

# Circuit Theory

[Wikibooks.org](https://en.wikibooks.org/)

March 16, 2013

On the 28th of April 2012 the contents of the English as well as German Wikibooks and Wikipedia projects were licensed under Creative Commons Attribution-ShareAlike 3.0 Unported license. An URI to this license is given in the list of figures on page 141. If this document is a derived work from the contents of one of these projects and the content was still licensed by the project under this license at the time of derivation this document has to be licensed under the same, a similar or a compatible license, as stated in section 4b of the license. The list of contributors is included in chapter Contributors on page 137. The licenses GPL, LGPL and GFDL are included in chapter Licenses on page 145, since this book and/or parts of it may or may not be licensed under one or more of these licenses, and thus require inclusion of these licenses. The licenses of the figures are given in the list of figures on page 141. This PDF was generated by the L<sup>A</sup>T<sub>E</sub>X typesetting software. The L<sup>A</sup>T<sub>E</sub>X source code is included as an attachment (`source.7z.txt`) in this PDF file. To extract the source from the PDF file, we recommend the use of <http://www.pdflabs.com/tools/pdftk-the-pdf-toolkit/> utility or clicking the paper clip attachment symbol on the lower left of your PDF Viewer, selecting **Save Attachment**. After extracting it from the PDF file you have to rename it to `source.7z`. To uncompress the resulting archive we recommend the use of <http://www.7-zip.org/>. The L<sup>A</sup>T<sub>E</sub>X source itself was generated by a program written by Dirk Hünninger, which is freely available under an open source license from [http://de.wikibooks.org/wiki/Benutzer:Dirk\\_Huenniger/wb2pdf](http://de.wikibooks.org/wiki/Benutzer:Dirk_Huenniger/wb2pdf). This distribution also contains a configured version of the `pdflatex` compiler with all necessary packages and fonts needed to compile the L<sup>A</sup>T<sub>E</sub>X source included in this PDF file.

# Contents

<b>1</b>	<b>Preface</b>	<b>3</b>
<b>2</b>	<b>Introduction</b>	<b>5</b>
<b>3</b>	<b>Basic Terminology</b>	<b>7</b>
3.1	Basic Terminology . . . . .	7
3.2	Summary . . . . .	8
<b>4</b>	<b>Variables and Standard Units</b>	<b>9</b>
4.1	Electric Charge (Coulombs) . . . . .	9
4.2	Voltage (Volts) . . . . .	9
4.3	Current (Amperes) . . . . .	10
4.4	Energy and Power . . . . .	11
<b>5</b>	<b>Electric Circuit Basics</b>	<b>13</b>
5.1	Circuits . . . . .	13
5.2	Ideal Wires . . . . .	13
5.3	Ideal Junctions or Nodes . . . . .	14
5.4	Measuring instruments . . . . .	14
5.5	Active Passive & ReActive . . . . .	15
5.6	Open and Short Circuits . . . . .	15
<b>6</b>	<b>Resistors and Resistance</b>	<b>17</b>
<b>7</b>	<b>Resistive Circuit Analysis Techniques</b>	<b>21</b>
<b>8</b>	<b>Source Transformations</b>	<b>23</b>
8.1	Source Transformations . . . . .	23
8.2	Black Boxes . . . . .	23
8.3	Thevenin's Theorem . . . . .	23
8.4	Norton's Theorem . . . . .	25
8.5	Circuit Transforms . . . . .	26
8.6	Open Circuit Voltage and Short Circuit Current . . . . .	27
8.7	Why Transform Circuits? . . . . .	28
<b>9</b>	<b>Maximum Power Transfer</b>	<b>29</b>
9.1	Maximum Power Transfer . . . . .	29
<b>10</b>	<b>Resistive Circuit Analysis Methods</b>	<b>33</b>
10.1	Analysis Methods . . . . .	33
10.2	Superposition . . . . .	33

10.3	Impulse Response . . . . .	34
10.4	Convolution . . . . .	35
<b>11</b>	<b>Energy Storage Elements</b>	<b>37</b>
<b>12</b>	<b>First-Order Circuits</b>	<b>39</b>
12.1	First Order Circuits . . . . .	39
12.2	RL Circuits . . . . .	39
12.3	RC Circuits . . . . .	40
12.4	First Order Solution . . . . .	41
<b>13</b>	<b>RLC Circuits</b>	<b>43</b>
13.1	Series RLC Circuit . . . . .	43
13.2	Resonance . . . . .	46
13.3	Conclusion . . . . .	47
<b>14</b>	<b>The Second-Order Circuit Solution</b>	<b>49</b>
14.1	Second-Order Solution . . . . .	49
14.2	Finding Differential Equations . . . . .	49
14.3	Zero-Input Response . . . . .	50
14.4	Characteristic Equation . . . . .	50
14.5	Damping . . . . .	51
14.6	Series RLC . . . . .	55
14.7	Parallel RLC . . . . .	56
14.8	Circuit Response . . . . .	57
<b>15</b>	<b>Mutual Inductance</b>	<b>59</b>
15.1	Magnetic Fields . . . . .	59
15.2	Mutual Inductance . . . . .	59
15.3	Ideal Transformers . . . . .	60
15.4	Modern Inductors . . . . .	61
15.5	Step Up and Step Down . . . . .	62
15.6	further reading . . . . .	63
<b>16</b>	<b>State-Variable Approach</b>	<b>65</b>
16.1	State Variables . . . . .	65
16.2	Aim of State Variables . . . . .	65
16.3	State Variable Selection . . . . .	65
16.4	System of Equations . . . . .	65
16.5	Further Reading . . . . .	66
<b>17</b>	<b>Sinusoidal Sources</b>	<b>67</b>
17.1	Steady State . . . . .	67
17.2	Sinusoidal Forcing Functions . . . . .	67
17.3	Other Terms . . . . .	67
17.4	Lead and Lag . . . . .	68
17.5	Sinusoidal Response . . . . .	68
17.6	Sinusoidal Conventions . . . . .	69
17.7	Sinusoidal Sources . . . . .	69

<b>18</b>	<b>Phasor Representation</b>	<b>71</b>
<b>19</b>	<b>Phasors</b>	<b>73</b>
19.1	Variables . . . . .	73
19.2	Passive circuit output is similar to input . . . . .	73
19.3	Phasor Transform . . . . .	74
19.4	Euler's Equation . . . . .	75
19.5	Phasor Symbols . . . . .	77
19.6	Not Vectors . . . . .	77
19.7	Cosine Convention . . . . .	79
19.8	Phasor Concepts . . . . .	79
19.9	Phasor Math . . . . .	79
19.10	Function transformation Derivation . . . . .	80
19.11	Transforming calculus operators into phasors . . . . .	81
19.12	Phasor Domain to Time Domain . . . . .	84
19.13	What is not covered . . . . .	84
19.14	Phasor Notation . . . . .	85
19.15	Phasor symbols . . . . .	85
<b>20</b>	<b>Differential Equations</b>	<b>87</b>
20.1	Phasors Generate the Particular Solution . . . . .	87
20.2	Phasors Don't Generate Integration Constants . . . . .	87
20.3	Differential Equations Review . . . . .	87
<b>21</b>	<b>Phasor Circuit Analysis</b>	<b>91</b>
21.1	Phasor Analysis . . . . .	91
21.2	Resistance, Impedance and Admittance . . . . .	91
21.3	Resistors . . . . .	92
21.4	Capacitors . . . . .	92
21.5	Inductors . . . . .	93
21.6	Impedances Connected in Series . . . . .	93
21.7	Impedances in Parallel . . . . .	94
21.8	Steps For Solving a Circuit With Phasors . . . . .	94
21.9	Network Function . . . . .	94
21.10	Gain . . . . .	95
21.11	Phase Shift . . . . .	95
<b>22</b>	<b>Phasor Theorems</b>	<b>97</b>
22.1	Circuit Theorems . . . . .	97
22.2	Ohm's Law . . . . .	97
22.3	Kirchoff's Laws . . . . .	97
22.4	Superposition . . . . .	98
22.5	Thevenin and Norton Circuits . . . . .	98
22.6	Maximum Power Transfer . . . . .	99
<b>23</b>	<b>Complex Power</b>	<b>101</b>
23.1	Laplace Transform . . . . .	101
23.2	Laplace Domain . . . . .	101

23.3	The Transform . . . . .	102
23.4	The Inverse Transform . . . . .	102
23.5	Transform Properties . . . . .	103
23.6	Initial Value Theorem . . . . .	103
23.7	Final Value Theorem . . . . .	104
23.8	Transfer Function . . . . .	104
23.9	Convolution Theorem . . . . .	105
23.10	Resistors . . . . .	106
23.11	Ohm's Law . . . . .	106
23.12	Capacitors . . . . .	107
23.13	Inductors . . . . .	108
23.14	Impedance . . . . .	108
23.15	References . . . . .	108
23.16	Laplace Circuit Solution . . . . .	108
23.17	Steps . . . . .	108
23.18	Fourier Transform . . . . .	110
23.19	Radial Frequency . . . . .	110
23.20	Fourier Domain . . . . .	111
23.21	Impedance and Reactance . . . . .	112
23.22	Frequency Domain Analysis . . . . .	112
23.23	Frequency Response . . . . .	113
23.24	Decibel . . . . .	114
23.25	Notes on Decibels . . . . .	115
23.26	Bode Plots . . . . .	115
23.27	Bode Plots by Different Methods . . . . .	116
23.28	Bode Analysis . . . . .	116
23.29	Log Magnitude Graph . . . . .	116
23.30	Phase Graph . . . . .	116
23.31	Poles and Zeros . . . . .	116
23.32	Bode Equation Format . . . . .	117
23.33	DC Gain . . . . .	118
23.34	Break Frequencies . . . . .	118
23.35	At Zero Radial Frequency . . . . .	119
23.36	At a Break Point . . . . .	119
23.37	Conclusions . . . . .	120
23.38	See Also . . . . .	121
23.39	Impedance . . . . .	121
23.40	Susceptance and Admittance . . . . .	121
23.41	Parallel Components . . . . .	121
23.42	Series Components . . . . .	122
23.43	Solving Circuits . . . . .	122
<b>24</b>	<b>Circuit Functions</b>	<b>123</b>
24.1	Circuit Functions . . . . .	123
24.2	The "Impulse Response" . . . . .	123
24.3	The "Network Function" . . . . .	123
24.4	The "Transfer Function" . . . . .	124
24.5	The "Frequency Response" . . . . .	124

<b>25</b>	<b>Phasor Arithmetic</b>	<b>125</b>
25.1	Phasor Arithmetic . . . . .	125
25.2	Forms . . . . .	125
25.3	Converting between Forms . . . . .	125
25.4	Addition . . . . .	126
25.5	Subtraction . . . . .	126
25.6	Multiplication . . . . .	126
25.7	Division . . . . .	127
25.8	Inversion . . . . .	127
25.9	Complex Conjugation . . . . .	127
<b>26</b>	<b>Decibels</b>	<b>129</b>
26.1	Definition . . . . .	129
26.2	Voltage Calculation . . . . .	129
26.3	Inverse Calculation . . . . .	130
26.4	Reference Units . . . . .	130
26.5	Decibel Arithmetic . . . . .	130
<b>27</b>	<b>Transform Tables</b>	<b>133</b>
<b>28</b>	<b>Resources</b>	<b>135</b>
28.1	Further Reading . . . . .	135
<b>29</b>	<b>Contributors</b>	<b>137</b>
	<b>List of Figures</b>	<b>141</b>
<b>30</b>	<b>Licenses</b>	<b>145</b>
30.1	GNU GENERAL PUBLIC LICENSE . . . . .	145
30.2	GNU Free Documentation License . . . . .	146
30.3	GNU Lesser General Public License . . . . .	147





# 1 Preface

This wikibook is going to be an introductory text about electric circuits. It will cover some the basics of electric circuit theory, circuit analysis, and will touch on circuit design. This book will serve as a companion reference for a 1st year of an Electrical Engineering undergraduate curriculum. Topics covered include AC and DC circuits, passive circuit components, phasors, and RLC circuits. The focus is on students of an electrical engineering undergraduate program. Hobbyists would benefit more from reading Electronics<sup>1</sup> instead.

*This book is not nearly completed, and could still be improved. People with knowledge of the subject are encouraged to contribute.*

The main editable text of this book is located at [http://en.wikibooks.org/wiki/Circuit\\_Theory](http://en.wikibooks.org/wiki/Circuit_Theory). The wikibooks version of this text is considered the most up-to-date version, and is the best place to edit this book and contribute to it.

---

<sup>1</sup> <http://en.wikibooks.org/wiki/Electronics>



## 2 Introduction



## 3 Basic Terminology

### 3.1 Basic Terminology

There are a few key terms that need to be understood at the beginning of this book, before we can continue. This is only a partial list of all terms that will be used throughout this book, but these key words are important to know before we begin the main narrative of this text.

#### Time domain

The time domain is described by graphs of power, voltage and current that depend upon time. The "Time domain" is simply another way of saying that our circuits change with time, and that the major variable used to describe the system is time. Another name is "Temporal".

#### Frequency domain

The frequency domain are graphs of power, voltage and/or current that depend upon frequency such as Bode plots<sup>1</sup>. Variable frequencies in wireless communication can represent changing channels or data on a channel. Another name is the "Fourier domain"<sup>2</sup>. Other domains that an engineer might encounter are the "Laplace domain" (or the "s domain" or "complex frequency domain"), and the "Z domain". When combined with the time, it is called a "Spectral" or "Waterfall"<sup>3</sup>.

#### Circuit Response

Circuits generally have inputs and outputs. In fact, it is safe to say that a circuit isn't useful if it doesn't have one or the other (usually both). Circuit response is the relationship between the circuit's input to the circuit's output. The circuit response may be a measure of either current or voltage.

#### Steady State

The final value, when all elements have a constant or periodic behaviour, is known as the steady-state value of the circuit. The circuit response at steady state (when things aren't changing) is also known as the "steady state response". The steady state solution is called the **particular solution**<sup>4</sup>.

#### Transient Response

A transient response occurs when:

- 
- 1 [http://en.wikibooks.org/wiki/%3Aw%3ABode\\_plot](http://en.wikibooks.org/wiki/%3Aw%3ABode_plot)
  - 2 [http://en.wikibooks.org/wiki/%3Aw%3AFourier\\_transform](http://en.wikibooks.org/wiki/%3Aw%3AFourier_transform)
  - 3 <http://en.wikipedia.org/wiki/Spectrogram>
  - 4 [http://en.wikibooks.org/wiki/Ordinary\\_Differential\\_Equations%2FNon\\_Homogenous\\_1](http://en.wikibooks.org/wiki/Ordinary_Differential_Equations%2FNon_Homogenous_1)

a circuit is turned on or off

a sensor responds to the physical world changes

static electricity is discharged

an old car with old spark plugs (before resistors were put in spark plugs) drives by

Transient means momentary, or a short period of time. Transient means that the energy in a circuit suddenly changes which causes the energy storage elements to react. The circuit's energy state is forced to change. When a car goes over a bump, it can fly apart, feel like a rock, or cushion the impact in a designed manner. The goal of most circuit design is to plan for transients, whether intended or not.

Transient solutions are determined using a **homogeneous solution**<sup>5</sup> technique.

## 3.2 Summary

When something changes in a circuit, there is a certain transition period before a circuit "settles down", and reaches its final value. The response that a circuit has before settling into its *steady-state response* is known as the *transient response*. Using using **Euler's formula**<sup>6</sup>, **complex numbers**<sup>7</sup>, **phasors**<sup>8</sup> and the **s-plane**<sup>9</sup>, a **homogeneous solution**<sup>10</sup> technique will be developed that captures the transient response by assuming the final state has no energy. In addition, a **particular solution**<sup>11</sup> technique will be developed that finds the final energy state. Added together, they predict the *circuit response*.

The related **Differential equation**<sup>12</sup> development of homogeneous and particular solutions will be avoided.

