bx}{\lambda_b d} \leq \frac{b}{d}x$,

$$\frac{1}{1-\frac{bx}{\lambda_b d}} < \frac{1}{1-\frac{1}{\lambda_b}} < 2b.$$

Using the inequalities

$$k! > \left(\frac{k}{e}\right)^k, \;\; \forall k > 0$$

we conclude that for $b \leq dn$ we have

$$\sum_{m>\ell} |E_b(m)x^m| \leq C_1^b b^{\ell+2} x^{\ell+1}.$$

Since the series $\sum_{b\geq 0} C_1^b b^{\ell+2} \frac{z^b}{b!}$ converges we conclude that

$$S_1(n) = O(x^{\ell+1}).$$

To estimate the second sum we choose $\lambda_b = 1$ in (2.7) and we deduce

$$E_b(m) \leq C_3^b.$$

Hence

$$\left|\sum_{b>dn} E_b(m)\frac{z^b}{b!}\right| \leq \frac{(C_3|z|)^b b^2}{b!} < (2C_3|z|)^2 \sum_{b>dn} \frac{(|C_3|z|)^{b-2}}{(b-2)!}.$$

Using Stirling's formula we deduce that for fixed z we have

$$\sum_{b>dn} \frac{(|C_3|z|)^{b-2}}{(b-2)!} < C_4(z) n^{-\ell-1}.$$

Hence

$$|S_2(n)| \leq C_4(z)(\ell+1)n^{-\ell-1}.$$

This completes the proof of (2.8) and of Theorem 1.3. □

3. Additional structural results

3.1. The case $d = 1$. Hence $\mathcal{Q}(t) = (t + \mu_1)$ so that

$$\int_0^\infty e^{-zt}(t+\mu_1)^n dt = e^{\mu_1 z}\int_0^\infty e^{-zt}t^n dt = e^{\mu_1 z}\frac{n!}{z^{n+1}}.$$

Hence in this case

$$\mathcal{F}_n(z) = e^{\mu_1 z}$$

and we deduce

$$A_0(z) = e^{\mu_1 z},\ \ A_k(z) = 0,\ \ \forall k \geq 1.$$

3.2. The case $d = 2$. This is a bit more complicated. We assume first that $\mu_1 = 0$ so that

$$\mathcal{Q}(t) = t^2 - \sigma^2.$$

Then

$$\mathcal{Q}(n,a) = \begin{cases} (-1)^k\sigma^{2(n-k)}\binom{n}{k} & \text{if } a = 2k \\ 0 & \text{if } a \text{ is odd,} \end{cases}$$

and we deduce

$$\begin{aligned}
\mathcal{F}_n(z) &= \sum_{b=0}^n \frac{(-1)^b\binom{n}{n-b}}{\binom{2n}{2n-2b}}\frac{(\sigma z)^{2b}}{(2b)!} = \sum_{b=0}^n \frac{n!(2n-2b)!}{(n-b)!(2n)!}\frac{(-1)^b(\sigma z)^{2b}}{b!} \\
&= \sum_{b=0}^n \frac{n(n-1)\cdots(n-b+1)}{2n(2n-1)\cdots(2n-2b+1)}\frac{(-1)^b(\sigma z)^{2b}}{b!} \\
&= \sum_{b=0}^n \frac{1}{2^{2b}}n^{-b}\frac{(1-1/n)\cdots(1-(b-1)/n)}{(1-1/(2n)\cdots(1-(2b-1)/(2n)}\frac{(-1)^b(\sigma z)^{2b}}{b!} \\
&= 1 - \frac{1}{2}n^{-1}\frac{1}{1-1/(2n)}\frac{(\sigma z^)2}{2!} \\
&\quad + \frac{1}{2^4}n^{-2}\frac{(1-1/n)}{(1-1/(2n))(1-2/(2n))(1-3/(2n))}\frac{(\sigma z)^4}{4!} + \cdots .
\end{aligned}$$

To obtain $A_k(z)$ we need to collect the powers n^{-k}. The above description shows that the coefficients of the monomials z^{2b} contain only powers n^{-k}, $k \geq b$. We conclude that $A_k(z)$ is a polynomial and

$$\deg_z A_k(z) \leq 2k.$$

Let us compute the first few of these polynomials. We have

$$\mathcal{F}_n(z) = 1 - \frac{1}{2}n^{-1}\left(1 + \frac{1}{2}n^{-1} + \cdots\right)\frac{(\sigma z)^2}{2!} + \frac{1}{2^4}n^{-2}\Big(1+\cdots\Big)\frac{(\sigma z)^4}{4!} + \cdots .$$

