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Promenade around Pascal Triangle – Number Motives

by
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Dedicated to the memory of Nicolae Popescu (1937-2010)
on the occasion of his 75th anniversary

Abstract

We survey classical results and recent developments, old and new problems, conjectures and ideas selected from the endless theme of iterated application of the fundamental rule of addition. The multitude of forms created by this spring, like the commandment *"Let there be light"* from the first day of creation, is emphasized by the role played by the prime numbers. The subject sounds harmoniously between poetry and astronomy or geometry, finding its origin in the East at Pingala (4-2nd century BC) and in the West at Apollonius of Perga (3rd century BC). Our work is divided in two parts: the present paper is mostly dedicated to playing with numbers, while the second one, which will follow in a companion paper, is based on geometrical motives.

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1 From history to content

Repeated application of simple rules often produce very complex objects. This fact is revealed in numerous situations (cf. Wolfram [Wol'02]). Pascal Arithmetic Triangle is generated recursively by two basic rules:

$$\text{(i) boundary conditions;} \qquad \text{(ii) insertion method.} \qquad (\mathcal{R})$$

To obtain a row: (i) copy the row above and attach a 1 at each endpoint; (ii) between any two consecutive elements that come from the previous row insert their sum; then delete the older entries. Small changes in (i) and (ii) may produce very different constructions. Although,

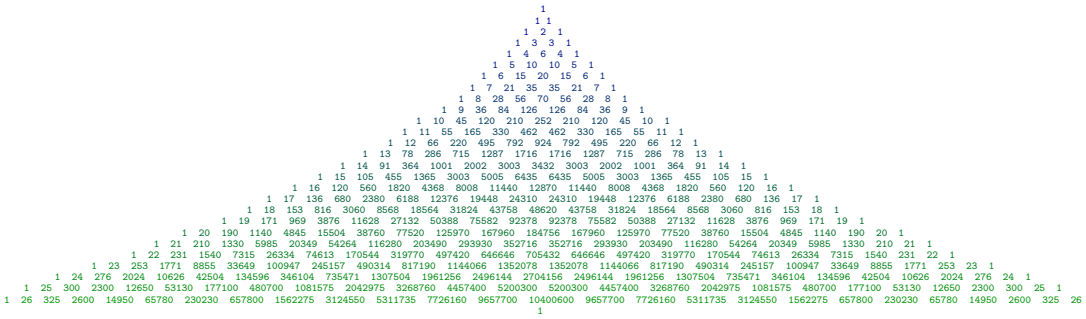

Figure 1: The first 27 rows of Pascal triangle with entries written in base 10.

they usually share remarkable arithmetic, geometric, topologic, and probabilistic properties. In each case, a 'kaleidoscope of mirrors', or the multitude of symmetries involved is perhaps the most appealing. We survey various such constructions, recent results and open questions.

Known for more than a millenium in Asia and Europe (cf. Burton [Bur'07]), the origins of the Pascal triangle are lost in the mist of time. In Chandaḥśāstra, the Hindu scholar Pingala has classified meters (chandas) or rhythm of poems that are closely allied to music (Bag [Bag'66]). He enumerated and counted the meters of a given length n that have exactly r syllables of a kind. In doing this, he obtained *Meruprastāra* (the stairway to the mythical mountain Meru). Then Halāyudha (cca. 975) in Mṛtasañjīvanī, a text of commentaries on Pingala's Chandaḥśāstra, clearly described Meruprastāra as what is today known as the arithmetic triangle of Pascal. Among those who considered the triangle before Pascal, we find: Al-Karaji (953-1029), Persia; Jia Xian (1010-1070), China; Al-Karaji (953-1029); Al-Samawal al-Maghribi (1030-1080) in Iraq, Marocco, Iran in his treatise *The brilliant in algebra*; Omar Khayyám (1048-1131), it is called *Khayyám triangle* in Iran; Yang Hui's (1238-1298) triangle in China; in 1303 was named the *Old Method* by Chu Chijie in Siyuan yujian (*Jade Mirror of the Four Unknown*). And in the West: Ramon Llull, a Majorcan theologian (1232–1316); Niccolo Tartaglia in *Generale Trattato* (1556) (the triangle bears his name in Italy); Michael Stifel in *Arithmetica Integra* (1544); in 1527, Petrus Apianus puts the triangle on the frontispiece of his book. De Moivre named it *Triangulum Arithmeticum Pascalianum* in 1730.

Pascal [Pas1665] wrote his *Traité du Triangle Arithmétique* in 1654. It was the result of a fruitful correspondence with Pierre de Fermat about calculating the odds in some games of chance. The relations between binomial coefficients proved by Pascal, later led Newton to the discovery of the binomial theorem for negative and fractional exponents, and Leibniz to the discovery of infinitesimal calculus[1]. Pascal triangle is extremely rich in displaying remarkable sequences of integers, triangular numbers, Fibonacci and Padovan numbers, squares, powers, Catalan, Bernoulli and Stirling numbers, etc. Consequently, there are so many relations, that 'when someone finds a new identity, there aren't many people who get excited about it any more, except the discoverer' (cf. Donald Knuth, *The art of computer science*, Vol. I, Chap. 1).

[1] 'On several occasions Leibniz was to declare that he was led to the invention of calculus more by studying Pascal's writings than anything else.' (cf. Burton [Bur'07, pages 412–413])

The following matrices [BC'04], [MP'09] nicely relate the triangle in the form used by Pascal with one close to what we use today. Let $L(\infty)_{i,j} = \binom{i}{j}$, $i, j \geq 0$, where, $\binom{i}{j} = 0$ if $i < j$.

$$L(\infty) = \begin{pmatrix} 1 & & & & & & & \\ 1 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ 1 & 3 & 3 & 1 & & & & \\ 1 & 4 & 6 & 4 & 1 & & & \\ 1 & 5 & 10 & 10 & 5 & 1 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \exp \begin{pmatrix} 0 & & & & & \\ 1 & 0 & & & & \\ 0 & 2 & 0 & & & \\ & 0 & 3 & 0 & & \\ & & & & \ddots & \end{pmatrix}$$

and

$$P(\infty) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & \mathbf{2} & 3 & 4 & 5 & \cdots \\ 1 & 3 & \mathbf{6} & 10 & 15 & \cdots \\ 1 & 4 & 10 & \mathbf{20} & 35 & \cdots \\ 1 & 5 & 15 & 35 & \mathbf{70} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Then

$$P(\infty) = L(\infty) \cdot L(\infty)^T; \quad \operatorname{tr}\big(P(n)\big) = \sum_{k=0}^{n-1} \frac{(2k)!}{(k!)^2} \quad \text{and} \quad \det(P(n)) = 1.$$

But looking further than pure identities, a more optimistic thought becomes appropriate, that of Jacobi Bernoulli [Ber1713, Part II, Chapter 3, page 88], who used the triangle to get into the intricacies of sums of consecutive p powers. He admired the exceptional properties of the triangle, which conceals within itself mysteries of combinations and top secrets of mathematics.