---

5 [http://en.wikibooks.org/wiki/Ordinary\\_Differential\\_Equations%2FHomogenous\\_1](http://en.wikibooks.org/wiki/Ordinary_Differential_Equations%2FHomogenous_1)  
6 [http://en.wikibooks.org/wiki/%3Aw%3AEuler%2527s\\_formula](http://en.wikibooks.org/wiki/%3Aw%3AEuler%2527s_formula)  
7 [http://en.wikibooks.org/wiki/Arithmetic\\_Course%2FTypes\\_of\\_Number%2FComplex\\_Number](http://en.wikibooks.org/wiki/Arithmetic_Course%2FTypes_of_Number%2FComplex_Number)  
8 <http://en.wikibooks.org/wiki/%3Aw%3APhasors>  
9 <http://en.wikibooks.org/wiki/%3Aw%3AS-plane>  
10 [http://en.wikibooks.org/wiki/Ordinary\\_Differential\\_Equations%2FHomogenous\\_1](http://en.wikibooks.org/wiki/Ordinary_Differential_Equations%2FHomogenous_1)  
11 [http://en.wikibooks.org/wiki/Ordinary\\_Differential\\_Equations%2FNon\\_Homogenous\\_1](http://en.wikibooks.org/wiki/Ordinary_Differential_Equations%2FNon_Homogenous_1)  
12 [http://en.wikibooks.org/wiki/Ordinary\\_Differential\\_Equations](http://en.wikibooks.org/wiki/Ordinary_Differential_Equations)

## 4 Variables and Standard Units

### 4.1 Electric Charge (Coulombs)

**Note:**

An electron has a charge of  
**-1.602×10E-19 C.**

**Electric charge**<sup>1</sup> is a physical property<sup>2</sup> of matter<sup>3</sup> that causes it to experience a force<sup>4</sup> when near other electrically charged matter. Electric Charge (symbol q) is measured in SI units called "**Coulombs**", which are abbreviated with the letter capital C.

We know that  $q = n \cdot e$ , where  $n$  = number of electrons and  $e = 1.6 \cdot 10^{-19}$ . Hence  $n = 1/e$  coulombs. A Coulomb is the total charge of  $6.24150962915265 \times 10^{18}$  electrons, thus a single electron has a charge of  $-1.602 \times 10^{-19}$ .

It is important to understand that this concept of "charge" is associated with static electricity. Charge, as a concept, has a physical boundary that is related to counting a group of electrons. "Flowing" electricity is an **entirely different situation**<sup>5</sup>. "Charge" and electrons separate. Charge moves at the speed of light while electrons move at the speed of **1 meter/hour**<sup>6</sup>. Thus in most circuit analysis, "charge" is an abstract concept unrelated to energy or an electron and more related to the flow of **information**<sup>7</sup>.

Electric charge is the subject of many fundamental laws, such as **Coulomb's Law**<sup>8</sup> and **Gauss' Law**<sup>9</sup> (static electricity) but is not used much in circuit theory.

### 4.2 Voltage (Volts)

**Voltage** is a measure of the work required to move a charge from one point to another in a electric field. Thus the unit "volt" is defined as a Joules (J) per Coulomb (C).

- 1 [http://en.wikibooks.org/wiki/%3Aw%3AElectric\\_charge](http://en.wikibooks.org/wiki/%3Aw%3AElectric_charge)
- 2 <http://en.wikibooks.org/wiki/%3Aw%3Aphysical%20property>
- 3 <http://en.wikibooks.org/wiki/%3Aw%3Amatter>
- 4 <http://en.wikibooks.org/wiki/%3Aw%3Aforce>
- 5 [http://en.wikibooks.org/wiki/%3Aw%3AElectric\\_charge%23Static\\_electricity\\_and\\_electric\\_current](http://en.wikibooks.org/wiki/%3Aw%3AElectric_charge%23Static_electricity_and_electric_current)
- 6 [http://en.wikibooks.org/wiki/%3Aw%3ADrift\\_velocity](http://en.wikibooks.org/wiki/%3Aw%3ADrift_velocity)
- 7 [http://en.wikibooks.org/wiki/%3Aw%3APhysical\\_information](http://en.wikibooks.org/wiki/%3Aw%3APhysical_information)
- 8 [http://en.wikibooks.org/wiki/%3Aw%3ACoulomb%2527s\\_law](http://en.wikibooks.org/wiki/%3Aw%3ACoulomb%2527s_law)
- 9 [http://en.wikibooks.org/wiki/%3Aw%3AGauss%2527s\\_law](http://en.wikibooks.org/wiki/%3Aw%3AGauss%2527s_law)

$$V = \frac{W}{q}$$

W represents work, q represents an amount of charge. Charge is a static electricity concept. The definition of a volt is shared between static and "flowing" electronics.

Voltage is sometimes called "electric potential", because voltage represents the a difference in Electro Motive Force (EMF) that can produce current in a circuit. More voltage means more potential for current. Voltage also can be called "Electric Pressure", although this is far less common.

Voltage is not measured in absolutes but in *relative* terms. The English language tradition obscures this. For example we say "What is the distance to New York?" Clearly implied is the relative distance from where we are standing to New York. But if we say "What is the voltage at \_\_\_\_\_?" What is the starting point?

Voltage is defined between two points. Voltage is relative to where 0 is defined. We say "The voltage from point A to B is 5 volts." It is important to understand EMF and voltage are two different things.

When the question is asked "What is the voltage at \_\_\_\_\_?", look for the ground symbol on a circuit diagram. Measure voltage from ground to \_\_\_\_\_. If the question is asked "What is the voltage from A to B?" then put the red probe on A and the black probe on B (not ground).

The absolute is referred to as "EMF" or Electro Motive Force. The difference between the two EMF's is a voltage.

### 4.3 Current (Amperes)

**Current** is a measurement of the flow of electricity. Current is measured in units called **Amperes** (or "Amps"). An ampere is "charge volume velocity" in the same way water current could be measured in "cubic feet of water per second." But current is a **base SI unit**<sup>10</sup>, a fundamental dimension of reality like space, time and mass. A coulomb or charge is not. A coulomb is actually defined in terms of the ampere. "Charge or Coulomb" is a **derived SI Unit**<sup>11</sup>. The coulomb is a fictitious entity left over from the **one fluid /two fluid**<sup>12</sup> philosophies of the 18th century.

This course is about flowing electrical energy that is found in all modern electronics. Charge volume velocity (defined by current) is a useful concept, but understand it has no basis in reality. Do not think of current as a bundle electrons carrying energy through a wire. Special relativity<sup>13</sup> and quantum mechanics<sup>14</sup> concepts are necessary to understand how

---

10 [http://en.wikibooks.org/wiki/%3A%3ASI\\_base\\_unit](http://en.wikibooks.org/wiki/%3A%3ASI_base_unit)

11 [http://en.wikibooks.org/wiki/%3A%3ASI\\_derived\\_units%23Derived\\_units\\_with\\_special\\_names](http://en.wikibooks.org/wiki/%3A%3ASI_derived_units%23Derived_units_with_special_names)

12 [http://en.wikibooks.org/wiki/%3A%3AHistory\\_of\\_electromagnetic\\_theory%23Late\\_18th\\_century](http://en.wikibooks.org/wiki/%3A%3AHistory_of_electromagnetic_theory%23Late_18th_century)

13 [http://en.wikibooks.org/wiki/Special\\_Relativity](http://en.wikibooks.org/wiki/Special_Relativity)

14 [http://en.wikibooks.org/wiki/Quantum\\_Mechanics](http://en.wikibooks.org/wiki/Quantum_Mechanics)



electrons move at **1 meter/hour**<sup>15</sup> through copper, yet electromagnetic energy moves at near the speed of light.

**Charge**<sup>a</sup> is similar to the **rest mass**<sup>b</sup> concept of relativity and generates the **U(1) symmetry**<sup>c</sup> of electromagnetism

<sup>a</sup> [http://en.wikibooks.org/wiki/%3A%3AElectric\\_charge%23Properties](http://en.wikibooks.org/wiki/%3A%3AElectric_charge%23Properties)

<sup>b</sup> [http://en.wikibooks.org/wiki/%3A%3ARest\\_mass](http://en.wikibooks.org/wiki/%3A%3ARest_mass)

<sup>c</sup> [http://en.wikibooks.org/wiki/%3A%3ACharge\\_%28physics%29](http://en.wikibooks.org/wiki/%3A%3ACharge_%28physics%29)

Amperes are abbreviated with an "A" (upper-case A), and the variable most often associated with current is the letter "i" (lower-case I). In terms of coulombs, an ampere is:

$$i = \frac{dq}{dt}$$

For the rest of this book, the lower-case J ( j ) will be used to denote an imaginary number, and the lower-case I ( i ) will be used to denote current.

Because of the widespread use of complex numbers in Electrical Engineering, it is common for electrical engineering texts to use the letter "j" (lower-case J) as the imaginary number, instead of the "i" (lower-case I) commonly used in math texts. This wikibook will adopt the "j" as the imaginary number, to avoid confusion.

## 4.4 Energy and Power

Electrical theory is about energy storage and the flow of energy in circuits. Energy is chopped up arbitrarily into something that doesn't exist but can be counted called a coulomb. Energy per coulomb is voltage. The velocity of a coulomb is current. Multiplied together, the units are energy velocity or power ... and the unreal "coulomb" disappears.

### 4.4.1 Energy

Energy is measured most commonly in Joules, which are abbreviated with a "J" (upper-case J). The variable most commonly used with energy is "w" (lower-case W). The energy symbol is w which stands for work. Work is something good that we, as humans value.

From a thermodynamics point of view, all energy consumed by a circuit is work ... all the heat is turned into work. Practically speaking, this can not be true. If it were true, computers would never consume any energy and never heat up.

The reason that all the energy going into a circuit and leaving a circuit is considered "work" is because from a thermodynamic point of view, electrical energy is ideal. All of it can

<sup>15</sup> [http://en.wikibooks.org/wiki/%3A%3ADrift\\_velocity](http://en.wikibooks.org/wiki/%3A%3ADrift_velocity)

be used. Ideally all of it can be turned into work. Most introduction to thermodynamics courses assume that electrical energy is completely organized (and has entropy of 0).

#### 4.4.2 Power

A corollary to the concept of energy being work, is that all the energy/power of a circuit (ideally) can be accounted for. The sum of all the power entering and leaving a circuit should add up to zero. No energy should be accumulated (theoretically). Of course capacitors will charge up and may hold onto their energy when the circuit is turned off. Inductors will create a magnetic field containing energy that will instantly disappear back into the source through the switch that turns the circuit off.

This course uses what is called the "passive<sup>16</sup>" sign convention for power. Energy put into a circuit by a power supply is negative, energy leaving a circuit is positive.

Power (the flow of energy) computations are an important part of this course. The symbol for power is  $w$  (for work) and the units are Watts or W.

---

<sup>16</sup> [http://en.wikibooks.org/wiki/%3Aw%3AElectric\\_power%23Passive\\_sign\\_convention](http://en.wikibooks.org/wiki/%3Aw%3AElectric_power%23Passive_sign_convention)

# 5 Electric Circuit Basics

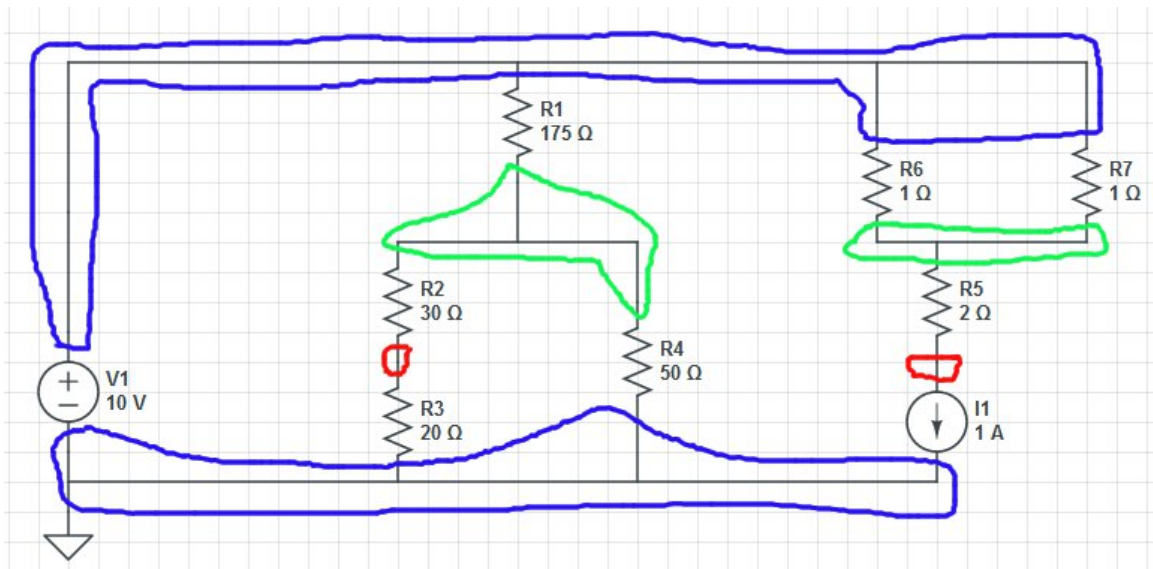
## 5.1 Circuits

**Circuits** (also known as "networks") are collections of circuit elements and wires. Wires are designated on a schematic as being straight lines. Nodes are locations on a schematic where 2 or more wires connect, and are usually marked with a dark black dot. Circuit Elements are "everything else" in a sense. Most basic circuit elements have their own symbols so as to be easily recognizable, although some will be drawn as a simple box image, with the specifications of the box written somewhere that is easy to find. We will discuss several types of basic circuit components in this book.

## 5.2 Ideal Wires

For the purposes of this book, we will assume that an ideal wire has zero total resistance, no capacitance, and no inductance. A consequence of these assumptions is that these ideal wires have infinite bandwidth, are immune to interference, and are — in essence — completely uncomplicated. This is not the case in real wires, because all wires have at least some amount of associated resistance. Also, placing multiple real wires together, or bending real wires in certain patterns will produce small amounts of capacitance and inductance, which can play a role in circuit design and analysis. This book will assume that all wires are ideal.

### 5.3 Ideal Junctions or Nodes



**Figure 1** Nodes are areas where the Electromotive Force is the same.

Nodes are also called "junctions" in this book in order to make a distinction between Node analysis, Kirchhoff's current law and discussions about a physical node itself. Here a physical node is discussed.

A junction is a group of wires that share the same electromotive force<sup>1</sup> (not voltage). Wires ideally have no resistance, thus all wires that touch wire to wire somewhere are part of the same node. The diagram on the right shows three big blue nodes, two smaller green nodes and two trivial (one wire touching another) nodes.

Sometimes a node is described as where two or more wires touch and students circle where wires intersect and call this a node. This only works on simple circuits.

One node has to be labeled ground in any circuit drawn before voltage can be computed or the circuit simulated. Typically this is the node having the most components connected to it. Logically it is normally placed at the bottom of the circuit logic diagram.

Ground is not always needed physically. Some circuits are floated<sup>2</sup> on purpose.

Node Quiz<sup>3</sup>

### 5.4 Measuring instruments

**Voltmeters and Ammeters** are devices that are used to measure the voltage across an element, and the current flowing through a wire, respectively.

<sup>1</sup> [http://en.wikibooks.org/wiki/%3Awikipedia%3AElectromotive\\_force](http://en.wikibooks.org/wiki/%3Awikipedia%3AElectromotive_force)

<sup>2</sup> [http://en.wikibooks.org/wiki/%3Aw%3AFloating\\_ground](http://en.wikibooks.org/wiki/%3Aw%3AFloating_ground)

<sup>3</sup> <http://en.wikibooks.org/wiki/%2FNodeQuiz%2F>

### 5.4.1 Ideal Voltmeters

An ideal voltmeter has an infinite resistance (in reality, several megaohms), and acts like an open circuit. A voltmeter<sup>4</sup> is placed across the terminals of a circuit element, to determine the voltage across that element. In practice the voltmeter siphons a enough energy to move a needle, cause thin strips of metal to separate or turn on a transistor so a number is displayed.

### 5.4.2 Ideal Ammeters

An ideal ammeter has zero resistance and acts like a short circuit. Ammeters require cutting a wire and plugging the two ends into the Ammeter. In practice an ammeter places a tiny resistor in a wire and measures the tiny voltage across it or the ammeter measures the magnetic field strength<sup>5</sup> generated by current flowing through a wire. Ammeters are not used that much because of the wire cutting, or wire disconnecting they require.