We deduce

$$A_0(z) = 1,\ \ A_1(z) = -\frac{1}{4}(\sigma z)^2,\ \ A_2(z) = -\frac{1}{8}(\sigma z)^2 + \frac{1}{2^4 4!}(\sigma z)^4.$$

If $\mu_1 \neq 0$ so that

$$\mathcal{Q}(t) = (t + r_1)(t + r_2),\quad r_1 + r_2 = 2\mu_1,$$

then we make the change in variables $t = s - \mu_1$ so that

$$\mathcal{Q}(t) = P(s) = s^2 - r^2, \quad \sigma^2 = (r_1 - \mu_1)^2 = \frac{(r_1 - r_2)^2}{4}.$$

Now observe that

$$4\mu_1^2 + (r_1 - r_2)^2 = (r_1 + r_2)^2 + (r_1 - r_2)^2 = 2(r_1^2 + r_2^2) = 4\mu_2$$

so that

$$\sigma^2 = \mu_2 - \mu_1^2.$$

Then

$$\mathcal{F}_n(\mathcal{Q}, z) = \frac{z^{2n+1}}{(2n)!}\int_0^\infty e^{-zt}\mathcal{Q}(t)^n = \frac{z^{2n+1}}{(2n)!}\int_0^\infty e^{-z(s-\mu_1)}P(s)^n ds = e^{\mu_1 z}\mathcal{F}_n(P, z).$$

We deduce

(3.1)

$$A_0(z) = e^{\mu_1 z}, \ \ A_1(z) = -\frac{e^{\mu_1 z}}{4}(\sigma z)^2, \ \ A_2(z) = e^{\mu_1 z}\Big(-\frac{1}{8}(\sigma z)^2 + \frac{1}{2^4 4!}(\sigma z)^4\Big).$$

For the couples mixing problem we have

$$\mathcal{Q}(t) = t^2 - 4t + 2$$

so that

$$\mu_1 = -\frac{4}{2} = -2, \ \ \sigma^2 = \frac{1}{4}(r_1 - r_2)^2 = \frac{1}{4}\Big((r_1 + r_2)^2 - 4r_1 r_2\Big) = \frac{1}{4}(16 - 8) = 2,$$

and we deduce

$$\delta_n = \mathcal{F}_n(\mathcal{Q}, z = 1) = e^{-2}\Big(1 - \frac{1}{2}n^{-1} - \frac{23}{96}n^{-2} + O(n^{-3})\Big). \tag{3.2}$$

3.3. The general case. Let us determine the coefficients $A_0(z)$ and $A_1(z)$ for general degree d. We use the definition

$$A_k(z) = \sum_{b\geq 0} E_b(k)\frac{z^b}{b!}.$$

For $|\vec{\alpha}| = b$

$$W_{\vec{\alpha}}(x) = W_{b,\alpha}(x) = \prod_{i=1}^{d}\left(\prod_{j=1}^{\alpha_i - 1}(1 - jx)\right) = \prod_{i=1}^{d}\left(1 - \left(\sum_{j=1}^{\alpha_i - 1} j\right)x + \cdots\right)$$

$$= 1 - \frac{1}{2}\left(\sum_{i=1}^{d}\alpha_i(\alpha_i - 1)\right)x + \cdots.$$

$$W_b(x/d) = \prod_{k=1}^{b-1}(1 + jx/d + \cdots) = 1 + \frac{b(b-1)}{2d}x + \cdots.$$

Next, compute the expectation of $R_b(x)$

$$E_b(R_b(x)) = E_b(\rho_b) - \frac{1}{2}E_b\left(\sum_{i=1}^{d}\alpha_i(\alpha_i - 1)\vec{r}^{\vec{\alpha}}\right)x + \cdots.$$

The multinomial formula implies

$$E_b(\rho_b) = \mu_1^b.$$

Next

$$E_b\left(\sum_{i=1}^d \alpha_i(\alpha_i-1)\vec{r}^{\vec{\alpha}}\right)=\frac{1}{d^b}\sum_{|\vec{\alpha}|=b}\binom{b}{\vec{\alpha}}\left(\sum_{i=1}^d \alpha_i(\alpha_i-1)\right)\vec{r}^{\vec{\alpha}}.$$

Now consider the partial differential operator

$$\mathcal{P}=\sum_{i=1}^d r_i^2\frac{\partial^2}{\partial r_i^2}.$$

Observe that the monomials $\vec{r}^{\vec{\alpha}}$ are eigenvectors of $\mathcal{P}$

$$\mathcal{P}\vec{r}^{\vec{\alpha}}=\left(\sum_{i=1}^d \alpha_i(\alpha_i-1)\right)\vec{r}^{\vec{\alpha}}.$$