2 The Shape of the game

The mountain shape in Figure 1 captures some magic of the triangle. One can use Stirling's formula to study the shape of the outer curve. More generally, let $b \geq 2$, n and S be fixed positive integers and denote by $d_b(m)$ the number of digits of m in base b. Then denote:

$$m_b(n,S) := \min_{\substack{a_1+\cdots+a_n=S \\ a_1,\ldots,a_n \in \mathbb{N}\setminus\{0\}}} \sum_{k=1}^{n} d_b(a_k) \quad \text{and} \quad M_b(n,S) := \max_{\substack{a_1+\cdots+a_n=S \\ a_1,\ldots,a_n \in \mathbb{N}\setminus\{0\}}} \sum_{k=1}^{n} d_b(a_k)\,.$$

Question 1. *Estimate $m_b(n,S)$ and $M_b(n,S)$.*

One may ask about the distribution of digits in Figure 1. The first digit of the binomial coefficients appears to satisfy Benford's law. The triangle has a different shape in another number system, such as the Chinese one used by Chu Chijie in 1303 or the Arabic one employed by Al-Samawal al-Maghribi.

Question 2. *Find the shape of the triangle in closed form.*

Problems similar to Questions 1 and 2 may be addressed for the shape of individual rows of the triangle (see Figure 2). The thickness of the lens shape figure that corresponds to the n-the row of the Pascal triangle is $\approx n$. Remark that if $n = 2m$ is even, the number that gives that thickness is $(m+1)C_m$, where C_m is the m-th Catalan number.

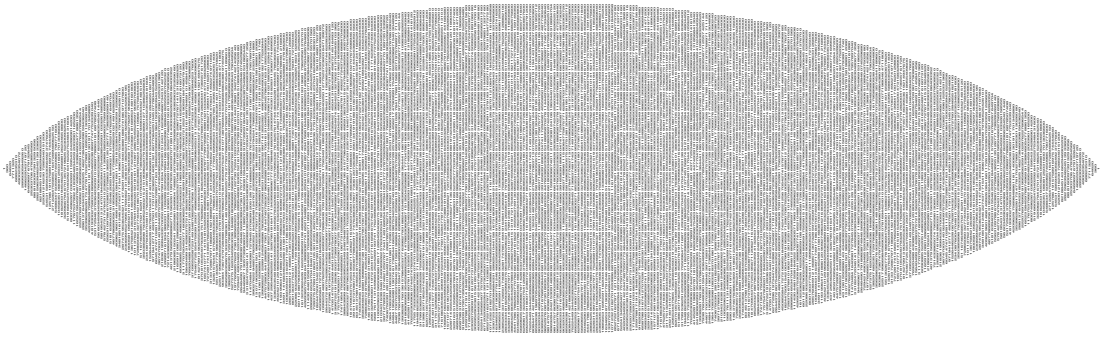

Figure 2: The 360th row of Pascal triangle. The binomial coefficients $\binom{360}{k}$, $0 \le k \le 360$, written in base 10 are displayed centered, and the figure is rotated by 90 degrees in the counterclockwise direction.

3 Singmaster Conjecture

Singmaster [Sng'71], [Sng'75] guesses that there are not too many points on any 'parabola' that cuts Pascal triangle passing only through entries greater than one of the same size. He proved that $N(t) = O\big(\log t\big)$ for $t \ge 2$, where

$$N(t) := \big\{m\colon\ m \text{ binomial coefficient, } t = m\big\},$$

and made the following conjecture.

Conjecture 1 (Singmaster)**.** *The function $N(t)$ is bounded.*

This seems very difficult to prove, although Singmaster believed that $N(t) \le 10$ or 12 should be the real margin. The best upper bound so far is:

$$N(t) = O\left(\frac{(\log t)(\log\log\log t)}{(\log\log t)^3}\right),$$

due to Kane [Kan'07], who improves on previous estimates of himself, Abbott, Erdös and Hanson.

As for lower bounds, the results obtained so far are sporadic: $N(2) = 1$, $N(3) = N(4) = N(5) = 2$, $N(6) = 3$; if $t < 10^{28}$ then $N(t) = 6$ for $t = 120, 210, 1540, 7140, 11628, 24310$; $N(3003) = 8$ since $\binom{3003}{1} = \binom{78}{2} = \binom{15}{5} = \binom{14}{6}$, the single known t for which $N(t) = 8$. Singmaster proved that $N(t) \ge 6$ infinitely often, but there are not known solutions of equations $N(t) = 5$ or 7.

4 Odds and Ends in Pascal Triangle – Sierpinski Gasket

Recognition of primes is an extremely difficult problem. The Pascal triangle encodes enough information to offer a genuine image of every positive integer. Looking at the entries in Pascal

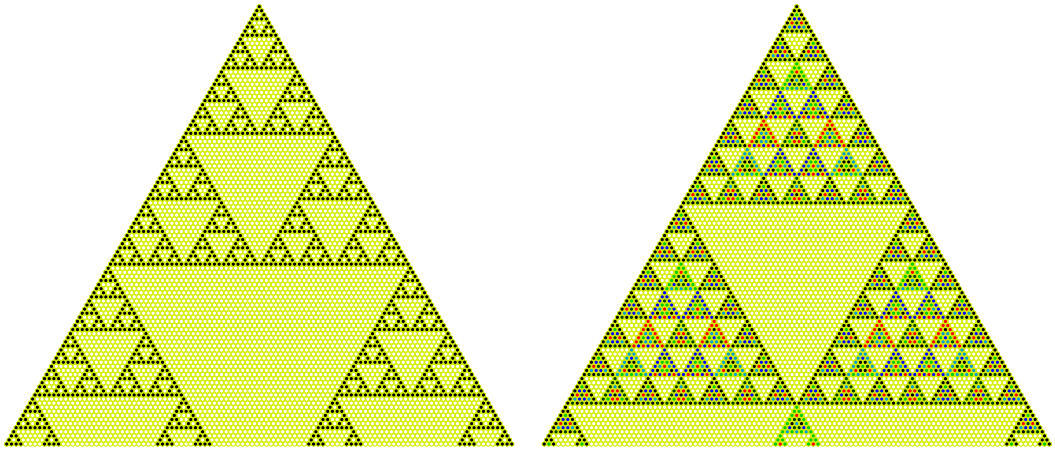

Figure 3: The first 107 rows of Pascal triangle modulo 2 on the left and modulo 7 on the right with residue classes drown in distinct colors.

triangle modulo n, for $n = 2, 3, 4, 5, \ldots$, using colors to distinguish distinct residue classes, one may see *the face of a prime*. It is striking to see the relative order in these images when n is prime (two examples are shown in Figure 3), as opposed to the 'chaos' when n is highly composite. A systematic study to discover some sort of 2-dimensional distribution is yet to be done.