## 5.5 Active Passive & ReActive

The elements which are capable of delivering energy or which are capable to amplify the signal are called "Active elements". All power supplies fit into this category.

The elements which will receive the energy and dissipate it are called "Passive elements". Resistors model these devices.

Reactive elements store and release energy into a circuit. Ideally they don't either consume or generate energy. Capacitors, and inductors fall into this category.

## 5.6 Open and Short Circuits

### 5.6.1 Open

No current flows through an open. Normally an **open** is created by a bad connector. Dust, bad solder joints, bad crimping, cracks in circuit board traces, create an **open**. Capacitors respond to DC by turning into **opens** after charging up. Uncharged inductors appear as **opens** immediately after powering up a circuit. The word **open** can refer to a problem description. The word **open** can also help develop an intuition about circuits.

Typically the circuit stops working with opens because 99% of all circuits are driven by voltage power sources. Voltage sources respond to an open with no current. Opens are the equivalent of clogs in plumbing .. which stop water from flowing.

On one side of the open, EMF will build up, just like water pressure will build up on one side of a clogged pipe. Typically a voltage will appear across the open.

---

<sup>4</sup> [http://en.wikibooks.org/wiki/%3Awikiversity%3ATutorial\\_on\\_preparing\\_the\\_multimeter\\_for\\_start\\_measuring\\_the\\_voltage](http://en.wikibooks.org/wiki/%3Awikiversity%3ATutorial_on_preparing_the_multimeter_for_start_measuring_the_voltage)

<sup>5</sup> [http://en.wikibooks.org/wiki/%3Aw%3AClamp\\_meter](http://en.wikibooks.org/wiki/%3Aw%3AClamp_meter)

### 5.6.2 Short

A voltage source responds to a **short** by delivering as much current as possible. An extreme example of this can be seen in this ball bearing motor video<sup>6</sup>. The motor appears as a short to the battery. Notice he only completes the short for a short time because he is worried about the car battery exploding.

Maximum current flows through a **short**. Normally a **short** is created by a wire, a nail, or some loose screw touching parts of the circuit unintentionally. Most component failures start with heat build up. The heat destroys varnish, paint, or thin insulation creating a **short**. The **short** causes more current to flow which causes more heat. This cycle repeats faster and faster until there is a puff of smoke and everything breaks creating an **open**. Most component failures start with a **short** and end in an **open** as they burn up. Feel the air temperature above each circuit component after power on. Build a memory of what normal operating temperatures are. Cold can indicate a short that has already turned into an open.

An uncharged capacitor initially appears as a **short** immediately after powering on a circuit. An inductor appears as a **short** to DC after charging up. The **short** concept also helps build our intuition, provides an opportunity to talk about electrical safety and helps describe component failure modes.

A **closed** switch can be thought of as short. Switches<sup>7</sup> are surprisingly complicated. It is in a study of switches that the term **closed** begins to dominate that of **short**.

---

6 <http://www.youtube.com/watch?v=g60okBMeTKo>

7 <http://en.wikibooks.org/wiki/%3Awikipedia%3ASwitch>

## 6 Resistors and Resistance

### 6.0.3 Resistors

Mechanical engineers seem to model everything with a spring. Electrical engineers compare everything to a **Resistor**. Resistors are circuit elements that resist the flow of current. When this is done a voltage appears across the resistor's two wires.

A pure resistor<sup>1</sup> turns electrical energy into heat. Devices similar to resistors turn this energy into light, motion, heat, and other forms of energy.



Figure 2

Current in the drawing above is shown entering the + side of the resistor. Resistors don't care which leg is connected to positive or negative. The + means where the positive or red probe of the volt meter is to be placed in order to get a positive reading. This is called the "positive charge<sup>2</sup>" flow sign convention. Some circuit theory classes (often within a physics oriented curriculum) are taught with an "electron flow" sign convention.

In this case, current entering the + side of the resistor means that the resistor is removing energy from the circuit. This is good. The goal of most circuits is to send energy out into the world in the form of motion, light, sound, etc.

---

<sup>1</sup> <http://en.wikibooks.org/wiki/%3A%3AResistor>

<sup>2</sup> [http://en.wikibooks.org/wiki/%3A%3AElectric\\_current%23Conventions](http://en.wikibooks.org/wiki/%3A%3AElectric_current%23Conventions)

### 6.0.4 Resistance

**Resistance**<sup>3</sup> is measured in terms of units called "Ohms" (volts per ampere), which is commonly abbreviated with the Greek letter  $\Omega$  ("Omega"). Ohms are also used to measure the quantities of *impedance* and *reactance*, as described in a later chapter. The variable most commonly used to represent resistance is "r" or "R".

Resistance is defined as:

$$r = \frac{\rho L}{A}$$

where  $\rho$  is the resistivity of the material, L is the length of the resistor, and A is the cross-sectional area of the resistor.

### 6.0.5 Conductance

**Conductance** is the inverse of resistance. Conductance has units of "Siemens" (S), sometimes referred to as mhos (ohms backwards, abbreviated as an upside-down  $\Omega$ ). The associated variable is "G":

$$G = \frac{1}{r}$$

Before calculators and computers, conductance helped reduce<sup>4</sup> the number of hand calculations that had to be done. Now conductance and it's related concepts of admittance<sup>5</sup> and susceptance can be skipped with matlab, octave, wolfram alpha and other computing tools. Learning one or more these computing tools is now absolutely necessary in order to get through this text.

---

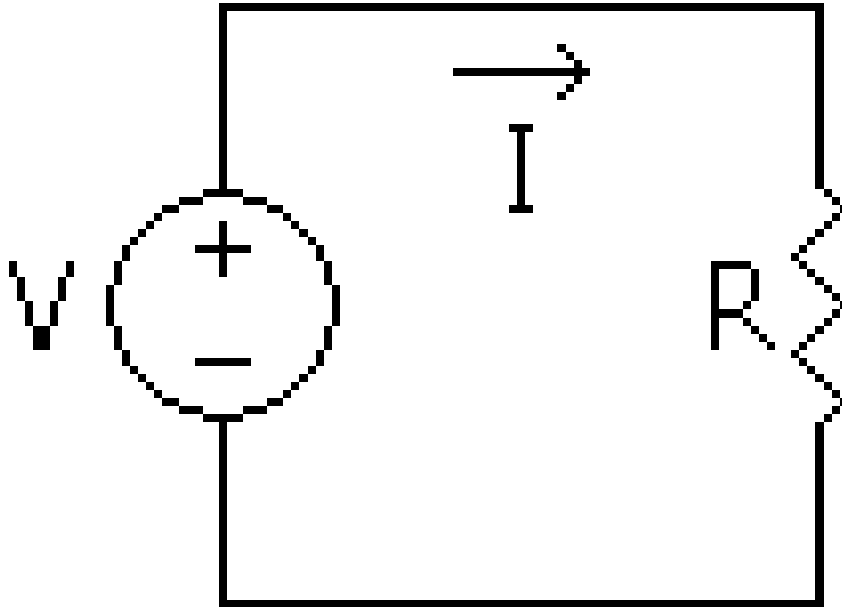
3 <http://en.wikibooks.org/wiki/%3AWikipedia%3AElectrical%20resistance>

4 <http://en.wikibooks.org/wiki/%3Aw%3ASmith%20chart>

5 <http://en.wikibooks.org/wiki/%3Awikipedia%3AAdmittance>



### 6.0.6 Resistor terminal relation



**Figure 3** A simple circuit diagram relating current, voltage, and resistance

The drawing on the right is of a battery and a resistor. Current is leaving the + terminal of the battery. This means this battery is turning chemical potential energy into electromagnetic potential energy and dumping this energy into the circuit. The flow of this energy or power is negative.

Current is entering the positive side of the resistor even though a + has not been put on the resistor. This means electromagnetic potential energy is being converted into heat, motion, light, or sound depending upon the nature of the resistor. Power flowing out of the circuit is given a positive sign.

The relationship of the voltage across the resistor  $V$ , the current through the resistor  $I$  and the value of the resistor  $R$  is related by ohm's law<sup>6</sup>:

**[Resistor Terminal Relation]**

<sup>6</sup> [http://en.wikibooks.org/wiki/%3Awikiversity%3A0hm%2527s\\_law](http://en.wikibooks.org/wiki/%3Awikiversity%3A0hm%2527s_law)

$$V = R * I$$

A resistor, capacitor and inductor all have only two wires attached to them. Sometimes it is hard to tell them apart. In the real world, all three have a bit of resistance, capacitance and inductance in them. In this unknown context, they are called two terminal devices. In more complicated devices, the wires are grouped into ports<sup>7</sup>. A two terminal device that expresses Ohm's law when current and voltage are applied to it, is called a resistor.

### 6.0.7 Resistor Safety

Resistors come in all forms. Most have a maximum power rating in watts. If you put too much through them, they can melt, catch on fire, etc.

### 6.0.8 Example

Suppose the voltage across a resistor's two terminals is 10 volts and the measured current through it is 2 amps. What is the resistance?

If  $v = iR$  then  $R = v/i = 10V/2A = 5ohms$

---

<sup>7</sup> [http://en.wikibooks.org/wiki/%3Awikipedia%3ATwo-port\\_network](http://en.wikibooks.org/wiki/%3Awikipedia%3ATwo-port_network)

## 7 Resistive Circuit Analysis Techniques



## 8 Source Transformations

### 8.1 Source Transformations

Independent current sources can be turned into independent voltage sources, and vice-versa, by methods called "Source Transformations." These transformations are useful for solving circuits. We will explain the two most important source transformations, **Thevenin's Source**, and **Norton's Source**, and we will explain how to use these conceptual tools for solving circuits.

### 8.2 Black Boxes

A circuit (or any system, for that matter) may be considered a **black box** if we don't know what is inside the system. For instance, most people treat their computers like a black box because they don't know what is inside the computer (most don't even care), all they know is what goes in to the system (keyboard and mouse input), and what comes out of the system (monitor and printer output).

Black boxes, by definition, are systems whose internals aren't known to an outside observer. The only methods that an outside observer has to examine a black box is to send input into the systems, and gauge the output.

### 8.3 Thevenin's Theorem

Let's start by drawing a general circuit consisting of a source and a load, as a block diagram:

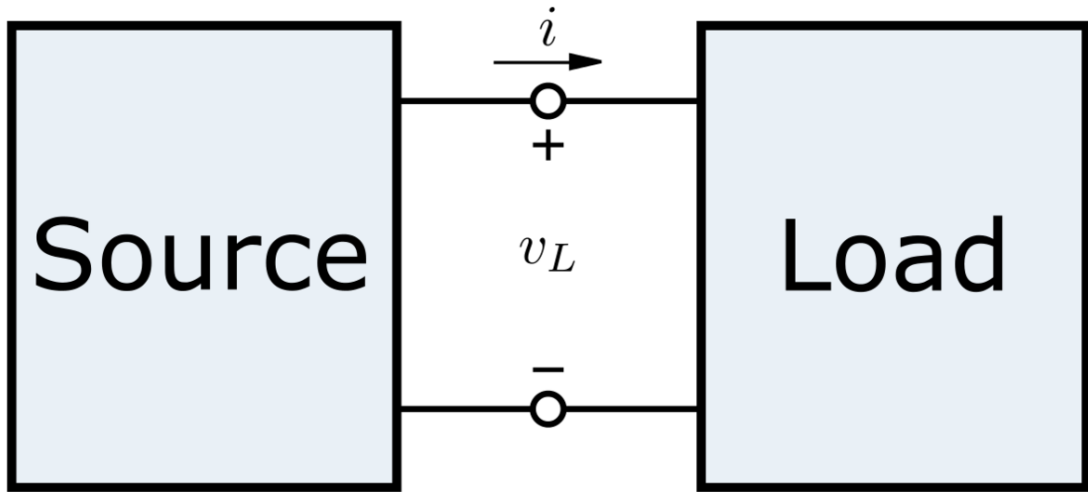
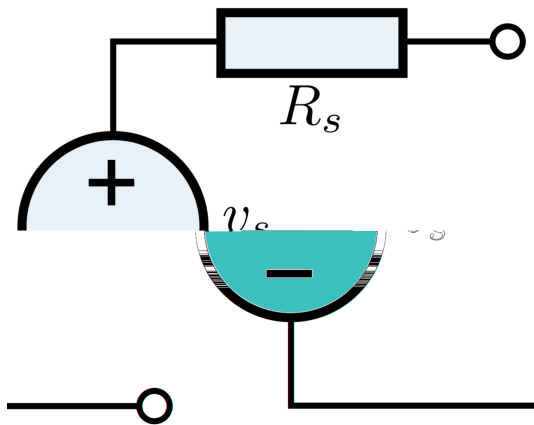
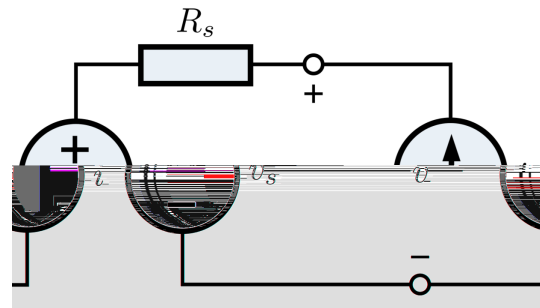


Figure 4

Let's say that the source is a collection of voltage sources, current sources and resistances, while the load is a collection of resistances only. Both the source and the load can be arbitrarily complex, but we can conceptually say that the source is directly equivalent to a single voltage source and resistance (figure (a) below).

Figure 5  
(a)Figure 6  
(b)

We can determine the value of the resistance  $R_s$  and the voltage source,  $v_s$  by attaching an independent source to the output of the circuit, as in figure (b) above. In this case we are using a current source, but a voltage source could also be used. By varying  $i$  and measuring  $v$ , both  $v_s$  and  $R_s$  can be found using the following equation:

$$v = v_s + iR_s$$

There are two variables, so two values of  $i$  will be needed. See Example 1<sup>1</sup> for more details. We can easily see from this that if the current source is set to zero (equivalent to an open circuit), then  $v$  is equal to the voltage source,  $v_s$ . This is also called the open-circuit voltage,  $v_{oc}$ .

This is an important concept, because it allows us to model what is inside a unknown (linear) circuit, just by knowing what is coming out of the circuit. This concept is known as **Thévenin's Theorem** after French telegraph engineer Léon Charles Thévenin<sup>2</sup>, and the circuit consisting of the voltage source and resistance is called the **Thévenin Equivalent Circuit**.

## 8.4 Norton's Theorem

Recall from above that the output voltage,  $v$ , of a Thévenin equivalent circuit can be expressed as

$$v = v_s + iR_s$$

Now, let's rearrange it for the output current,  $i$ :

$$i = -\frac{v_s}{R_s} + \frac{v}{R_s}$$

This is equivalent to a KCL description of the following circuit. We can call the constant term  $v_s/R_s$  the source current,  $i_s$ .

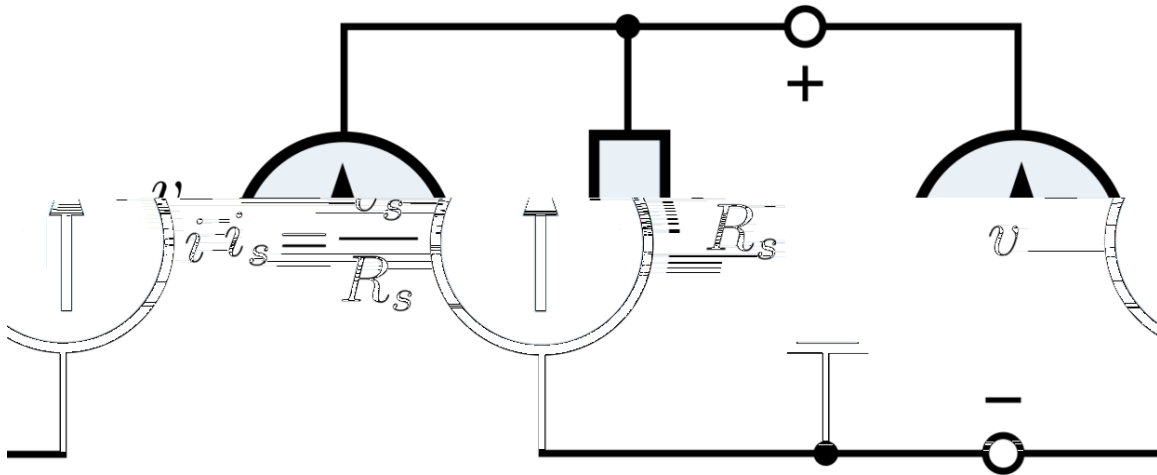


Figure 7

<sup>1</sup> <http://en.wikibooks.org/wiki/%2FExamples%23Example1>

<sup>2</sup> <http://en.wikipedia.org/wiki/L%27E9on%20Charles%20Th%27E9venin>

The equivalent current source and the equivalent resistance can be found with an independent source as before (see Example 2<sup>3</sup>).