We deduce

$$E_b\left(\sum_{i=1}^d \alpha_i(\alpha_i-1)\vec{r}^{\vec{\alpha}}\right)=\frac{1}{2d^b}\mathcal{P}S(\vec{r})^b=\frac{1}{2}\mathcal{P}\mu_1^b.$$

Hence

$$E_b(R_b(x)=\mu_1^b-\frac{1}{2}(\mathcal{P}\mu_1^b)x+\cdots$$

and we deduce

$$\begin{aligned}E_b(x)&=\left(\mu_1^b-\frac{1}{2}(\mathcal{P}\mu_1^b)x+\cdots\right)\left(1+\frac{b(b-1)}{2d}x+\cdots\right)\\&=\mu_1^b+\frac{1}{2}\left(\frac{b(b-1)}{d}\mu_1^b-\mathcal{P}\mu_1^b\right)x+\cdots.\end{aligned}$$

We deduce $A_0(z)=e^{\mu_1 z}$

$$A_1(z)=\frac{\mu_1^2}{2d}\sum_{b=2}^\infty\frac{z^b}{(b-2)!}-\frac{1}{2}\mathcal{P}e^{\mu_1 z}=\frac{\mu_1^2z^2}{2d}e^{\mu_1 z}-\frac{1}{2}\mathcal{P}e^{\mu_1 z}.$$

We can simplify the answer some more.

$$\mathcal{P}\mu_1^b=\frac{1}{d^b}\mathcal{P}S(x)^b=\frac{b(b-1)}{d^b}\left(\sum_{i=1}^d r_i^2\right)S(x)^{b-2}=\frac{b(b-1)}{d}\mu_2\mu_1^{b-2}.$$

We conclude that

$$\mathcal{P}e^{\mu_1 z}=\frac{\mu_2z^2}{d}\sum_{b\geq 2}\frac{(\mu_1 z)^{b-2}}{(b-2)!}=\frac{\mu_2z^2}{d}e^{\mu_1 z}.$$

Hence

$$A_0(z)=e^{\mu_1 z},\ \ A_1(z)=\frac{e^{\mu_1 z}}{2d}(\mu_1^2-\mu_2)z^2. \tag{3.3}$$

For $d=2$ we recover part of the formulæ (3.1).

3.4. Proof of the structure theorem. Clearly we can assume $d > 1$. We imitate the strategy used in the case $d = 2$. Thus, after the change in variables $t \to t - \mu_1$ we can assume that $\mu_1 = 0$ so that $\mathcal{Q}(t)$ has the special form[1]

$$\mathcal{Q}(t) = t^d + a_{d-2}t^{d-2} + \cdots + a_0.$$

Set

$$T(n,b) := \frac{\mathcal{Q}(n, dn-b)}{\binom{dn}{dn-b}}.$$

This is a power series in $x = n^{-1}$,

$$T(n,b) = T_b(x)\,|_{x=n^{-1}}, \quad T_b(x) = \sum_{k\geq 0} T_b(k)x^k.$$

We have

$$A_k(z) = \sum_{b\geq 0} T_b(k)\frac{z^b}{b!},$$

and we need to prove that A_k is a polynomial for every k. We denote by $\ell(b)$ the order of the first nonzero coefficient of $T_b(x)$,

$$\ell(b) = \min\{k \geq 0;\ \ T_b(k) \neq 0\}.$$

To prove the desired conclusion it suffices to show that

$$\lim_{b\to\infty} \ell(b) = \infty. \tag{3.4}$$

For every multiindex $\vec{\beta} = (\beta_d, \beta_{d-2}, \ldots, \beta_1, \beta_0)$ we set

$$L(\vec{\beta}) = d\beta_d + (d-2)\beta_{d-2} + \cdots + \beta_1.$$

Let $\vec{a} := (1, a_{d-2}, \ldots, a_1, a_0) \in \mathbb{C}^d$ and

$$\mathcal{B}_n := \big\{\vec{\beta} \in \mathbb{Z}^d_{\geq 0};\ \ |\vec{\beta}| = n,\ \ L(\vec{\beta}) = dn - b\big\}.$$

We have

$$T(n,b) = \frac{1}{\binom{dn}{dn-b}} \cdot \sum_{\vec{\beta}\in\mathcal{B}_n} \binom{n}{\vec{\beta}}\vec{a}^{\,\vec{\beta}}. \tag{3.5}$$

Now observe that for every multiindex $\vec{\beta} \in \mathcal{B}_n$ we have

$$2\beta_{d-2} + 3\beta_{d-3} + \cdots + (d-1)\beta_1 + d\beta_0 = d|\vec{\beta}| - L(\vec{\beta}) = b.$$

In particular we deduce

$$\beta_j \leq \frac{b}{d-j} \leq \frac{b}{2},\ \ \forall 0 \leq j \leq d-2 \tag{3.6}$$

and

$$2\beta_d + b = 2\beta_d = 2\beta_{d-2} + 3\beta_{d-3} + \cdots + (d-1)\beta_1 + d\beta_0$$

$$\geq 2\beta_d + 2\beta_{d-2} + \cdots + 2\beta_1 + 2\beta_0 = 2n$$

---
[1]

so that

$$n - \beta_d \leq \frac{b}{2}. \tag{3.7}$$

These simple observations have several important consequences.