Pascal triangle modulo prime powers brought the attention of many mathematicians. Among them are Legendre, Cauchy, Gauss, Kummer, Hermite, Hensel, Lucas, Granville [Gra'95]. Singmaster[Sng'80] studied the distribution of entries that are equal to zero in the Pascal triangle modulo n. Actually, the most attention by far was given to the Pascal triangle modulo 2, the *Sierpinski triangle*. Glaisher [Gla1899] proved that the odd numbers in any row of the Pascal triangle is always a power of 2. For higher powers of 2 and similar problems for other primes the reader is referred to Granville [Gra'92], [Gra'97], Huard et al. [HSW'97], [HSW'98], Miheț[Mih'10], Rowland [Row'11a], [Row'11b] and Shevelev [She'11]. Another intriguing property that relates the Pascal triangle modulo 2 with Fermat numbers was discovered by Hewgill [Hew'77]. Let $c(n, j) := \binom{n}{j} \pmod 2$ and let $F_j := 2^{2^j}$ be the Fermat numbers for $j = 0, 1, \ldots$ Then

$$\sum_{j=0}^{n} c(n, j)2^j = F_0^{d_0} F_1^{d_1} \cdots F_r^{d_r}, \tag{4.1}$$

where $d_0, \ldots, d_r \in \{0, 1\}$ are the digits of n in base 2, that is $n = d_0 + 2d_1 + \cdots + d_r 2^r$ and $d_r \neq 0$. Reading the rows of the Pascal triangle modulo 2 as numbers written in base 2, we

obtain the expressions from (4.1). These numbers are:

$$1,\ 3,\ 5,\ 15,\ 17,\ 51,\ 85,\ 255,\ 257,\ 771,\ldots \tag{4.2}$$

The 32-nd is $2^{32}-1 = 4294967295$, which is the product of 3, 5, 17, 257 and 65537, the only known Fermat primes (cf. Cosgrave [Cos'99], Dubner and Gallot [DG'01]). Gauss proved that the regular N-gons constructible with the ruler and the compass are those for which N is a square free product of Fermat primes multiplied by a power of 2. Gardner [Gar'77] and Watkins [CG'96], [KLS'01] observed that the known constructible N-gons with N odd are those for which N is one of the first 32 numbers of the sequence (4.2). Close resemblance to Glaisher's result are relations between binomial coefficients and Bernoulli numbers, such as

$$\sum_{j=0}^{n} B_j \binom{n}{j} = B_n\,.$$

In a work of Lehmer [Leh'35], one finds:

$$\sum_{j=0}^{n} B_{6j} \binom{6n+3}{6j} = 2n+1$$

and

$$\sum_{j=0}^{n} B_{6j+2} \binom{6n+5}{6j+2} = \frac{1}{3}(6n+5)\,.$$

Kummer and later Vandiver [Van'39] obtained similar relations modulo powers of primes, which have various applications (cf. [Car'68]). Their congruences employ sums of binomial coefficients with weights Euler numbers or Bernoulli numbers. Later, Carlitz (see [Car'68] and the references therein) obtained more general results of this kind. Sierpinski triangle, first outlined in mathematical form by Sierpinski [Sie'15] appeared beforehand in the XIII-th century Cosmati mosaics in cathedrals of Rome region (see Wolfram [Wol'02, page 43]). There are several ways to define the Sierpinski triangle (one of them being nondeterministic[2]). The most common one is to start with a given triangle, delete from its interior the triangle determined by the midpoints of the edges, then do the same thing with the three remaining triangles, and repeat the process forever. The set of remaining points is the *Sierpinski Gasket* (Figure 3, left). It is a basic type of fractal, and its dimension δ_S is easy to find: doubling its size, it replicates three times the original triangle, so $3 = 2^{\delta_S}$, and $\delta_S = \log 3/\log 2 \approx 1.58496$. Fraenkel and Kontorovich [FK'07] recover Sierpinski triangle and its connection to binomial coefficients in the context of a p-variety XY with a Nim-product and p-sieve with p prime. Sierpinski gasket is a fertile ground of investigation, where one uses tools from analysis, `pde`, harmonic function theory, number theory, see Ben-Bassat et al. [BST'99], Needleman et al. [NSTY'04], Ben-Gal et al. [BSSY'06], Hinoa and Kumagai [HK'06], Teplyaev [Tep'07], DeGrado et al. [DRS'09], image recognition and processing, theoretical or applied physics Huzler [Huz'08], Daerden and Vanderzande [DV'98], Pradhan et al. [PCRM'03], Belrose [Bel'04].

[2]Take a triangle and label its vertices by the numbers 1, 2, 3. Pick a point P in its interior. Cast a 3-face die. Draw the midpoint of the segment determined by P and the chosen vertex, and let P be this drawn point. Repeat the process endlessly. The collection of drowned points is the Sierpinski triangle.

5 Cousins of Pascal Triangle

Replacing the 1-s on the edges of the triangle with different sequences produces interesting outcomes. Hosoya [Hos'76] proposes a Fibonacci triangle where each entry is the sum of the previous two numbers in either the row or the column:

$$HF(\infty) = \begin{pmatrix} \mathbf{1} & 1 & 2 & 3 & 5 & 8 & \cdots \\ 1 & \mathbf{1} & 2 & 3 & 5 & 8 & \cdots \\ 2 & 2 & \mathbf{4} & 6 & 10 & 16 & \cdots \\ 3 & 3 & 6 & \mathbf{9} & 15 & 24 & \cdots \\ 5 & 5 & 10 & 15 & \mathbf{25} & 40 & \cdots \\ 8 & 8 & 16 & 24 & 40 & \mathbf{64} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Actually, this is the multiplication table of Fibonacci numbers. Here the sum of entries a_{jk} with $j + k = n$, are the first convolved Fibonacci numbers (see Koshy [Kos'01] and Klavzar and Peterin [KP'07]). Falcón and Plaza [FP'07] study a Pascal 2-triangle using the k-Fibonacci sequence, while Fahr and Ringel [FR'12] change the rule of insertion to get Fibonacci partition triangles.

The sequences on the boundary can be quite arbitrary. Given $\mathbf{u} = (u_0, \ldots, u_{n-1})$ and $\mathbf{v} = (v_0, \ldots, v_{n-1})$, with $u_0 = v_0$, put $P(j, 0) = u_j$, $P(j, j) = v_j$ for $0 \le j \le n$ and

$$P(j, k) = P(j - 1, k - 1) + P(j - 1, k), \qquad \text{for } 2 \le j \le n, \quad 1 \le k \le n - 1.$$

Then $P_{\mathbf{u},\mathbf{v}} = (P(j, k))_{j,k}$ is the *generalized Pascal triangle* with $\mathbf{u}$ and $\mathbf{v}$ the generating edges.

One can also consider triangles with no boundary conditions. Steinhaus' definition is as follows. Let $\mathbf{x} = (x_0, \ldots, x_{n-1})$ and $\partial\mathbf{x} := (x_0 + x_1, \ldots, x_{n-2} + x_{n-1})$. Then, recursively $\partial^0\mathbf{x} = \mathbf{x}$, $\partial^1\mathbf{x} = \partial\mathbf{x}$, and $\partial^i\mathbf{x} = \partial\partial^{i-1}\mathbf{x}$. This generates the shrinking triangle $S(\mathbf{x}) = (\mathbf{x}, \partial\mathbf{x}, \ldots, \partial^{n-1}\mathbf{x})$, whose lines are shorter and shorter, the last one, $\partial^{n-1}\mathbf{x}$, containing just one number. One problem is to characterize the $\mathbf{x}$-s based on the shape of the produced triangles. Molluzzo [Mol'76] and J. Chappelon [Cha'08] considered *Steinhaus triangles* with entries in $\mathbb{Z}_m$. Most questions on Steinhaus or generalized Pascal triangles refer to triangles with components in $\mathbb{F}_2$, also named *Boolean triangles*. Steinhaus [Ste'58], Harborth and Hurlbert [Har'72], [HH'05] estimate the number of triangles that are *balanced* on the number of entries. Kutyreva and Malyshev [KM'06] estimate the number of Boolean Pascal triangles of size n containing a positive proportion of ones. Eliahou et al. [EH'04], [EH'05], [EMR'07] investigate binary sequences that generate triangles with symmetries, while Brunat and Maureso [BM'11] give explicit formulae for the number of these binary triangles having rotational and dihedral symmetries.