When the above circuit (the **Norton Equivalent Circuit**, after Bell Labs engineer E.L. Norton<sup>4</sup>) is disconnected from the external load, the current from the source all flows through the resistor, producing the requisite voltage across the terminals,  $v_{oc}$ . Also, if we were to short the two terminals of our circuit, the current would all flow through the wire, and none of it would flow through the resistor (current divider rule). In this way, the circuit would produce the short-circuit current  $i_{sc}$  (which is exactly the same as the source current  $i_s$ ).

## 8.5 Circuit Transforms

We have just shown turns out that the Thévenin and Norton circuits are just different representations of the same black box circuit, with the same Ohm's Law/KCL equations. This means that we cannot distinguish between Thévenin source and a Norton source from outside the black box, and that we can directly equate the two as below:

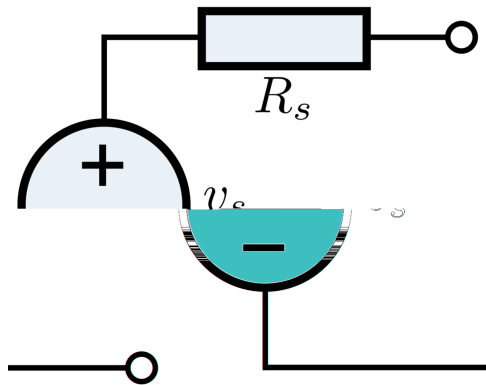


Figure 8

≡

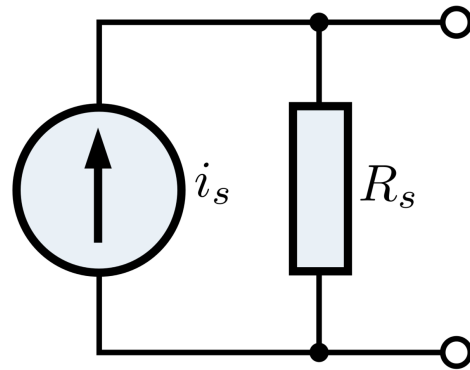


Figure 9

We can draw up some rules to convert between the two:

- The values of the resistors in each circuit are conceptually identical, and can be called the equivalent resistance,  $R_{eq}$ :

$$R_{s_n} = R_{s_t} = R_s = R_{eq}$$

- The value of a Thévenin voltage source is the value of the Norton current source times the equivalent resistance (Ohm's law):

$$v_s = i_s r$$

<sup>3</sup> <http://en.wikibooks.org/wiki/%2FExamples%23Example%202>

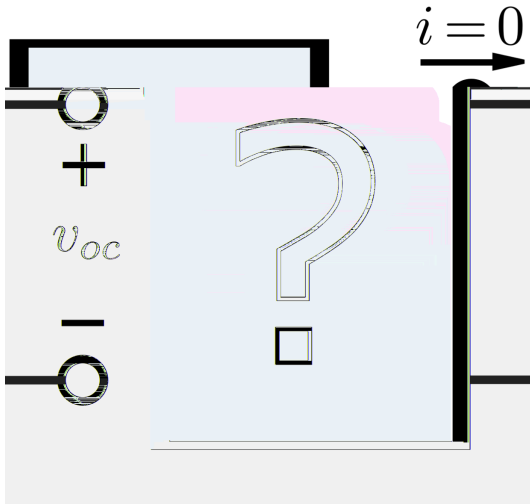
<sup>4</sup> [http://en.wikipedia.org/wiki/Edward\\_Lawry\\_Norton](http://en.wikipedia.org/wiki/Edward_Lawry_Norton)



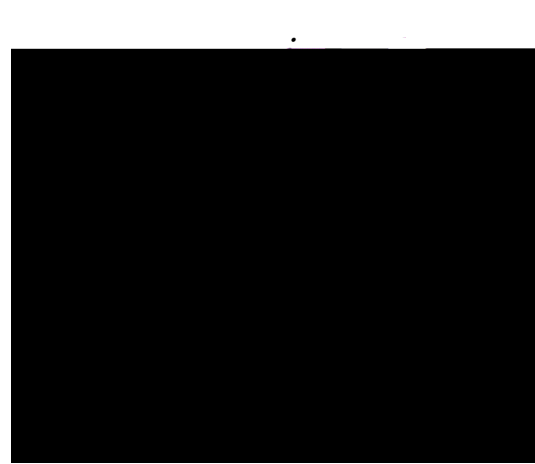
If these rules are followed, the circuits will behave identically. Using these few rules, we can transform a Norton circuit into a Thévenin circuit, and vice versa. This method is called **source transformation**. See Example 3<sup>5</sup>.

## 8.6 Open Circuit Voltage and Short Circuit Current

The open-circuit voltage,  $v_{oc}$  of a circuit is the voltage across the terminals when the current is zero, and the short-circuit current  $i_{sc}$  is the current when the voltage across the terminals is zero:



**Figure 10**  
*The open circuit voltage*



**Figure 11**

*The short circuit current*

We can also observe the following:

- The value of the Thévenin voltage source is the open-circuit voltage:

$$v_s = v_{oc}$$

- The value of the Norton current source is the short-circuit current:

$$i_s = i_{sc}$$

We can say that, generally,

$$R_{eq} = \frac{v_{oc}}{i_{sc}}$$

<sup>5</sup> <http://en.wikibooks.org/wiki/%2FExamples%23Example%203>

## 8.7 Why Transform Circuits?

How are Thevenin and Norton transforms useful?

Describe a black box characteristics in a way that can predict its reaction to any load.

Find the current through and voltage across any device by removing the device from the circuit! This can instantly make a complex circuit much simpler to analyze.

Stepwise simplification of a circuit is possible if voltage sources have a series impedance and current sources have a parallel impedance.

## 9 Maximum Power Transfer

### 9.1 Maximum Power Transfer

Often we would like to transfer the most power from a source to a load placed across the terminals as possible. How can we determine the optimum resistance of the load for this to occur?

Let us consider a source modelled by a Thévenin equivalent (a Norton equivalent will lead to the same result, as the two are directly equivalent), with a load resistance,  $R_L$ . The source resistance is  $R_s$  and the open circuit voltage of the source is  $v_s$ :

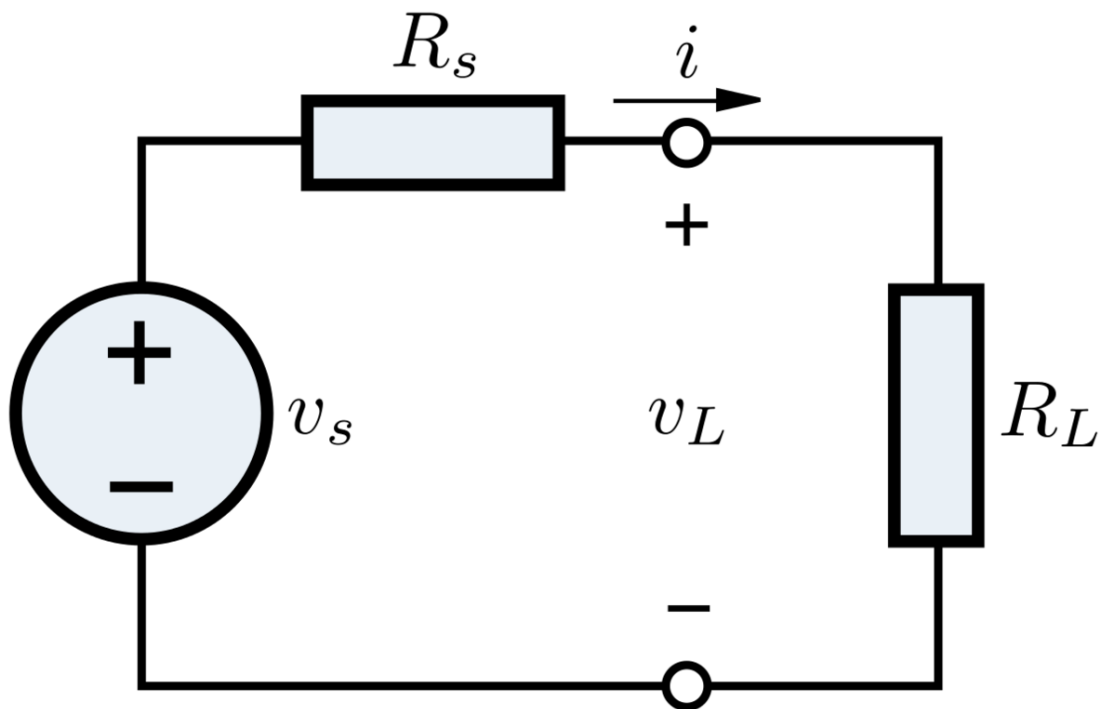


Figure 12

The current in this circuit is found using Ohm's Law:

$$i = \frac{v_s}{R_s + R_L}$$

The voltage across the load resistor,  $v_L$ , is found using the voltage divider rule:

$$v_L = v_s \frac{R_L}{R_s + R_L}$$

We can now find the power dissipated in the load,  $P_L$  as follows:

$$P_L = v_L i = \frac{R_L v_s^2}{(R_s + R_L)^2}$$

We can now rewrite this to get rid of the  $R_L$  on the top:

$$P_L = \frac{v_s^2}{\left(\frac{R_s}{\sqrt{R_L}} + \sqrt{R_L}\right)^2} = \frac{v_s^2}{R_s \left(\frac{\sqrt{R_s}}{\sqrt{R_L}} + \frac{\sqrt{R_L}}{\sqrt{R_s}}\right)^2}$$

Assuming the source resistance is not changeable, then we obtain maximum power by minimising the bracketed part of the denominator in the above equation. It is an elementary mathematical result that  $x + x^{-1}$  is at a minimum when  $x=1$ . In this case, it is equal to 2. Therefore, the above expression is minimum under the following condition:

$$\frac{\sqrt{R_s}}{\sqrt{R_L}} = 1$$

This leads to the condition that:

$$\left\{ \begin{array}{l} \text{style=" border: solid 2px #D6D6FF; padding: 1em;" valign="top"} \\ \text{align="left"} | R_L = R_s \end{array} \right\}$$

We will get maximum power out of the source if the load resistance is identical to the internal source resistance. This is the **Maximum Power Transfer Theorem**.

### 9.1.1 Efficiency

The efficiency,  $\eta$  of the circuit is the proportion of all the energy dissipated in the circuit that is dissipated in the load. We can immediately see that at maximum power transfer to the load, the efficiency is 0.5, as the source resistor has half the voltage across it. We can also see that efficiency will increase as the load resistance increases, even though the power transferred will fall.

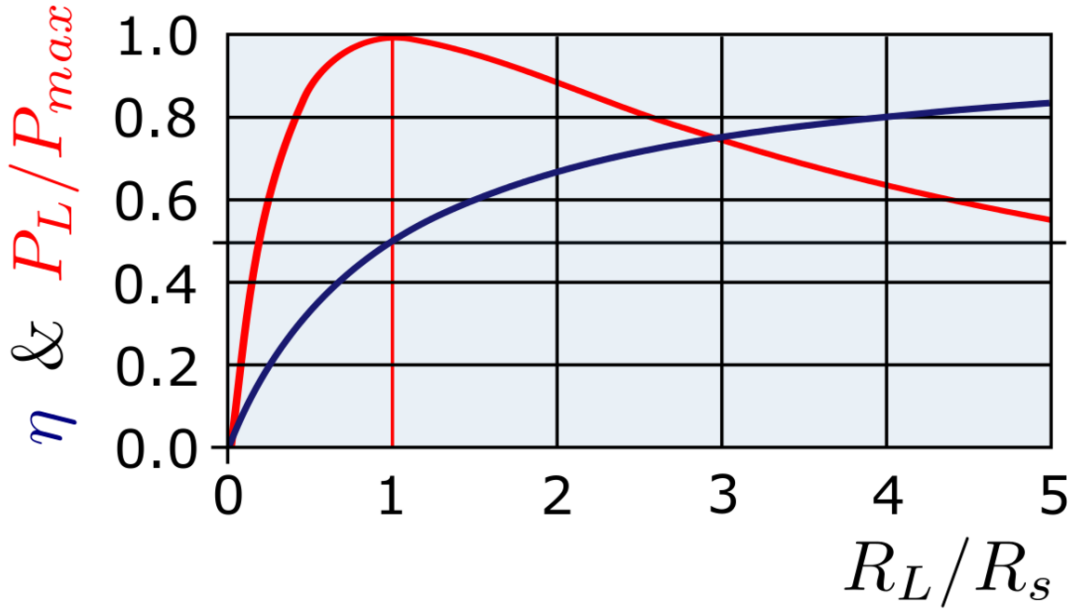
The efficiency can be calculated using the following equation:

$$\eta = \frac{P_L}{P_L + P_s}$$

where  $P_s$  is the power in the source resistor. This can be found using a simple modification to the equation for  $P_L$ :

$$P_s = \frac{v_s^2}{R_L \left( \frac{\sqrt{R_s}}{\sqrt{R_L}} + \frac{\sqrt{R_L}}{\sqrt{R_s}} \right)^2}$$

The graph below shows the power in the load (as a proportion of the maximum power,  $P_{max}$ ) and the efficiency for values of  $R_L$  between 0 and 5 times  $R_s$ .



**Figure 13**

It is important to note that under conditions of maximum power transfer as much power is dissipated in the source as in the load. This is not a desirable condition if, for example, the source is the electricity supply system and the load is your electric heater. This would mean that the electricity supply company would be wasting half the power it generates. In this case, the generators, power lines, etc. are designed to give the lowest source resistance possible, giving high efficiency. The maximum power transfer condition is used in (usually high-frequency) communications systems where the source resistance can not be made low, the power levels are relatively low and it is paramount to get as much signal power as possible to the receiving end of the system (the load).



# 10 Resistive Circuit Analysis Methods

## 10.1 Analysis Methods

When circuits get large and complicated, it is useful to have various methods for simplifying and analyzing the circuit. There is no perfect formula for solving a circuit. Depending on the type of circuit, there are different methods that can be employed to solve the circuit. Some methods might not work, and some methods may be very difficult in terms of long math problems. Two of the most important methods for solving circuits are **Nodal Analysis**, and **Mesh Current Analysis**. These will be explained below.

## 10.2 Superposition

One of the most important principals in the field of circuit analysis is the principal of **superposition**. It is valid only in linear circuits.

*The **superposition principle** states that the total effect of multiple contributing sources on a linear circuit is equal to the sum of the individual effects of the sources, taken one at a time.*

What does this mean? In plain English, it means that if we have a circuit with multiple sources, we can "turn off" all but one source at a time, and then investigate the circuit with only one source active at a time. We do this with every source, in turn, and then add together the effects of each source to get the total effect. Before we put this principle to use, we must be aware of the underlying mathematics.

### 10.2.1 Necessary Conditions

Superposition can only be applied to **linear** circuits; that is, all of a circuit's sources hold a linear relationship with the circuit's responses. Using only a few algebraic rules, we can build a mathematical understanding of superposition. If  $f$  is taken to be the response, and  $a$  and  $b$  are constant, then:

$$f(ax_1 + bx_2) = f(ax_1) + f(bx_2)$$

In terms of a circuit, it clearly explains the concept of superposition; each input can be considered individually and then summed to obtain the output. With just a few more algebraic properties, we can see that superposition cannot be applied to non-linear circuits.