First, observe that they imply that there exists an integer $N(b)$ *which depends only* b *and* d, such that

$$|\mathcal{B}_n| \leq N(b), \;\; \forall n > 0.$$

Thus the sum (3.5) has fewer than $N(b)$ terms.

Next, if we set $|a| := \max_{0\leq j\leq d-2} |a_j|$ then, we deduce

$$|\vec{a}^{\,\vec{\beta}}| \leq |a|^{\beta_0+\cdots+\beta_{d-2}} \leq |a|^{\frac{b(d-1)}{2}} = C_5(b).$$

Finally, using the identity

$$\binom{n}{\vec{\beta}} = \binom{n}{\beta_d}\cdot\binom{n-\beta_d}{\beta_{d-2}}\binom{n-\beta_d-\beta_{d-2}}{\beta_{d-3}}\cdots$$

the inequalities (3.7) and $\binom{m}{k} \leq 2^m$, $\forall m \geq k$ we deduce

$$\binom{n}{\vec{\beta}} \leq \binom{n}{\beta_d}\cdot 2^{\frac{b(d-1)}{2}} \leq 2^{\frac{b(d-1)}{2}}\binom{n}{\lfloor b/2\rfloor+1} \leq C_6(b)n^{\lfloor b/2\rfloor+1}, \;\; \forall n \gg b.$$

Hence

$$\sum_{|\vec{\beta}|=n, L(\vec{\beta})=dn-b}\left|\binom{n}{\vec{\beta}}\vec{a}^{\,\vec{\beta}}\right| \leq N(b)C_5(b)C_6(b)n^{\lfloor b/2\rfloor+1} = C_7(b)n^{\lfloor b/2\rfloor+1}.$$

On the other hand

$$\frac{1}{\binom{dn}{dn-b}} \leq C_8(b)n^{-b}$$

so that

$$|T(n,b)| = |T_b(n^{-1})| \leq C_9(b)n^{\lfloor b/2\rfloor+1-b} \leq C_9(b)n^{1-b/2}.$$

This shows

$$T_b(k) = 0, \;\; \forall k \leq b/2-1$$

so that

$$\ell(b) \geq b/2-1 \to \infty \;\text{ as } b \to \infty.$$

□

Remark 3.1. We can say a bit more about the structure of the polynomials

$$B_k(\mu_1,\dots,\mu_d,z) \in R_d = \mathbb{C}[\mu_1,\dots,\mu_d,z], \;\; k>0.$$

If we regard B as a polynomial in $r_1,\dots,r_d$ we see that it vanishes precisely when $r_1=\cdots=r_d$. Note that

$$r_1=\cdots=r_d=r \Longleftrightarrow \mathcal{Q}(t)=(t+r)^d.$$

On the other hand

$$\sum_k t^k\mu_k = \frac{1}{d}\sum_{i=1}^d\sum_{k\geq 0}(r_it)^k = \frac{1}{d}\sum_{i=1}^d\frac{1}{1-r_it} \overset{(s:=1/t)}{=} \frac{s}{d}\sum_{i=1}^d\frac{1}{s+\mu_i} = \frac{s}{d}\frac{\mathcal{Q}'(s)}{\mathcal{Q}(s)}.$$

If $\mathcal{Q}(s) = (s + r)^d$ we deduce

$$\frac{s}{d}\frac{\mathcal{Q}'(s)}{\mathcal{Q}(s)} = \frac{s}{s+r} = \frac{1}{1-rt} = \sum_{k\geq 0}(rt)^k.$$

This implies that

$$r_1 = \cdots = r_d \Longleftrightarrow \mu_i^j = \mu_j^i, \ \ \forall 1 \leq i, j \leq k \Longleftrightarrow \mu_j = \mu_1^j, \ \ \forall 1 \leq j \leq d.$$

The ideal I in R_d generated by the binomials $\mu_1^j - \mu_j$ is prime since $R_d/I \cong \mathbb{C}[\mu_1, z]$. Using the Hilbert *Nullstelensatz* we deduce that B_k must belong to this ideal so that we can write

$$B_k(\mu_1, \ldots, \mu_d, z) = A_{2k}(\mu, z)(\mu_1^2 - \mu_2) + \cdots + A_{dk}(\mu, z)(\mu_1^d - \mu_d).$$

□

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