6 Euclid-Mullin sequences

The sequence $\mathcal{P}$ of prime numbers, like the Pascal arithmetic triangle, is also generated by a simple rule (Eratostene's sieve). Gallagher [Gal'76], [Gal'81] showed that primes appear like in a Poissonian process. Goldston and Ledoan [GL'12] show Poisson distribution for individual spacings between neighboring primes. The result is implicitly contained in an earlier work of

Odlyzko, Rubinstein and Wolf [ORW'99] on jumping champions. All these results are proved assuming the prime k-tuple conjecture of Hardy and Littlewood.

Subsequences of primes constructed recursively are natural places to look for symmetries. If p_1, p_2, …, p_r is a list of known primes, then the integer $p_1 \cdot p_2 \cdots p_r + 1$ is not divisible by any of p_1, p_2, …, p_r, so it is either prime or divisible by a different prime. In any case, a new prime p_{r+1} may be added to the list. Choosing various selection rules of primes from the set of divisors of $n_r = p_1 \cdot p_2 \cdots p_r + 1$, we get a large class of sequences of primes (cf. Caldwell and Gallot [CG'01]). For example, one may work with $\mathrm{spf}(n_r)$, the smallest prime factor of n_r, or $\mathrm{gpf}(n_r)$, the greatest prime factor of n_r. These choices produce the first Euclid-Mullin sequence:

$$2, 3, 7, 43, 13, 53, 5, 6221671, 38709183810571, 139, 2801, 11, 17, 5471, 52662739, \ldots \tag{6.1}$$

and the second Euclid-Mullin sequence:

$$2, 3, 7, 43, 139, 50207, 340999, 2365347734339, 4680225641471129, \ldots \tag{6.2}$$

These sequences are not monotonic, but their members do not repeat. Also, they are infinite, but since factoring large numbers is a very difficult task, only few terms of each of them are known: the first 47 members of the first one and 13 of the second one. Mullin [Mul'63] asked if the sequence (6.1) contains all primes, while Cox and Van der Poorten [CV'68] found a few primes (5, 11, 13, 17, and a some others) that are absent from (6.2) and conjectured that there are infinitely many of them. This conjecture was proved by Booker [Boo'12]. Sloane's Encyclopedia of Sequences, started in 1964, today in the form of OEIS foundation [OEIS], and the references therein collect more information about the Euclid-Mullin sequences and some others that are related to them.

7 Baby-Fractal sequences of numbers

Kimberling [Kim'95] defines a *fractal sequence* of numbers as one that contains itself as a proper subsequence. For example, the sequence

$$1, 1, 2, 1, 3, 2, 4, 1, 5, 3, 6, 2, 7, 4, 8, 1, 9, 5, 10, 3, 11, 6, 12, 2, 13, 7, 14, 4, 15, 8, \ldots$$

is fractal because we get the same sequence after we delete from it the first appearance of all positive integers. This sequence appears in a card sorting algorithm of Kimberling and Shultz [KS'97], [Kim'97]. Simple examples of fractal sequences of numbers are constant or cyclic sequences, or sequences that repeat periodically except for finitely many terms. We call them *baby-fractal sequences*. The '$3n+1$' type problems, iterated applications of absolute differences (Thwaites [Thw'96a]) and the gpf-sequences produce examples galore of such sequences.

7.1 The '$3n+1$' conjecture

For any positive integer n, let $C(n) = n/2$, for n even and $C(n) = 3n + 1$, for n odd. For example, starting with $n = 7$, repeated application of $C(\cdot)$ gives:

$$7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, \mathbf{4}, \mathbf{2}, \mathbf{1}, 4, 2, 1, \ldots$$

The '$3n+1$' conjecture asserts that for any $n \geq 1$, the sequence $\left\{C^{(k)}(n)\right\}_{k\geq 1}$ eventually enters into the cycle $1, 4, 2$. The number of iterations necessary to reach the cycle may be large even when n is small (for example, starting with $n = 27$, it takes 111 steps to reach 1). Collatz [Cox'71] mentions the '$3n+1$' problem for the first time in a mathematics journal, but the problem entered the mathematical folklore in the early 1950's. It was first posed by Collatz in 1937 and independently by Thwaites (cf. Wirsching [Wir'98]). Thwaites [Thw'85] indicates even the day and the hour of his discovery, while Collatz [Col'86] says that because he couldn't solve it, he did not publish anything, but the problem was publicized in seminars in different countries after he told it to Helmut Hasse. In spite of numerous tries to solve it, the conjecture is wide open. Wirsching [Wir'98], Chamberland [Cha'03b] and Lagarias [Lag'11], [Lag'12] wrote surveys on the problem. Functions analogue to $C(\cdot)$ and similar problems gather into a large category, but a general statement on periodicity of the generated sequences looks similar to one about a Turing machine, which falls beyond the undecidability line. This was proved by Conway [Con'72], while Kurtz and Simon [KS'07] showed that a certain generalization of the '$(3n+1)$'-problem is undecidable.

7.2 The greatest prime factor, a resourceful tool and a startling phenomenon

Let $\mathrm{spf}(n)$ be *the smallest prime factor* and $\mathrm{gpf}(n)$ *the greatest prime factor* of any integer $n \geq 2$. These functions are simple to define but very deep in nature. In recent studies, mostly, the $\mathrm{gpf}(\cdot)$ function was employed and many generated sequences are like baby fractals, terminating in surprising cycles. Back and Caragiu [BC'10] combined the Fibonacci growing rule with the $\mathrm{gpf}(\cdot)$ function in the role of a molifier. The new terms are obtained by a fixed combination of neighbor terms. In this sense, the Fibonacci growing rule resembles exactly the Pascal-rule, since $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ and $F_0 = 0$, $F_1 = 1$. The general idea is to consider recursive sequences of primes produced by the linear formula:

$$p_j = \mathrm{gpf}(a_0 + a_1 p_{j-1} + a_2 p_{j-2} + \cdots + a_r p_{j-r})\,. \tag{7.1}$$

(Here, the coefficients $a_0, \ldots, a_r$ are non negative integers.) The case $r = 1$ and $a_0 > 0$ where investigated by Caragiu and Scheckelhoff [CS'06] and by Caragiu and Back [BC'09]. In this case, they showed that when a_0 divides a_1 the generated sequences are always ultimately periodic. Furthermore, based also on computer investigations, they conjecture that this property characterizes all these sequences that do not necessarily satisfy the divisibility constrain on the coefficients.