In this example, the response  $y$  is equal to the square of the input  $x$ , i.e.  $y=x^2$ . If  $a$  and  $b$  are constant, then:

$$y = (ax_1 + bx_2)^2 \neq (ax_1)^2 + (bx_2)^2 = y_1 + y_2$$

Note that this is only one of an infinite number of counter-examples...

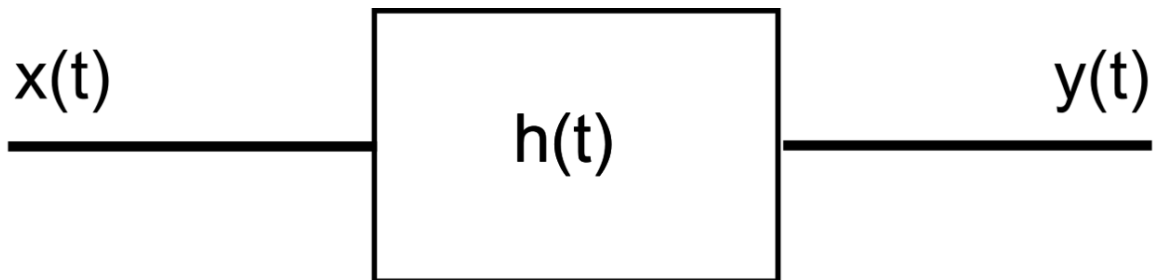
### 10.2.2 Step by Step

Using superposition to find a given output can be broken down into four steps:

1. Isolate a source - Select a source, and set all of the remaining sources to zero. The consequences of "turning off" these sources are explained in Open and Closed Circuits<sup>1</sup>. In summary, turning off a voltage source results in a short circuit, and turning off a current source results in an open circuit. (Reasoning - no current can flow through a open circuit and there can be no voltage drop across a short circuit.)
2. Find the output from the isolated source - Once a source has been isolated, the response from the source in question can be found using any of the techniques we've learned thus far.
3. Repeat steps 1 and 2 for each source - Continue to choose a source, set the remaining sources to zero, and find the response. Repeat this procedure until every source has been accounted for.
4. Sum the Outputs - Once the output due to each source has been found, add them together to find the total response.

## 10.3 Impulse Response

An **impulse response** of a circuit can be used to determine the output of the circuit:



**Figure 14**

The output  $y$  is the **convolution**  $h * x$  of the input  $x$  and the impulse response:

[Convolution]

---

<sup>1</sup> Chapter 5.6.2 on page 16



$$y(t) = (h * x)(t) = \int_{-\infty}^{+\infty} h(t-s)x(s)ds$$

If the input,  $x(t)$ , was an **impulse** ( $\delta(t)$ ), the output  $y(t)$  would be equal to  $h(t)$ .

By knowing the impulse response of a circuit, any source can be plugged-in to the circuit, and the output can be calculated by convolution.

## 10.4 Convolution

The **convolution operation** is a very difficult, involved operation that combines two equations into a single resulting equation. Convolution is defined in terms of a definite integral, and as such, solving convolution equations will require knowledge of integral calculus. This wikibook will not require a prior knowledge of integral calculus, and therefore will not go into more depth on this subject than a simple definition, and some light explanation.

### 10.4.1 Definition

The convolution  $a * b$  of two functions  $a$  and  $b$  is defined as:

$$(a * b)(t) = \int_{-\infty}^{\infty} a(\tau)b(t-\tau)d\tau$$

Remember:

Asterisks mean **convolution**, not **multiplication**

The asterisk operator is used to denote convolution. Many computer systems, and people who frequently write mathematics on a computer will often use an asterisk to denote simple multiplication (the asterisk is the multiplication operator in many programming languages), however an important distinction must be made here: **The asterisk operator means convolution.**

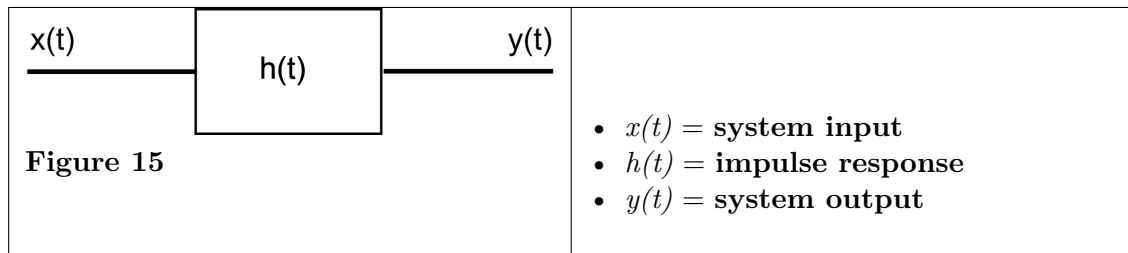
### 10.4.2 Properties

Convolution is commutative, in the sense that  $a * b = b * a$ . Convolution is also *distributive* over addition, i.e.  $a * (b + c) = a * b + a * c$ , and *associative*, i.e.  $a * (b * c) = (a * b) * c$ .

### 10.4.3 Systems, and convolution

Let us say that we have the following block-diagram system:

---



Where  $x(t)$  is the input to the circuit,  $h(t)$  is the circuit's impulse response, and  $y(t)$  is the output. Here, we can find the output by convoluting the impulse response with the input to the circuit. Hence we see that the impulse response of a circuit is not just the ratio of the output over the input. In the frequency domain however, component in the output with frequency  $\omega$  is the product of the input component with the same frequency and the transition function at that frequency. The moral of the story is this: *the output to a circuit is the input convolved with the impulse response.*

## 11 Energy Storage Elements



# 12 First-Order Circuits

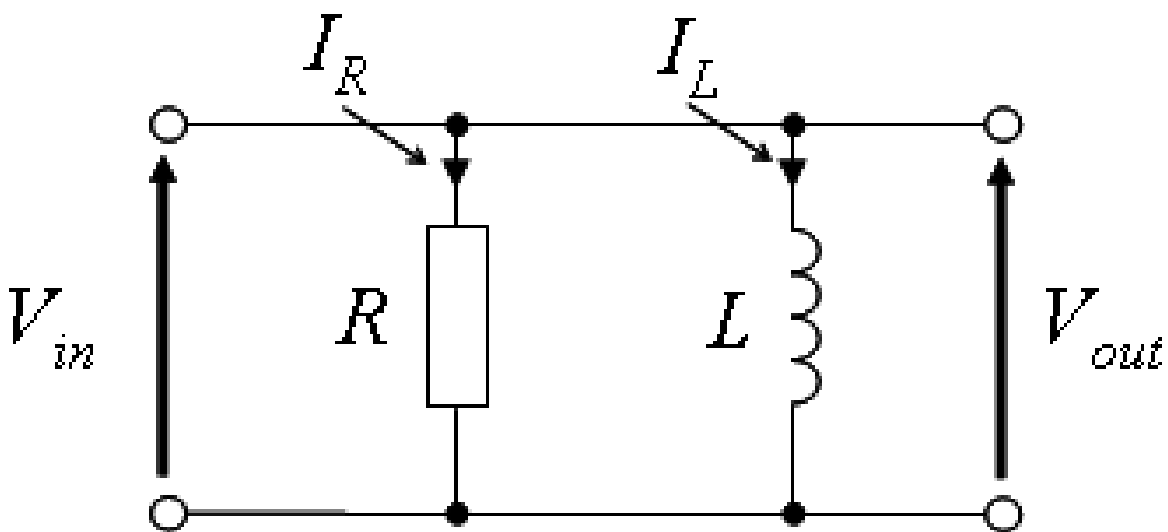
## 12.1 First Order Circuits

First order circuits are circuits that contain only one energy storage element (capacitor or inductor), and that can therefore be described using only a first order differential equation. The two possible types of first-order circuits are:

1. RC (resistor and capacitor)
2. RL (resistor and inductor)

RL and RC circuits is a term we will be using to describe a circuit that has either a) resistors and inductors (RL), or b) resistors and capacitors (RC).

## 12.2 RL Circuits



**Figure 16** An RL parallel circuit

An RL Circuit has at least one resistor (R) and one inductor (L). These can be arranged in parallel, or in series. Inductors are best solved by considering the current flowing through the inductor. Therefore, we will combine the resistive element and the source into a Norton Source Circuit. The Inductor then, will be the external load to the circuit. We remember the equation for the inductor:

$$v(t) = L \frac{di}{dt}$$

w:RL circuit<sup>1</sup>

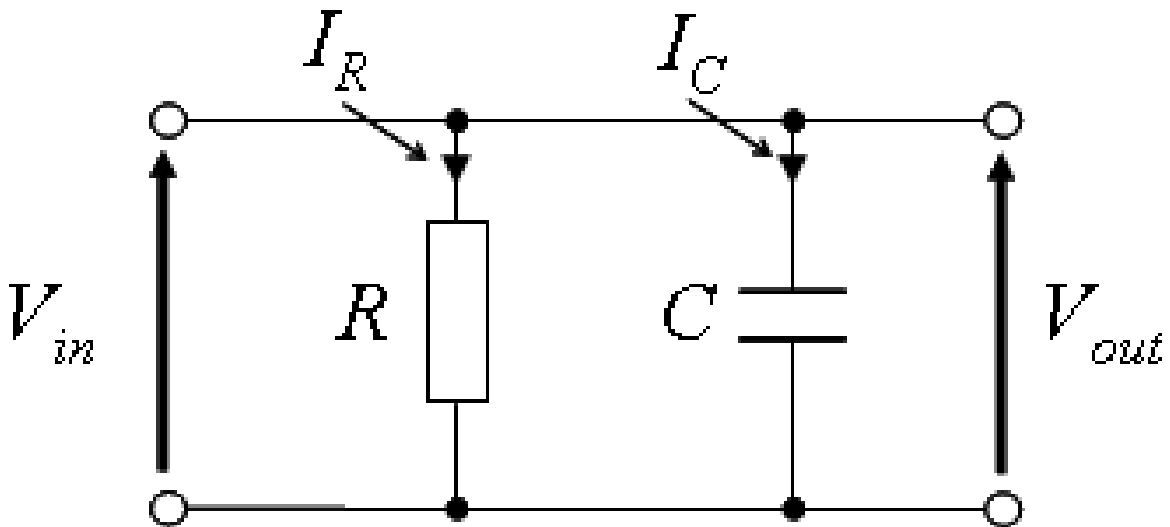
If we apply KCL on the node that forms the positive terminal of the voltage source, we can solve to get the following differential equation:

$$i_{source}(t) = \frac{L}{R_n} \frac{di_{inductor}(t)}{dt} + i_{inductor}(t)$$

We will show how to solve differential equations in a later chapter.

## 12.3 RC Circuits

w:RC circuit<sup>2</sup>



**Figure 17** A parallel RC Circuit

An RC circuit is a circuit that has both a resistor (R) and a capacitor (C). Like the RL Circuit, we will combine the resistor and the source on one side of the circuit, and combine them into a thevenin source. Then if we apply KVL around the resulting loop, we get the following equation:

$$v_{source} = RC \frac{dv_{capacitor}(t)}{dt} + v_{capacitor}(t)$$

---

<sup>1</sup> <http://en.wikipedia.org/wiki/RL%20circuit>

<sup>2</sup> <http://en.wikipedia.org/wiki/RC%20circuit>

## 12.4 First Order Solution

### 12.4.1 Series RL

The differential equation of the series RL circuit

$$L \frac{dI}{dt} + IR = 0$$

$$\frac{dI}{dt} = -I \frac{R}{L}$$

$$\frac{1}{I} dI = -\frac{R}{L} dt$$

$$\int \frac{1}{I} dI = -\frac{R}{L} \int dt$$

$$\ln I = -\frac{R}{L} t + C$$

$$I = e^{(-\frac{R}{L} t + C)}$$

$$I = Ae^{(-\frac{R}{L} t)} \cdot A = e^C$$

t	I(t)
0	A
1 $\frac{R}{L}$	36% A
2 $\frac{R}{L}$	A
3 $\frac{R}{L}$	A
4 $\frac{R}{L}$	A
5 $\frac{R}{L}$	1% A

### 12.4.2 Series RC

The differential equation of the series RC circuit

$$C \frac{dV}{dt} + \frac{V}{R} = 0$$

$$\frac{dV}{dt} = -V \frac{1}{RC}$$

$$\frac{1}{V} dV = -\frac{1}{RC} dt$$

$$\int \frac{1}{V} dV = -\frac{1}{RC} \int dt$$

$$\ln V = -\frac{1}{RC} t + C$$

$$V = e^{(-\frac{1}{RC} t + C)}$$

$$V = Ae^{(-\frac{1}{RC} t)} \cdot A = e^C$$

t	V(t)
0	A
1 $\frac{1}{RC}$	36% A
2 $\frac{1}{RC}$	A
3 $\frac{1}{RC}$	A
4 $\frac{1}{RC}$	A

<b>t</b>	<b>V(t)</b>
$5 \frac{1}{RC}$	1% A

### 12.4.3 Time Constant

The series RL and RC has a Time Constant

$$T = \frac{L}{R}$$

$$T = \frac{RC}{1}$$

In general, from an engineering standpoint, we say that the system is at steady state ( Voltage or Current is almost at Ground Level ) after a time period of five Time Constants.



## 13 RLC Circuits

w:RLC Circuit<sup>1</sup>

### 13.1 Series RLC Circuit

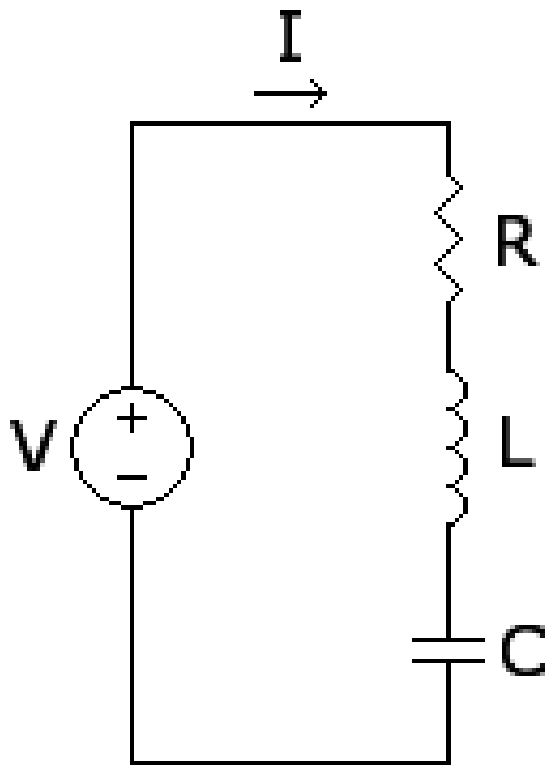


Figure 18

#### 13.1.1 Second Order Differential Equation

$$L \frac{dI}{dt} + IR + \frac{1}{C} \int I dt = 0$$

---

<sup>1</sup> <http://en.wikipedia.org/wiki/RLC%20Circuit>

$$\frac{d^2 I}{dt^2} + \frac{R}{L} \frac{dI}{dt} + \frac{I}{LC} = 0$$

The characteristic equation is

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s = -\alpha \pm \sqrt{\alpha^2 - \beta^2}$$

Where

$$\alpha = \frac{R}{2L}$$

$$\beta = \frac{1}{\sqrt{LC}}$$

When  $\sqrt{\alpha^2 - \beta^2} = 0$

$$\alpha^2 = \beta^2; R = 2\sqrt{\frac{L}{C}}$$

The equation only has one real root .  $s = -\alpha = -\frac{R}{2L}$

The solution for  $I(t) = Ae^{(-\frac{R}{2L}t)}$

The I - t curve would look like

When  $\sqrt{\alpha^2 - \beta^2} > 0$

$$\alpha^2 > \beta^2 . R > \frac{L}{C}$$

The equation only has two real root .  $s = -\alpha \pm \sqrt{\alpha^2 - \beta^2}$

The solution for  $I(t) = e^{-\alpha + \sqrt{\alpha^2 - \beta^2}t} + e^{-\alpha - \sqrt{\alpha^2 - \beta^2}t} = e^{-\alpha}e^{j(\sqrt{\alpha^2 - \beta^2}t)} + e^{-j(\sqrt{\alpha^2 - \beta^2}t)}$

The I - t curve would look like

When  $\sqrt{\alpha^2 - \beta^2} < 0$

$$\alpha^2 < \beta^2 . R < \frac{L}{C}$$

The equation has two complex root .  $s = -\alpha \pm j\sqrt{\beta^2 - \alpha^2}$

The solution for  $I(t) = e^{(-\alpha + j\sqrt{\beta^2 - \alpha^2}t)} + e^{(-\alpha - j\sqrt{\beta^2 - \alpha^2}t)} = e^{-\alpha}e^{j(\sqrt{\beta^2 - \alpha^2}t)} + e^{-j(\sqrt{\beta^2 - \alpha^2}t)}$

The I - t curve would look like

### 13.1.2 Damping Factor

The damping factor is the amount by which the oscillations of a circuit gradually decrease over time. We define the damping ratio to be:

Circuit Type	Series RLC	Parallel RLC
Damping Factor	$\zeta = \frac{R}{2L}$	$\zeta = \frac{1}{2RC}$
Resonance Frequency	$\omega_o = \frac{1}{\sqrt{LC}}$	$\omega_o = \frac{1}{\sqrt{LC}}$

Compare The Damping factor with The Resonance Frequency give rise to different types of circuits: **Overdamped**, **Underdamped**, and **Critically Damped**.