The gpf-Fibonacci sequences are those generated by $p_j = \mathrm{gpf}(p_{j-1} + p_{j-2})$, for $j \geq 2$. For instance, if $p_0 = 509$, $p_1 = 673$, the first elements are:

$$509, 673, 197, 29, 113, 71, 23, 47, \mathbf{7}, \mathbf{3}, \mathbf{5}, \mathbf{2}, 7, 3, 5, 2, \ldots$$

Back and Caragiu [BC'10] proved that if the first two terms of a gpf-Fibonacci sequence are distinct, then it eventually enters into the 4-cycle $7, 3, 5, 2$. Similarly, the gpf-Tribonacci sequences are given by $p_j = \mathrm{gpf}(p_{j-1} + p_{j-2} + p_{j-3})$, for $j \geq 3$. Two examples of gpf-Tribonacci sequences are:

$$9431, 563, 523, 809, 379, \mathbf{59}, \mathbf{43}, \mathbf{37}, \mathbf{139}, \mathbf{73}, \mathbf{83}, 59, 43, 37, \ldots\ldots \tag{7.2}$$

and

$$\begin{gathered}\mathbf{31},\mathbf{13},\mathbf{7},\mathbf{17},\mathbf{37},\mathbf{61},\mathbf{23},\mathbf{11},\mathbf{19},\mathbf{53},\mathbf{83},\mathbf{31},\mathbf{167},\mathbf{281},\mathbf{479},\mathbf{103},\mathbf{863},\\ \mathbf{17},\mathbf{983},\mathbf{23},\mathbf{31},\mathbf{61},\mathbf{23},\mathbf{23},\mathbf{107},\mathbf{17},\mathbf{7},\mathbf{131},31,13,7,\dots\end{gathered} \tag{7.3}$$

Again, both sequences (7.2) and (7.3) enter into cycles. The length of the first one is 6, while the length of the second one is 28. This phenomenon is widely spread, but these two cycles are quite rare. In fact, most astonishingly, in the large class of gpf-Tribonacci sequences, only 4 distinct cycles where discovered so far. The other two have length 100 and 212. Back and Caragiu observed that almost 99% of gpf-Tribonacci sequences with $p_0, p_1, p_2 \le 1000$ enter into one of the longer cycles and the one of length 100 is encountered about three times more often than the one of length 212. They state the following conjecture.

Conjecture 2 (Back, Caragiu [BC'10])**.** *All recurrent sequences of primes defined by relation* (7.1) *are ultimately periodic.*

A multidimensional version of the problem with vector analogues of gpf-sequences is formulated by Caragiu, Sutherland and Zaki [CSZ'11]. Using bounds on the spacings between consecutive primes, Caragiu, Zaki and one of the authors [CZZ'12] found the rate of growth of the related infinite order recurrent sequences of primes defined by $q_j = \operatorname{gpf}(q_1+q_2+\cdots+q_{j-1}), \quad \text{for } j \ge 2$. They showed that $q_j = j/2 + O\big(j^{0.525}\big)$.

7.3 Ultimately periodic sequences in Ducci games

Let $d \ge 2$ be a positive integer, let $\mathbb{N}_d$ be the set of sequences with nonnegative integer entries that are periodic of period d, and define the application $\phi_d\colon \mathbb{N}_d \to \mathbb{N}_d$, $\phi_d(a_0, a_1, \dots) := (a_0', a_1', \dots)$, where $a_j' = |a_j - a_{j-1}|$ for $j \ge 0$. Let $\mathbf{a} \in \mathbb{N}_d$ and denote by $\|\mathbf{a}\|$ the largest element of $\mathbf{a}$. Since in $\mathbb{N}_d$ there are finitely many sequences whose components are $\le \|\mathbf{a}\|$, it follows that starting with any $\mathbf{a} \in \mathbb{N}_d$ and applying ϕ_d repeatedly, produces a sequence that eventually enters into a cycle. The first interesting fact is that the length of each cycle is 1 if and only if d is a power of 2. Thus,

$$\phi_{2^k}^{(n)}(\mathbf{a}) = (0, 0, \dots), \quad \text{for any } k \in \mathbb{N},\ \mathbf{a} \in \mathbb{N}_{2^k} \text{ and sufficiently large } n. \tag{7.4}$$

Pairing neighbor numbers placed around a circular table and taking absolute differences recursively is just a different setting of the same problem. The original Ducci game begins with only four numbers, and the evolution may be viewed in a Diffy box. Start by placing the numbers on the corners of a square. Then, for the next generation, the absolute differences of neighbor numbers are recorded on the middle of the edges. This produces a new square, on which the operation is repeated. The iterated process eventually ends with 4 zeros. During the action a graph is generated and one may consider a more developed game, by taking the absolute differences of neighbor numbers on the other edges. In the same way, a similar game may be played on different graphs. This leads tangentially to Golomb rulers problems, code theory and even worldly applications (Malkevitch [Mal'12]).

The origin of the problem is not completely clear (cf. Thwaites [Thw'96b] and Behn et al. [BKP'05]). The article of Ciamberlini and Marengoni [CM'37] seems to be the first published

source on the subject. Ciamberlini and Marengoni begin their work by stating that Prof. Ducci has communicated the problem to them long before. This might had happened in the XIXth century, as both Enrico Ducci (1864–1940) and Corrado Ciamberlini (1861–1944) lived at the time. Anyhow, Thomas et al. [CT'04] and [CST'05] place the first reference to Ducci games at the end of the XIXth century. But for this, their cited support is Honsberger [Hon'98], whose single reference from page 73 attributes the cyclic quadruples game to an observation made by professor E. Ducci in the 1930's. The problem was discovered repeatedly, and several independent proofs where given (see Ciamberlini and Marengoni [CM'37], Meyers [Mey'82], Pompilli [Pom'96], Thwaites [Thw'96b], Andriychenko and Chamberland [AC'00], Crâşmaru and the authors[CCZ'00], Chamberland [Cha'03a] and the references therein).

Following a note of Campbell [Cam'96] that advertized Thwaites conjectures [Thw'96a] and being unaware of the other previous results, Crâşmaru and the authors [CCZ'00] gave three distinct proofs of problem (7.4), the second £100–conjecture of Thwaites. Regarding the cycles, it turns out that the essence of the evolution function ϕ is captured by its restriction to $\mathbb{U}_d$, the subset of all the elements of $\mathbb{N}_d$ with components in $\{0,1\}$. The advantage is that the evolution function restricted in this way is additive. Then the j-th component of $\phi^{(n)}_{|\mathbb{U}_d}(a_0, a_1, \ldots)$ is equal to

$$\sum_{k=0}^{n} \binom{n}{j} a_{j+k} \pmod 2.$$

Moreover, one only needs to understand the transforms of the unitary sequence $\mathbf{e_0}$, whose components are all equal to zero, except those of rank divisible by d, which are equal to one. In this case we have

$$\phi^{(n)}(\mathbf{e_0}) = \big(S_d(n,0),\, S_d(n,-1),\, S_d(n,-2),\, \ldots\big) \pmod 2, \tag{7.5}$$

where

$$S_d(n,r) = \sum_{\substack{1\le k\le n \\ k\equiv r \pmod d}} \binom{n}{k}.$$

Expression (7.5) is used in [CCZ'00] to investigate the general problem of finding the length of the cycles for arbitrary d. It is shown that the periods (multiples of the length of cycles) depend on the order of 2 modulo the largest odd factor of d. Short periods occur when $d = 2^p - 1$ is a Mersenne prime (47 such primes are known so far: 3, 7, 31, 127, 8191, ..., $2^{43\,112\,609} - 1$). In this case d is the length of the cycle. Long cycles may happen when d is prime and 2 is a primitive root modulo d. (For example 3, 5, 11, 13, 19, 29, 37, ... and Artin's conjecture states that there are infinitely many such primes.) In these cases $d\big(2^{\frac{d-1}{2}} - 1\big)$ is a period. Anyhow, very long cycles do occur, because known partial results on Artin's conjecture allow to deduce that there are infinitely many primes d for which the length of the cycle is larger than $2^{d^{1/4}}$.

A more general evolution function is defined as follows. Let $\mathcal{S} = \{0,1\}^d$. Denote by $\rho(\mathbf{x})$ the circular rotation to the right of the vector $\mathbf{x} \in \mathcal{S}$ (e.g., for $d = 5$, $\rho(1,0,1,0,0) = (0,1,0,1,0)$) and $\cdot\colon \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ the componentwise addition modulo 2. Let s and $\alpha_1, \ldots, \alpha_s$ be positive integers and define $\theta\colon \mathcal{S} \to \mathcal{S}$ by

$$\theta(\mathbf{x}) = \rho^{(\alpha_1)}(\mathbf{x}) \cdots \rho^{(\alpha_s)}(\mathbf{x}).$$

Since $\theta(\mathbf{x})$ has an accentuated chaotic character, Crâşmaru [Cra'01] initiated the construction of a cryptosystem based on the fact that there is an effective way to calculate $\theta(\mathbf{x})$. For $s = 2$, $\alpha_1 = 0$ and $\alpha_2 = 1$ the function $\theta(\mathbf{x})$ replicates Ducci's evolution function. There is an efficient algorithm to calculate $\theta^{(n)}(\mathbf{x})$ in $O_s(\log n)$ steps (cf. [CCZ'00]). In the case $\theta(\mathbf{x}) = \mathbf{x}\rho(\mathbf{x})$, the procedure is based on the formula

$$\theta^{(n)}(\mathbf{x}) = \mathbf{x} \prod_{R \subset \mathcal{P}(\mathcal{R}_n)} \rho^{\left(\sum_{r\in R} r\right)}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{S}, \tag{7.6}$$

where $n = 2^{l_0} + 2^{l_1} + \cdots + 2^{l_\mu}$ with $l_0 < \cdots < l_\mu$ is the representation of n in base 2, and

$$\mathcal{R}_n = \big\{ r \colon r \equiv 2^{l_i} \pmod d,\ \ 0 \le r \le d-1,\ \text{ for some } \ 0 \le i \le \mu \big\}.$$

For any $m \in \{1, \dots, d\}$ denote

$$\nu_{n,d}(m) = \#\{\mathcal{R} \subset \mathcal{R}_n \colon \sum_{r\in\mathcal{R}} r \equiv m \pmod d\}.$$

The representation (7.6) and the fact that $(0, \dots, 0)$ and $(1, \dots, 1) \in \mathcal{S}$ are the only fixed points of $\rho(\mathbf{x})$ yield a criterion of periodicity, which states that a positive integer n is a period for $\theta(\mathbf{x}) = \mathbf{x}\rho(\mathbf{x})$ if and only if the numbers $\nu_{k,d}(m)$, $1 \le m \le d$, have the same parity (cf. [CCZ'00, Corollary 3]). Furthermore, computer experiments show that when n is the length of the shortest period, the numbers $\nu_{n,d}(m)$ are most of the time equal.

Conjecture 3 ([CCZ'00])**.** *Suppose d is prime, s is the order of 2 mod d, s is even and $n = d(2^{s/2} - 1)$. Then*

$$\nu_{n,d}(1) = \nu_{n,d}(2) = \cdots = \nu_{n,d}(d-1) = \nu_{n,d}(d) + 2.$$

As consequence of Conjecture 3, we get precise values for the length of the periods of Ducci's evolution function $\theta(\mathbf{x}) = \mathbf{x}\rho(\mathbf{x})$. The stretch of the initial iterations before the games enter into the cycle and various results on the length of cycles in the n-Ducci game for particular values and for n satisfying divisibility constrains were obtained by Webb [Web'82], Ludington [Lud'88], Ehrlich [Ehr'90], Ludington-Young [LY'90] and [LY'99], Creely [Cre'88], Calkin et al. [CST'05], Lidman and Thomas [LT'07], Brown and Merzel [BM'07]. Some generalizations of the Ducci game with weights, with p-adic integers or with algebraic numbers are considered in Chamberland [Cha'03a], Breuer [Bre'07] and [Bre'10], Baxter and Caragiu [BC'07], Caragiu, Zaki and the second author [CZZ'11], while the close relation between the length of the cycles and Pascal triangle modulo 2 is studied by Glaser and Schöffl [GS'95], and Breuer [Bre'98].

8 p-adic functions and convergents to e

The convergents of continued fractions of linear fractional transformations involving the Euler number e and the special exponentials $e^{2/h}$ reveal augmented symmetries. Farther, running across the sequence of denominators of convergents of the continued fraction of e, one finds

that their divisibility properties exhibit attractive 'supercongruences' modulo powers of primes from a distinguished set $\mathcal{B}$ (cf. Berndt et al. [BKZ'12]), which are encoded by six remarkable p-adic functions. These functions satisfy certain functional equations and are represented by *binomial coefficient series*, whose coefficients also carry the signature of the entries from the Pascal triangle.

Since 1873, when Hermite [Coh'06] proved that e is not an algebraic number, continued fractions became an important instrument not only in diophantine approximation, but also in finding the barrier between transcendence and algebraicity. There is a regularity in the continued fraction of e, more precisely $e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, \ldots]$. Hurwitz (cf. Perron [Per'54, §33]) proved a general statement, which roughly asserts that if the terminal components of the continued fraction for a number r consists of a few arithmetic progressions, then $(ar + b)/(cr + d)$, were $a, b, c, d \in \mathbb{Z}$ and $ad - bc \neq 0$ has the same property. For example, we have:

$$\frac{1}{2e-5} = [2, 3, 2, 3, 1, 2, 1, 3, 4, 3, 1, 4, 1, 3, 6, 3, 1, 6, 1, 3, 8, 3, 1, 8, 1, 3, \ldots] \tag{8.1}$$

and

$$\frac{-36e+98}{78e-212} = [5, 2, 5, 1, 2, 5, 1, 2, 1, 5, 4, 5, 1, 4, 5, 1, 4, 1, 5, 6, 5, 1, 6, 5, 1, 6, 1, 5, \ldots] \,. \tag{8.2}$$