### 13.1.3 Bandwidth

[Bandwidth]

$$\Delta\omega = 2\zeta$$

For series RLC circuit:

$$\Delta\omega = 2\zeta = \frac{R}{L}$$

For Parallel RLC circuit:

$$\Delta\omega = 2\zeta = \frac{1}{RC}$$

### 13.1.4 Quality Factor

[Quality Factor]

$$Q = \frac{\omega_o}{\Delta\omega} = \frac{\omega_o}{2\zeta}$$

For Series RLC circuit:

$$Q = \frac{\omega_o}{\Delta\omega} = \frac{\omega_o}{2\zeta} = \frac{L}{R\sqrt{LC}} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

For Parallel RLC circuit:

$$Q = \frac{\omega_o}{\Delta\omega} = \frac{\omega_o}{2\zeta} = \frac{RC}{\sqrt{LC}} = R\sqrt{\frac{C}{L}}$$

### 13.1.5 Stability

Because inductors and capacitors act differently to different inputs, there is some potential for the circuit response to approach infinity when subjected to certain types and amplitudes of inputs. When the output of a circuit approaches infinity, the circuit is said to be **unstable**. Unstable circuits can actually be dangerous, as unstable elements overheat, and potentially rupture.

A circuit is considered to be stable when a "well-behaved" input produces a "well-behaved" output response. We use the term "Well-Behaved" differently for each application, but generally, we mean "Well-Behaved" to mean a finite and controllable quantity.

## 13.2 Resonance

### 13.2.1 With $R = 0$

When  $R = 0$ , the circuit reduces to a series LC circuit. When the circuit is in resonance, the circuit will vibrate at the resonant frequency.

$$Z_L = Z_C$$

$$\omega L = \frac{1}{\omega C}$$

$$\omega = \frac{1}{\sqrt{LC}}$$

$$f = \frac{1}{2\pi} \frac{1}{\sqrt{LC}}$$

*The circuit vibrates and has the capability of producing a Standing Wave when  $R = 0$ ,  $L = C$*

### 13.2.2 With $R \neq 0$

When  $R \neq 0$  and the circuit operates in resonance .

The frequency dependent components  $L$ ,  $C$  cancel out ie  $Z_L - Z_C = 0$  so that the total impedance of the circuit is  $Z_R + Z_L + Z_C = R + [Z_L - Z_C] = R + 0 = R$

The current of the circuit is  $I = \frac{V}{R}$

The Operating Frequency is  $\omega = \frac{1}{\sqrt{LC}}$

If the current is halved by doubling the value of resistance then

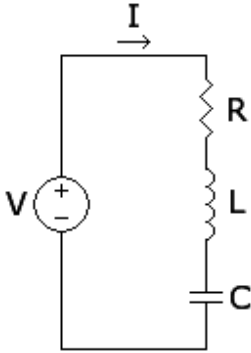
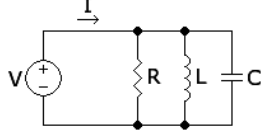
$$I = \frac{V}{2R}$$

Circuit will be stable over the range of frequencies from  $\omega_1 - \omega_2$

The circuit has the capability to select bandwidth where the circuit is stable . Therefore, it is best suited for Tuned Resonance Select Bandwidth Filter

Once using L or C to tune circuit into resonance at resonance frequency  $f = \frac{1}{2\pi\sqrt{LC}}$  The current is at its maximum value  $I = \frac{V}{R}$  . Reduce current above  $I = \frac{V}{2R}$  circuit will respond to narrower bandwidth than  $\omega_1 - \omega_2$ . Reduce current below  $I = \frac{V}{2R}$  circuit will respond to wider bandwidth than  $\omega_1 - \omega_2$ .

### 13.3 Conclusion

Circuit	General	Series RLC	Parallel RLC
Circuit		 <p><b>Figure 19</b></p>	 <p><b>Figure 20</b></p>
Impedance	$Z$	$Z = (j\omega)^2 + (j\omega)\frac{R}{L} + \frac{1}{LC}$	$Z = \frac{1}{\frac{1}{RLC} \frac{1}{(j\omega)^2 + j\omega\frac{1}{RC} + \frac{1}{LC}}}$
Roots	$\lambda$	$\lambda = -\zeta \pm \sqrt{\zeta^2 - \omega_o^2}$	$\lambda = -\zeta \pm \sqrt{\zeta^2 - \omega_o^2}$
$I(t)$	$Ae^{\lambda_1 t} + Be^{\lambda_2 t}$	$Ae^{\lambda_1 t} + Be^{\lambda_2 t}$	$Ae^{\lambda_1 t} + Be^{\lambda_2 t}$
Damping Factor	$\zeta$	$\zeta = \frac{R}{2L}$	$\zeta = \frac{1}{2RC}$
Resonant Frequency	$\omega_o$	$\omega_o = \frac{1}{\sqrt{LC}}$	$\omega_o = \frac{1}{\sqrt{LC}}$
Band Width	$\Delta\omega = 2\zeta$	$\frac{R}{L}$	$\frac{1}{CR}$
Quality factor	$Q = \frac{\omega_o}{\Delta\omega} = \frac{\omega_o}{2\zeta}$	$Q = \frac{L}{R\sqrt{LC}} = \frac{1}{R}\sqrt{\frac{L}{C}}$	$Q = \frac{CR}{\sqrt{LC}} = R\sqrt{\frac{C}{L}}$



# 14 The Second-Order Circuit Solution

## 14.1 Second-Order Solution

This page is going to talk about the solutions to a second-order, RLC circuit. The second-order solution is reasonably complicated, and a complete understanding of it will require an understanding of differential equations. This book will not require you to know about differential equations, so we will describe the solutions without showing how to derive them. The derivations may be put into another chapter, eventually.

The aim of this chapter is to develop the **complete response** of the second-order circuit. There are a number of steps involved in determining the complete response:

1. Obtain the differential equations of the circuit
2. Determine the resonant frequency and the damping ratio
3. Obtain the characteristic equations of the circuit
4. Find the roots of the characteristic equation
5. Find the natural response
6. Find the forced response
7. Find the complete response

We will discuss all these steps one at a time.

## 14.2 Finding Differential Equations

A **Second-order circuit** cannot possibly be solved until we obtain the second-order differential equation that describes the circuit. We will discuss here some of the techniques used for obtaining the second-order differential equation for an RLC Circuit.

### Note

Parallel RLC Circuits are easier to solve in terms of current. Series RLC circuits are easier to solve in terms of voltage.

### 14.2.1 The Direct Method

The most direct method for finding the differential equations of a circuit is to perform a nodal analysis, or a mesh current analysis on the circuit, and then solve the equation for the input function. The final equation should contain only derivatives, no integrals.

### 14.2.2 The Variable Method

If we create two variables,  $g$  and  $h$ , we can use them to create a second-order differential equation. First, we set  $g$  and  $h$  to be either inductor currents, capacitor voltages, or both. Next, we create a single first order differential equation that has  $g = f(g, h)$ . Then, we write another first-order differential equation that has the form:

$$\frac{dh}{dt} = Kg \text{ or } \frac{1}{K} \frac{dh}{dt} = g$$

Next, we substitute in our second equation into our first equation, and we have a second-order equation.

## 14.3 Zero-Input Response

The **zero-input** response of a circuit is the state of the circuit when there is no forcing function (no current input, and no voltage input). We can set the differential equation as such:

$$\frac{d^2i}{dt^2} + 2\zeta \frac{di}{dt} + \omega_o^2 i(t) = 0$$

This gives rise to the characteristic equation of the circuit, which is explained below.

## 14.4 Characteristic Equation

The characteristic equation of an RLC circuit is obtained using the "Operator Method" described below, with zero input. The characteristic equation of an RLC circuit (series or parallel) will be:

$$s^2i + \frac{R}{L}si + \frac{1}{LC}i = 0$$

The roots to the characteristic equation are the "solutions" that we are looking for.

### 14.4.1 Finding the Characteristic Equation

This method of obtaining the characteristic equation requires a little trickery. First, we create an operator  $s$  such that:

$$sx = \frac{dx}{dt}$$

Also, we can show higher-order operators as such:

$$s^2x = \frac{d^2x}{dt^2}$$



Where  $x$  is the voltage (in a series circuit) or the current (in a parallel circuit) of the circuit source. We write 2 first order differential equations for the inductor currents and/or the capacitor voltages in our circuit. We convert all the differentiations to  $s$ , and all the integrations (if any) into  $(1/s)$ . We can then use Cramer's rule to solve for a solution.

### 14.4.2 Solutions

The solutions of the characteristic equation are given in terms of the resonant frequency and the damping ratio:

[Characteristic Equation Solution]

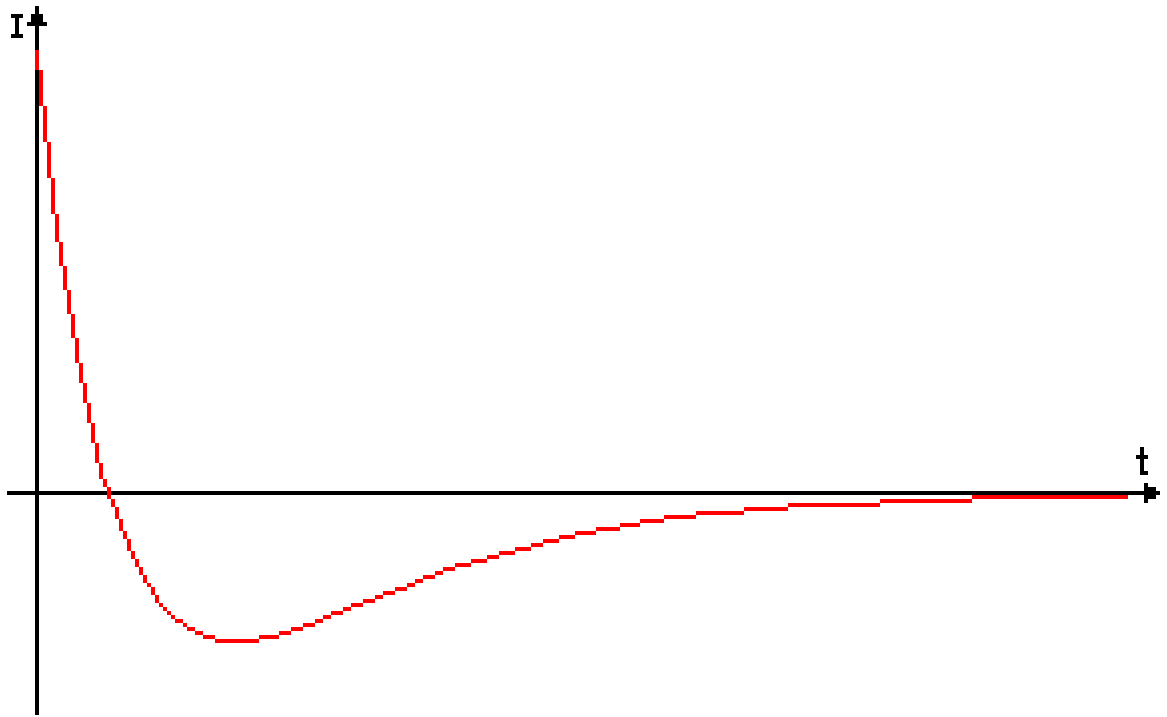
$$s = -\zeta \pm \sqrt{\zeta^2 - \omega_o^2}$$

If *either* of these two values are used for  $s$  in the assumed solution  $x = Ae^{st}$  and that solution completes the differential equation then it can be considered a valid solution. We will discuss this more, below.

## 14.5 Damping

The solutions to a circuit are dependant on the type of **damping** that the circuit exhibits, as determined by the relationship between the damping ratio and the resonant frequency. The different types of damping are Overdamping, Underdamping, and Critical Damping.

### 14.5.1 Overdamped



**Figure 21** RLC series Over-Damped Response

A circuit is called **Overdamped** when the following condition is true:

$$\alpha > \omega_0$$

In this case, the solutions to the characteristic equation are two distinct, positive numbers, and are given by the equation:

$$I(t) = Ae^{s_1 t} + Be^{s_2 t}, \text{ where}$$

$$s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

In a parallel circuit:

$$\alpha = 1/(2RC)$$

$$\omega_0 = 1/\sqrt{LC}$$

In a series circuit:

$$\alpha = R/(2L)$$

$$\omega_0 = 1/\sqrt{LC}$$

Overdamped circuits are characterized as having a very large settling time, and possibly a large steady-state error.

### 14.5.2 Underdamped

A Circuit is called **Underdamped** when the damping ratio is less than the resonant frequency.

$$\zeta < \omega_0$$

In this case, the characteristic polynomial's solutions are complex conjugates. This results in oscillations or *ringing* in the circuit. The solution consists of two conjugate roots:

$$\lambda_1 = -\zeta + i\omega_c$$

and

$$\lambda_2 = -\zeta - i\omega_c$$

where

$$\omega_c = \sqrt{\omega_o^2 - \zeta^2}$$

The solutions are:

$$i(t) = Ae^{(-\zeta + i\omega_c)t} + Be^{(-\zeta - i\omega_c)t}$$

for arbitrary constants  $A$  and  $B$ . Using Euler's formula, we can simplify the solution as:

$$i(t) = e^{-\zeta t} [C \sin(\omega_c t) + D \cos(\omega_c t)]$$

for arbitrary constants  $C$  and  $D$ . These solutions are characterized by *exponentially decaying sinusoidal response*. The higher the **Quality Factor** (below), the longer it takes for the oscillations to decay.