A similar result occurs for generalized continued fractions. Inspired by Angel's [Ang'10] investigation of a family of generalized continued fractions that converge to rational numbers, Gottfried Helms found, by computer experiments, a few examples that converge to rational expressions involving e. In particular, the expressions that are analogous to relations (8.1) and (8.2) are:

$$\frac{1}{2e-5} = 2 + \frac{1}{3+}\,\frac{2}{4+}\,\frac{3}{5+}\,\frac{4}{6+}\,\cdots$$

and

$$\frac{-36e+98}{78e-212} = 5 + \frac{3}{6+}\,\frac{4}{7+}\,\frac{5}{8+}\,\frac{6}{9+}\,\cdots$$

The result of Hurwitz applies as well to some exponentials $r = e^{a/h}$, with a, h positive integers. The following expressions were already known by Euler, Stieltjes and Hurwitz:

$$e^{\frac{1}{q}} = [\overline{1, (2j-1)q-1, 1}]_{j=1}^{\infty} = [1, q-1, 1, 1, 3q-1, 1, 1, 5q-1, 1, 1, 7q-1, \ldots] \tag{8.3}$$

for $q \geq 1$, and

$$e^{\frac{2}{2q+1}} = [\overline{1, (1+6j)q+3j, (12+24j)q+6+12j, (5+6j)q+2+3j, 1}]_{j=0}^{\infty}, \tag{8.4}$$

for $q \geq 0$. Thakur [Tha'96] further extended Hurvitz's result. He found the patterns displayed by the simple continued fractions for the analogues of the fractional transform $(ae^{2/h} + b)/(ce^{2/h} + d)$ in function fields $\mathbb{F}_q[t]$. We remark that relations (8.4) and (8.3) are special, in the sense that no such relations are known for e^3 or for other powers of e. Also, the number of arithmetic progressions needed to express the tail of (8.3) and (8.4) is small (three, respectively five), but their fractional transforms may need many. For example, 94 arithmetic progressions are needed to describe the tail of the continued fraction of $3e^{1/3} - 1/3$.

Ramanujan, and later Davis, found best possible diophantine approximations of $e^{2/a}$ (see Berndt et al. [BKZ'12] for a detailed description). Sondow and Schlam's investigations [Son'06], [SS'08], [SS'10], compare convergents of the simple continued fraction of e to the partial sums of the Taylor series of e. Berndt, Kim and the second author [BKZ'12] proved that at most $O_a\big(\log n\big)$ of the first convergents of $e^{2/a}$ may coincide with partial fractions of the series

$$e^{2/a} = \sum_{j=0}^{\infty} \frac{2^j}{a^j\, j!}\,. \tag{8.5}$$

Furthermore, as a consequence of a finer analysis in the case $a = 2$, they proved that equality occurs only twice, which settles a conjecture of Sondow [Son'06]. This required the study of the symmetries of the sequence $\{q_j\}_{j\in\mathbb{Z}}$ of the denominators of convergents. (Here, for $j \le 0$ the definition of q_j is made naturally, by the same recursive defining relation of the convergents for $j \ge 0$.) An important role is played by their images modulo powers of primes selected from a distinguished set of primes: $\mathcal{B} := \big\{p\colon\ q_{j+6p} \equiv q_j \pmod p,\ \text{for any } j \in \mathbb{Z}\big\}$. There are other equivalent ways to define the set $\mathcal{B}$, whose first elements are:

$$\mathcal{B} := \big\{3, 7, 11, 17, 47, 53, 61, 67, 73, 79, 89, 101, 139, \dots\big\}\,.$$

Based in part on numerical data, in [BKZ'12] the following conjecture is made:

Conjecture 4 (Berndt, Kim, Zaharescu [BKZ'12])**.**

$$\lim_{x\to\infty} \frac{\#\{p \in \mathcal{B}\colon\ p \le x\}}{\pi(x)} = \frac{1}{e}\,.^{3}$$

The divisibility and the congruence properties of the sequence $\{q_j\}_{j\in\mathbb{Z}}$ are captured by a few functions $f_r\colon \mathbb{Z} \to \mathbb{Z}$ defined by $f_r(l) := q_{6l-r}$, for $l \in \mathbb{Z}$. These functions are 1-Lipschitzian with respect to the p-adic absolute value $|\cdot|_p$ on the field of p-adic numbers $\mathbb{Q}_p$, normalized by $|p|_p = 1/p$. Consequently, by a result of Mahler [Mah'58], they can be represented in a *Pascal series*

$$f_r(x) = a_{r,0} + a_{r,1}x + a_{r,2}\frac{x(x-1)}{2} + \cdots + a_{r,j}\binom{x}{j} + \cdots\,,$$

where

$$a_{r,j} = \sum_{k=0}^{j} (-1)^k \binom{j}{k} f(j-k)\,,$$

that is, $a_{r,0} = f_r(0) = q_{-r}$, $a_{r,1} = f_r(1) - f_r(0) = q_{-r+6} - q_{-r}$, $a_{r,2} = f_r(2) - 2f_r(1) + f_r(0) = q_{-r+12} - 2q_{-r+6} + q_{-r}$, ... This is reminiscent of *continued fractions* of Hurwitz type, whose definition involves binomial coefficients and the iterated difference operator (cf. Perron [Per'54, §32]). Of these functions, only six of them $f_r(x)$ for $r = 1, \dots, 6$ are essential. This sextuple of functions is intrinsically linked by a few functional relations. Moreover, each $f_r(x)$ has a unique extension by continuity to a function defined on $\mathbb{Z}_p$, the ring of integers in $\mathbb{Q}_p$, and the

[3] Unrelated to $\mathcal{B}$, but based on similar heuristic arguments, the proportion of missed residue classes modulo p by the set of factorials $\{1!, 2!, \dots, p!\}$ is conjectured by Richard K. Guy to have the same limit [CVZ'00].

extensions satisfy the same functional equations. Among the properties of any function $f_r(x)$ that are reckoned as interesting to investigate, Berndt et al. [BKZ'12] highlight the natural questions:

Question 3 (BKZ2012)**.** (a) *Is $f_r(x)$ differentiable?* (b) *Does $f_r(x)$ has any zeros except certain trivial ones?*

We remark that Question 3 a. is equivalent to the limit $a_{r,j}/j \to 0$ as $j \to \infty$.

9 A whopping class of fractals

The authors [CZ'12] consider triangles generated by a multiplicative rule. These are similar to Pascal arithmetic triangle, except that the length of the new lines appears as decaying. In reality, triangles may have any size, since we view them as chunks cut off from an infinite triangle, if the first generating row is so. Let us consider the following function

$$Z(a,b) := \frac{ab}{\gcd(a,b)^2}\,.$$

Starting with a given sequence of integers, the subsequent generations are obtained as follows: under two consecutive terms, say a and b, in the next generation put $Z(a,b)$. For example, Figure 4 shows the triangle of order $n = 10$ that starts on the first line with the sequence of natural numbers. More precisely, this is triangle

$$T_{\mathbb{N}}(n) := \big\{a_{j,k} \colon 1 \le j \le k \le n\big\}, \tag{9.1}$$

where $a_{1,k} = k$ for $k \ge 1$ and $a_{j+1,k} = Z(a_{j,k-1}, a_{j,k})$, for $j \ge 1$, $k \ge 2$.

1		2		3		4		5		6		7		8		9		10	
	2		6		12		20		30		42		56		72		90		
		3		2		15		6		35		12		63		20			
			6		30		10		210		420		84		1260				
				5		3		21		2		5		15					
					15		7		42		10		3						
						105		6		105		30							
							70		70		14								
								1		5									
									5										

Figure 4: Triangle $T_{\mathbb{N}}(10)$, that is, the cut-off triangle of order 10 generated by the repetead application of the $Z(\cdot,\cdot)$ rule on the sequence of positive integers.

This is reminiscent of the Gilbreath's Conjecture from 1958 (for which an incorrect proof was given by Proth in 1878). In that case, the first line of the triangle from Figure 5 lists the primes and the following generations are obtained by taking the absolute difference of neighbor numbers and iterating this operation. The conjecture states that the left edge of that triangle contains only ones.

2		3		5		7		11		13		17		19		23		29						
	1		2		2		4		2		4		2		4		6							
		1		0		2		2		2		2		2		2								
			1		2		0		0		0		0		0									
				1		2		0		0		0		0										
					1		2		0		0		0											
						1		2		0		0												
							1		2		0													
								1		2														
									1															

Figure 5: Triangle generated by iterated absolute differences staring with the sequence of the first 10 primes.

Odlyzko [Odl'93] confirmed Gilbreath's earlier checks and verified the conjecture for triangles of order $< \pi(10^{13})$. He had also offered heuristics to support it for analogue triangles produced by starting with many other sequences whose spacings are sufficiently random and not too large. The analogous property of the triangle from Figure 4 is that the maximal powers of all primes that divide the numbers situated on the left edge are ones.

Conjecture 5. *The left edge of the triangle $T_{\mathbb{N}}(n)$, for $n \geq 1$, contains only square free numbers.*

Spectacular properties of the $Z(\cdot,\cdot)$-generated triangles are revealed by the scans of their p-localized versions. Each of these renderings may be considered as a *face* of the corresponding prime, an analogue facet of the Pascal arithmetic triangle modulo p. The collection of these p-generated images build on the unique 'p-print face' and inherent character. For instance, for the prime $p = 3$, Figure 6 is just a bit from this collection. There, darker colors indicate higher powers. Notice that each number m that is part of the triangle carries a 'potential energy' proportional to the maximum power of p in m. This energy radiates in the subsequent generations along the 'force lines' on the left and on the right, and downward. Also, when forces coming from opposite directions meet, their strength cancels like in a physical dynamic clash. These carry flavors from the world of billiards, in which techniques involving Farey series play a significant role in solving problems that involve successive insertions (see Boca et al. [BGZ'03a], [BGZ'03b], [BZ'07], [BG'09], [Boca'10] and Alkan et al. [ALZ'06]). The propagation of the potential energy continues endlessly as long as the sequence from the first line does not terminate. This does not happen when the $Z(\cdot,\cdot)$ rule is applied repeatedly on the n-th row of the Pascal triangle. In that case, triangles are radically distinctive, changing abruptly with the change of n. Looking at their localizations, even modulo the same p, one hardly notices any similarity in figures for successive values of n. This is a result of mixing the additive insertion rule used in the construction of the Pascal triangle with the multiplicative fibers distilled by the $Z(\cdot,\cdot)$ rule.

Let $T_{\mathcal{B}}(n)$ be the $Z(\cdot,\cdot)$-generated triangles, whose first line are the binomial coefficients: $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$. The 5-local renderings of $T_{\mathcal{B}}(n)$ with $n = 24, 74, 124, 374, 624$ show that

they are entirely flat. This is a consequence of the fact that for these values of n, no binomial coefficient $\binom{n}{k}$, for $0 \leq k \leq n$, is divisible by 5.

Question 4. *For fixed p, are there infinitely many $n > 1$ for which the p-localized triangle $T_{\mathcal{B}}(n)$ is completely flat?*

We conclude our promenade by tempting the reader to explore patterns such as those that start to reveal in Figure 6., where the constraints imposed by the small size of the triangle lead to the appearance of a central darker figure which looks like a dwarf.

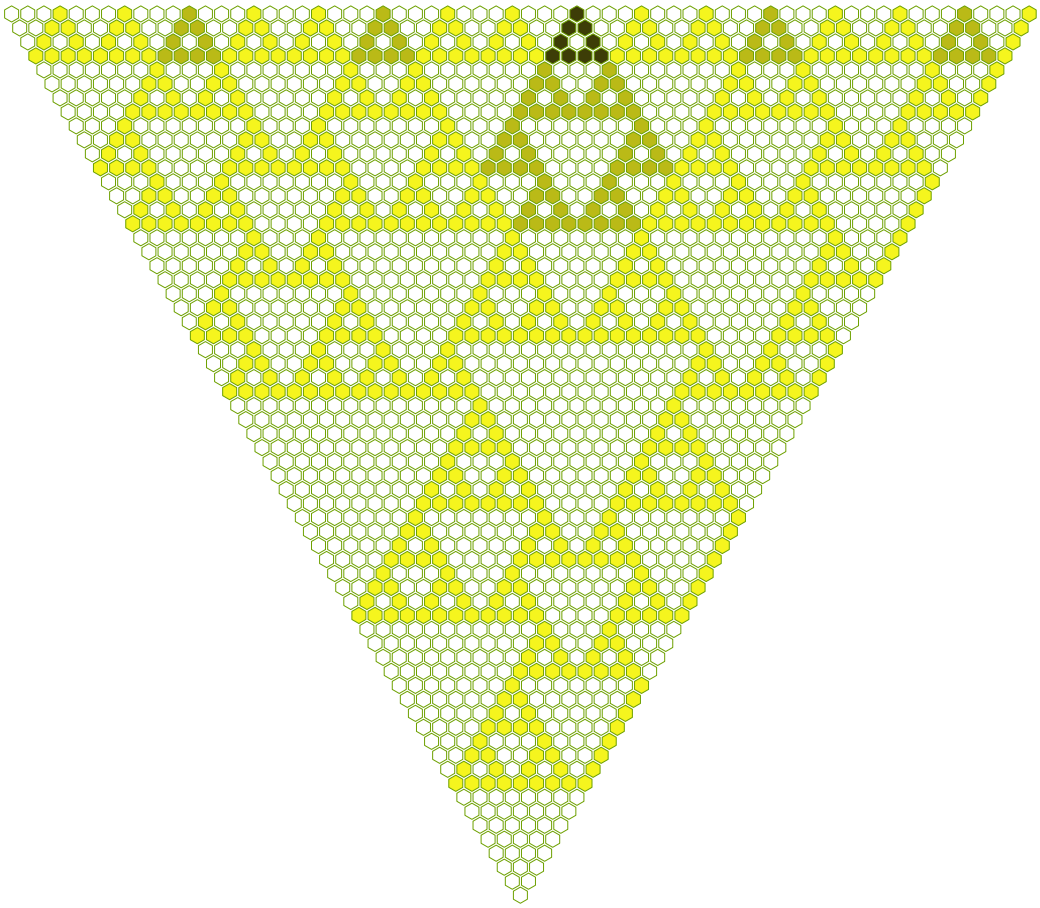

Figure 6: The 3-adic scan of order 64 of the Fibonacci–$Z(\cdot,\cdot)$-generated triangle.

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