### 14.5.3 Critically Damped



**Figure 22** RLC series Critically Damped

A circuit is called **Critically Damped** if the damping factor is equal to the resonant frequency:

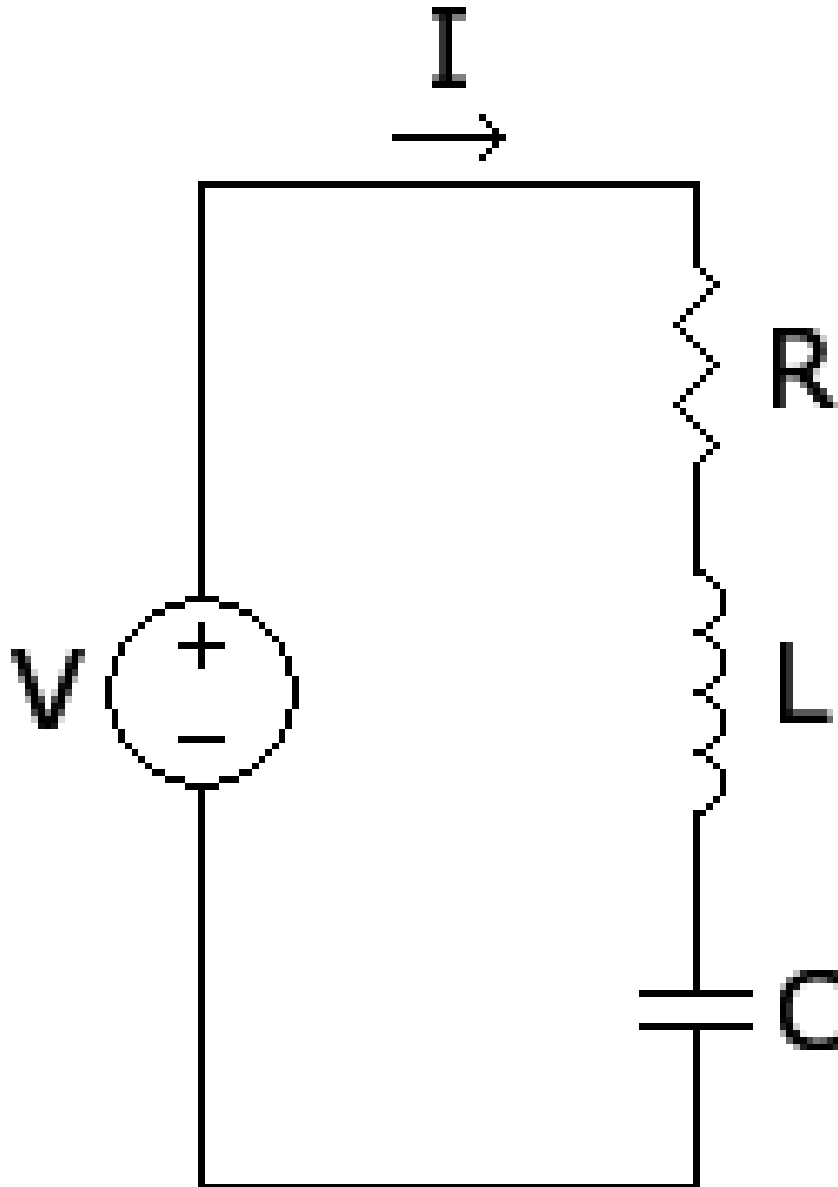
$$\zeta = \omega_0$$

In this case, the solutions to the characteristic equation is a double root. The two roots are identical ( $\lambda_1 = \lambda_2 = \lambda$ ), the solutions are:

$$I(t) = (A + Bt)e^{\lambda t}$$

for arbitrary constants  $A$  and  $B$ . Critically damped circuits typically have low overshoot, no oscillations, and quick settling time.

## 14.6 Series RLC



**Figure 23** A series RLC circuit.

The differential equation to a simple series circuit with a constant voltage source  $V$ , and a resistor  $R$ , a capacitor  $C$ , and an inductor  $L$  is:

$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i = 0$$

The characteristic equation then, is as follows:

$$Ls^2 + Rs + \frac{1}{C} = 0$$

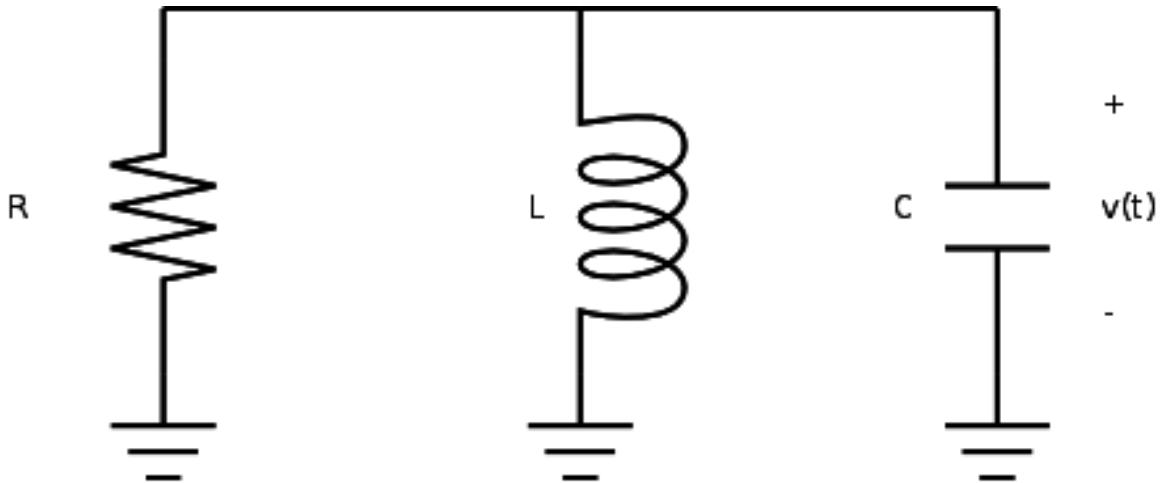
With the two roots:

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

and

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

## 14.7 Parallel RLC



**Figure 24** A parallel RLC Circuit.

The differential equation to a parallel RLC circuit with a resistor  $R$ , a capacitor  $C$ , and an inductor  $L$  is as follows:

$$C \frac{d^2 v}{dt^2} + \frac{1}{R} \frac{dv}{dt} + \frac{1}{L} v = 0$$

Where  $v$  is the voltage across the circuit. The characteristic equation then, is as follows:

$$Cs^2 + \frac{1}{R}s + \frac{1}{L} = 0$$

With the two roots:

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

and

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

## 14.8 Circuit Response

Once we have our differential equations, and our characteristic equations, we are ready to assemble the mathematical form of our circuit response. RLC Circuits have differential equations in the form:

$$a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0x = f(t)$$

Where  $f(t)$  is the forcing function of the RLC circuit.

### 14.8.1 Natural Response

The **natural response** of a circuit is the response of a given circuit to zero input (i.e. depending only upon the initial condition values). The natural Response to a circuit will be denoted as  $x_n(t)$ . The natural response of the system must satisfy the unforced differential equation of the circuit:

[Unforced function]

$$a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0x = 0$$

We remember this equation as being the "zero input response", that we discussed above. We now define the natural response to be an exponential function:

$$x_n = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

Where  $s$  are the roots of the characteristic equation of the circuit. The reasons for choosing this specific solution for  $x_n$  is based in differential equations theory, and we will just accept it without proof for the time being. We can solve for the constant values, by using a system of two equations:

$$x(0) = A_1 + A_2$$

$$\frac{dx(0)}{dt} = s_1 A_1 + s_2 A_2$$

Where  $x$  is the voltage (of the elements in a parallel circuit) or the current (through the elements in a series circuit).

### 14.8.2 Forced Response

The **forced response** of a circuit is the way the circuit responds to an input forcing function. The Forced response is denoted as  $x_f(t)$ .

Where the forced response must satisfy the forced differential equation:

[Forced function]

$$a_2 \frac{d^2 x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = f(t)$$

The forced response is based on the input function, so we can't give a general solution to it. However, we can provide a set of solutions for different inputs:

Input Form	Output Form
$K$ (constant)	$A$ (constant)
$M \sin(\omega t)$	$A \sin(\omega t) + B \cos(\omega t)$
$M e^{-at}$	$A e^{-at}$

### 14.8.3 Complete Response

The **Complete response** of a circuit is the sum of the forced response, and the natural response of the system:

[Complete Response]

$$x_c(t) = x_t(t) + x_s(t)$$

Once we have derived the complete response of the circuit, we can say that we have "solved" the circuit, and are finished working.



# 15 Mutual Inductance

## 15.1 Magnetic Fields

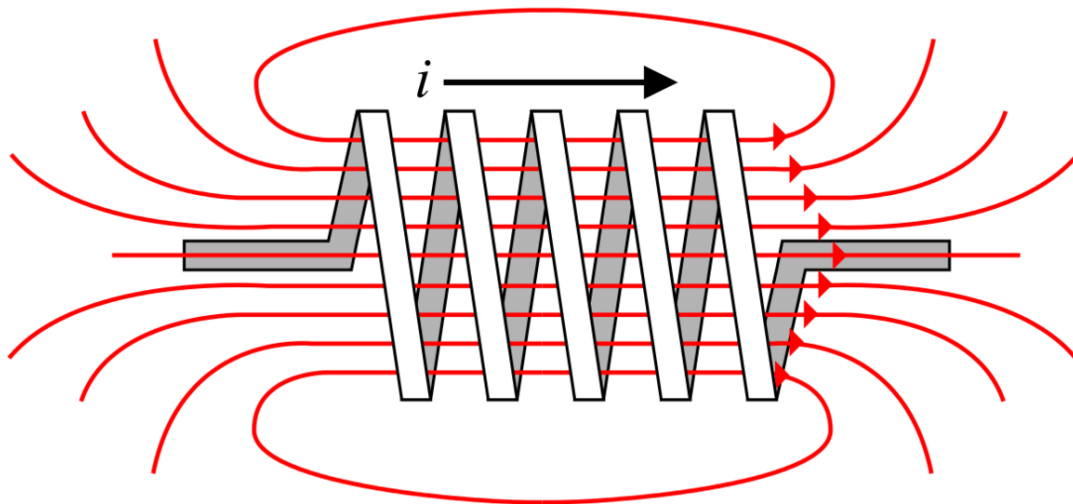


Figure 25

Inductors store energy in the form of a magnetic field. The magnetic field of an inductor actually extends outside of the inductor, and can be affected (or can affect) another inductor close by. The image above shows a magnetic field (red lines) extending around an inductor.

## 15.2 Mutual Inductance

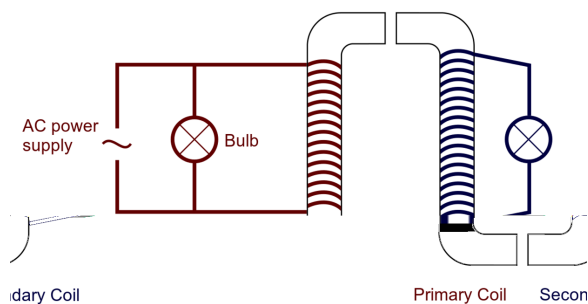


Figure 26

If we accidentally or purposefully put two inductors close together, we can actually transfer voltage and current from one inductor to another. This property is called **Mutual Inductance**. A device which utilizes mutual inductance to alter the voltage or current output is called a **transformer**.

The inductor that creates the magnetic field is called the *primary coil*, and the inductor that picks up the magnetic field is called the *secondary coil*. Transformers are designed to have the greatest mutual inductance possible by winding both coils on the same *core*. (In calculations for inductance, we need to know which materials form the path for magnetic flux. *Air core* coils have low inductance; Cores of iron or other magnetic materials are better 'conductors' of magnetic flux.)

The voltage that appears in the secondary is caused by the *change* in the shared magnetic field, each time the current through the primary changes. Thus, transformers work on A.C. power, since the voltage and current change continuously.

### 15.3 Ideal Transformers

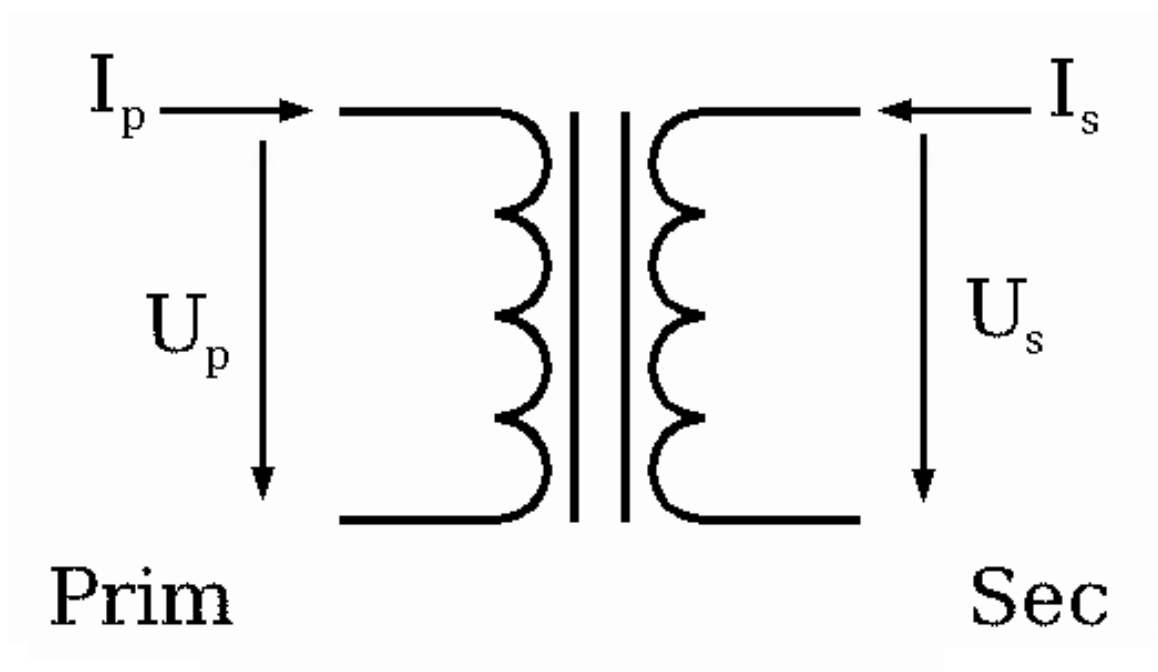
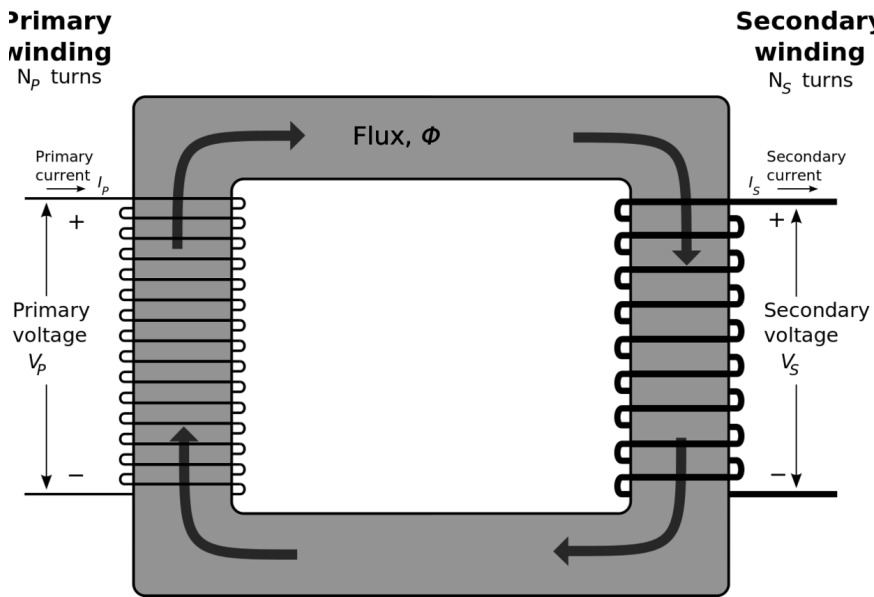


Figure 27

## 15.4 Modern Inductors



**Figure 28**

When the coils of number of turns  $N_1$  conducts current . There exists a Magnetic Field  $B$  on the coil . Changes of  $B$  will generates an Induced Voltage on the turns of coil  $N_1$  and  $N_2$  as shown

$$-\xi_p = N_p \frac{dB}{dt}$$

$$-\xi_s = N_s \frac{dB}{dt}$$

The ratio of  $-\xi_2$  over  $-\xi_1$

$$-\xi_p / -\xi_s = \frac{N_p}{N_s}$$

If Input voltage at coil of turn  $N_p = -\xi_p$  and the Output voltage will be

$$\frac{V_s}{V_p} = -\xi_s / -\xi_p = \frac{N_s}{N_p}$$

$$V_s = V_p \frac{N_s}{N_p}$$

Thus, this device is capable of Increase, Decrease and Conduct Voltage just by changing the turn ratio of the coils

Therefore, the output voltage can be

- Increased or Step Up by increasing number of turns of coil  $N_s$  greater than  $N_p$
- Decreased or Step Down by Decreasing number of turns of coil  $N_s$  less than  $N_p$
- Buffered by setting number of turns of coil  $N_s$  equal to  $N_p$

The following photo shows several examples of the construction of inductors and transformers. At the upper right is a toroidal core type (toroid is the mathematical term for a donut

shape). This shape very efficiently contains the magnetic flux, so less power (or signal) is lost to heating up the core.

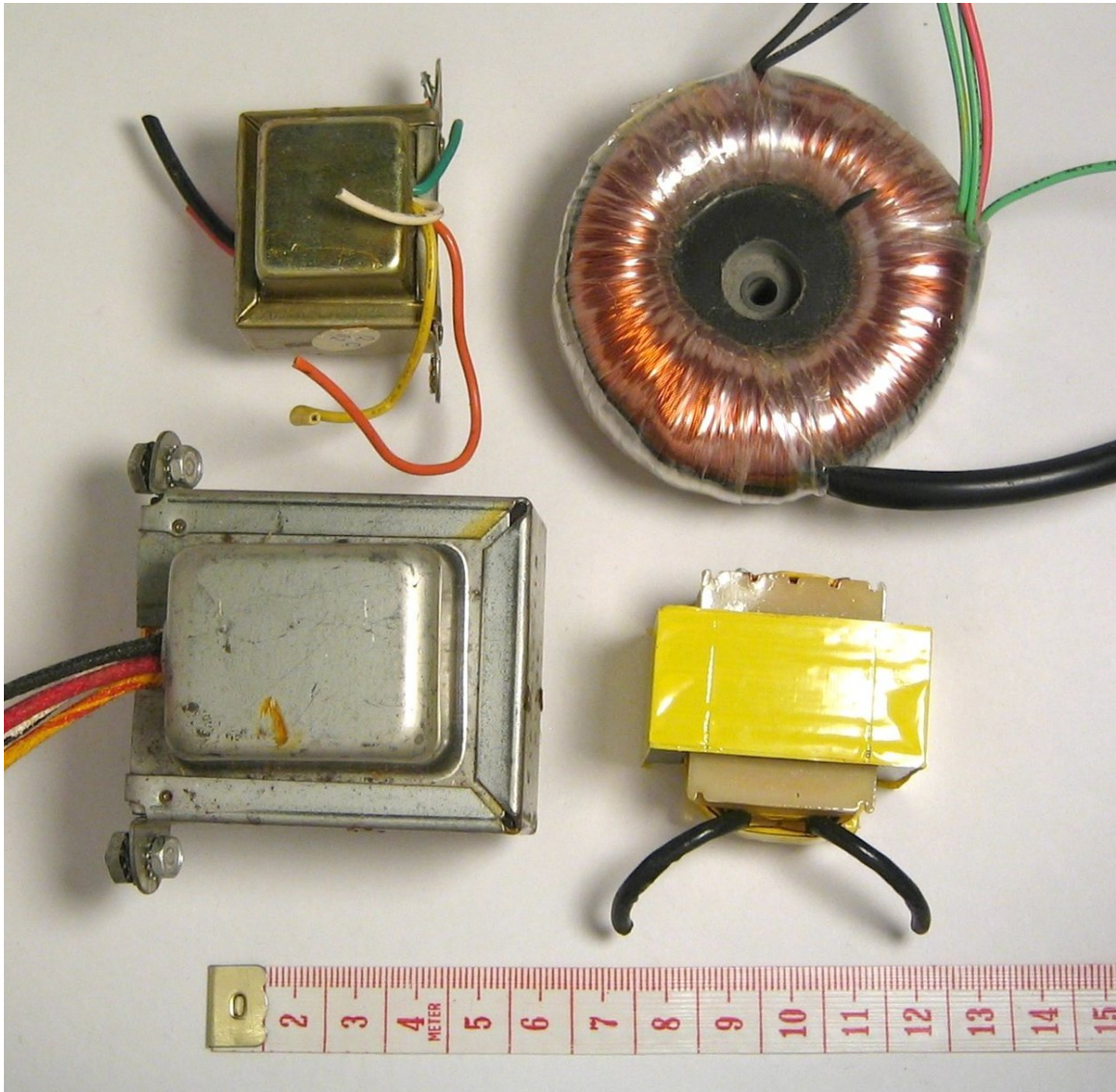


Figure 29

### 15.5 Step Up and Step Down

The terms 'step-up' and 'step-down' are used to compare the secondary (output) voltage to the voltage supplied to the primary.

Many transformers are specially designed to operate exclusively as step-up or step-down. While an ideal transformer could simply be 'turned around', we find that many actual transformers are built to perform best at certain ranges of voltage and current.

For example, a power transformer may be used to step down household AC (about 120 Volts) to 24V for home heating controls, etc. The output current is higher than the primary current in this example, so the transformer is made with a heavier gauge of wire in its secondary windings.

In transformers that deal with very high voltages, special attention is paid to insulation. The windings that deal with thousands of volts must resist arcing and other problems we do not see at home.

Finally, some transformers in electronic equipment are designed for a task known as 'impedance matching', rather than for specific in/out voltages. This function is explained in literature covering audio and radio topics.

## 15.6 further reading

- Electronics/Transformers<sup>1</sup>

---

<sup>1</sup> <http://en.wikibooks.org/wiki/Electronics%2FTransformers>



# 16 State-Variable Approach

## 16.1 State Variables

A more modern approach to circuit analysis is known as the **state variable method**, which we will attempt to describe here. We use variables called **state variables** to describe the current state of the energy storage elements (capacitors and inductors). Here, we are using the word "state" to mean "condition" or "status" of the elements.

## 16.2 Aim of State Variables

What is the goal of using state variables? We have all sorts of other methods for solving and describing circuits, so why would we introduce another method for dealing with circuits?

An answer to both these questions lay in the theory of the state variable method: the state variable approach attempts to describe a circuit using a system of first-order differential equations instead of a single, higher-order equation. A system of first-order equations can then be easily manipulated using linear algebra techniques, and can be solved through brute-force methods such as **cramers rule**.

## 16.3 State Variable Selection

We choose state variables to be either the voltage across a capacitor, or the current through an inductor. If our circuit has multiple irreducible capacitors or inductors, we assign a state variable to each one.

## 16.4 System of Equations

Once we have assigned our state variables, we are tasked with finding a first-order differential equation that describes each one individually. Once we have all of these equations, we can set them up in matrix form, and use Cramers rule, or another analysis method to solve the system.

## 16.5 Further Reading

- Control Systems/State-Space Equations<sup>1</sup>

---

<sup>1</sup> <http://en.wikibooks.org/wiki/Control%20Systems%2FState-Space%20Equations>



# 17 Sinusoidal Sources

## 17.1 Steady State

"Steady State" means that we are not dealing with turning on or turning off circuits in this section. We are assuming that the circuit was turned on a very long time ago and it is behaving in a pattern. We are computing what the pattern will look like. The "complex frequency" section models turning on and off a circuit with an exponential.

## 17.2 Sinusoidal Forcing Functions

Let us consider a general AC forcing function:

$$v(t) = M \sin(\omega t + \phi)$$

In this equation, the term  $M$  is called the "Magnitude", and it acts like a scaling factor that allows the peaks of the sinusoid to be higher or lower than  $\pm 1$ . The term  $\omega$  is what is known as the "Radial Frequency". The term  $\phi$  is an offset parameter known as the "Phase".

Sinusoidal sources can be current sources, but most often they are voltage sources.

## 17.3 Other Terms

There are a few other terms that are going to be used in many of the following sections, so we will introduce them here:

### Period

The period of a sinusoidal function is the amount of time, in seconds, that the sinusoid takes to make a complete wave. The period of a sinusoid is always denoted with a capital  $T$ . This is not to be confused with a lower-case  $t$ , which is used as the independent variable for time.

### Frequency

Frequency is the reciprocal of the period, and is the number of times, per second, that the sinusoid completes an entire cycle. Frequency is measured in Hertz (Hz). The relationship between frequency and the Period is as follows:

$$f = \frac{1}{T}$$

Where  $f$  is the variable most commonly used to express the frequency.

### Radian Frequency

Radian frequency is the value of the frequency expressed in terms of Radians Per Second, instead of Hertz. Radian Frequency is denoted with the variable  $\omega$ . The relationship between the Frequency, and the Radian Frequency is as follows:

$$\omega = 2\pi f$$

### Phase

The phase is a quantity, expressed in radians, of the time shift of a sinusoid. A sinusoid phase-shifted  $\phi = +2\pi$  is moved forward by 1 whole period, and looks exactly the same. An important fact to remember is this:

$$\sin\left(\frac{\pi}{2} - t\right) = \cos(t) \text{ or } \sin(t) = \cos\left(t - \frac{\pi}{2}\right)$$

Phase is often expressed with many different variables, including  $\phi, \psi, \theta, \gamma$  etc... This wikibook will try to stick with the symbol  $\phi$ , to prevent confusion.

## 17.4 Lead and Lag

A circuit element may have both a voltage across its terminals and a current flowing through it. If one of the two (current or voltage) is a sinusoid, then the other must also be a sinusoid (remember, voltage is the derivative of the current, and the derivative of a sinusoid is always a sinusoid). However, the sinusoids of the voltage and the current may differ by quantities of magnitude and phase.

If the current has a lower phase angle than the voltage the current is said to **lag** the voltage. If the current has a higher phase angle than the voltage, it is said to **lead** the voltage. Many circuits can be classified and examined using lag and lead ideas.

## 17.5 Sinusoidal Response

Reactive components (capacitors and inductors) are going to take energy out of a circuit like a resistor and then pump some of it back into the circuit like a source. The result is initially a mess. But after a while (5 time constants), the circuit starts behaving in a pattern. The capacitors and inductors get in a rhythm that reflects the driving sources. If the source is sinusoidal, the currents and voltages will be sinusoidal. This is called the "particular" or "steady state" response. In general:

$$A_{in} \cos(\omega_{in}t + \phi_{in}) \rightarrow A_{out} \cos(\omega_{out}t + \phi_{out})$$

What happens initially, what happens if the capacitor is initially charged, what happens if sources are switched in and out of a circuit is that there is an energy imbalance. A voltage or current source might be charged by the initial energy in a capacitor. The derivative

of the voltage across an Inductor might instantaneously switch polarity. Lots of things are happening. We are going to save this for later. Here we deal with the steady state or "particular" response first.

## 17.6 Sinusoidal Conventions

For the purposes of this book we will generally use cosine functions, as opposed to sine functions. If we absolutely need to use a sine, we can remember the following trigonometric identity:

$$\cos(\omega t) = \sin(\pi/2 - \omega t)$$

We can express all sine functions as cosine functions. This way, we don't have to compare apples to oranges per se. This is simply a convention that this wikibook chooses to use to keep things simple. We could easily choose to use all  $\sin( )$  functions, but further down the road it is often more convenient to use cosine functions instead by default.

## 17.7 Sinusoidal Sources

There are two primary sinusoidal sources: wall outlets and oscillators<sup>1</sup>. Oscillators are typically crystals that electrically vibrate and are found in devices that communicate or display video such as TV's, computers, cell phones, radios. An electrical engineer or tech's working area will typically include a "function generator"<sup>2</sup> which can produce oscillations at many frequencies and in shapes that are not just sinusoidal.

RMS or Root mean square<sup>3</sup> is a measure of amplitude that compares with DC magnitude in terms of power, strength of motor, brightness of light, etc. The trouble is that there are several types of AC amplitude:

- peak
- peak to peak
- average
- RMS

Wall outlets are called AC<sup>4</sup> or alternating current. Wall outlets are sinusoidal voltage sources that range from 100 RMS volts, 50 Hz to 240 RMS volts 60 Hz world wide. RMS, rather than peak (which makes more sense mathematically), is used to describe magnitude for several reasons:

- historical reasons related to the competition between Edison (DC power) and Tesla (Sinusoidal or AC power)

---

1 [http://en.wikibooks.org/wiki/%3Aw%3AElectronic\\_oscillator](http://en.wikibooks.org/wiki/%3Aw%3AElectronic_oscillator)  
2 [http://en.wikibooks.org/wiki/%3Aw%3AFunction\\_generator](http://en.wikibooks.org/wiki/%3Aw%3AFunction_generator)  
3 <http://en.wikibooks.org/wiki/%3Aw%3ARoot%20mean%20square>  
4 <http://en.wikibooks.org/wiki/%3Aw%3AAlternating%20current>

- effort to compare/relate AC (wall outlets) to DC (cars, batteries) .. 100 RMS volts is approximately 100 DC volts.
- average sinusoidal is zero
- meter movements (physical needles moving<sup>5</sup> on measurement devices) were designed to measure both DC and RMS AC

RMS is a type of average:  $p_{\text{rms}} = \sqrt{\frac{1}{T_2 - T_1} \int_{T_1}^{T_2} [p(t)]^2 dt}$

Electrical power delivery<sup>6</sup> is a complicated subject that will not be covered in this course. Here we are trying to define terms, design devices that use the power and understand clearly what comes out of wall outlets.

---

<sup>5</sup> <http://en.wikibooks.org/wiki/%3Aw%3AAmmeter%23Types>

<sup>6</sup> [http://en.wikibooks.org/wiki/%3Aw%3AThree-phase\\_electric\\_power](http://en.wikibooks.org/wiki/%3Aw%3AThree-phase_electric_power)

## 18 Phasor Representation



# 19 Phasors

## 19.1 Variables

Variables are defined the same way. But there is a difference. Before variables were either "known" or "unknown." Now there is a sort of in between.

At this point the concept of a constant function (a number) and a variable function (varies with time) needs to be reviewed. See this <sup>1</sup> /student professor dialogue. Knowns are described in terms of functions, unknowns are computed based upon the knowns and are also functions.

For example:

$$v(t) = M_v \cos(\omega t + \phi_v) \text{ voltage varying with time}$$

Here  $v(t)$  is the symbol for a function. It is assigned a function of the symbols  $M_v, \omega, \phi_v$  and  $t$ . Typically time is not ever solved for.

Time remains an unknown. Furthermore all power, voltage and current turn into equations of time. Time is not solved for. Because time is everywhere, it can be eliminated from the equations. Integrals and derivatives turn into algebra and the answers can be purely numeric (before time is added back in).

At the last moment, time is put back into voltage, current and power and the final solution is a function of time.

Most of the math in this course has these steps:

1. describe knowns and unknowns in the time domain, describe all equations
2. change knowns into phasors, eliminate derivatives and integrals in the equations
3. solve numerically or symbolically for unknowns in the phasor domain
4. transform unknowns back into the time domain

## 19.2 Passive circuit output is similar to input

If the input to a linear circuit is a sinusoid, then the output from the circuit will be a sinusoid. Specifically, if we have a voltage sinusoid as such:

$$v(t) = M_v \cos(\omega t + \phi_v)$$

---

<sup>1</sup> <http://en.wikibooks.org/wiki/%2Fstudent%20professor%20%2F>

Then the current through the linear circuit will also be a sinusoid, although its magnitude and phase may be different quantities:

$$i(t) = M_i \cos(\omega t + \phi_i)$$

Note that both the voltage and the current are sinusoids with the same radial frequency, but different magnitudes, and different phase angles. Passive circuit elements cannot change the frequency of a sinusoid, only the magnitude and the phase. Why then do we need to write  $\omega$  in every equation, when it doesn't change? For that matter, why do we need to write out the  $\cos( )$  function, if that never changes either? The answers to these questions is that *we don't need to write these things every time*. Instead, engineers have produced a short-hand way of writing these functions, called "phasors".

### 19.3 Phasor Transform

Phasors are a type of "transform." We are transforming the circuit math so that time disappears. Imagine going to a place where time doesn't exist.

We know that every function can be written as a series of sine waves of various frequencies and magnitudes added together. (Look up fourier transform animation). The entire world can be constructed from sin waves<sup>2</sup>. Here, one sine wave is looked at, the repeating nature ( $\omega$ ) is stripped away. What's left is a phasor. Since time is made of circles, and if we consider just one of these circles, we can move to a world where time doesn't exist and circles are "things". Instead of the word "world", use the word "domain" or "plane" as in two dimensions.

Math in the Phasor domain is almost the same as DC circuit analysis. What is different is that inductors and capacitors have an impact that needs to be accounted for.

The transform into the Phasors plane or domain and transforming back into time is based upon Euler's equation. It is the reason you studied imaginary numbers in past math class.

---

2 [http://en.wikibooks.org/wiki/%3Aw%3AMatter\\_wave](http://en.wikibooks.org/wiki/%3Aw%3AMatter_wave)



## 19.4 Euler's Equation

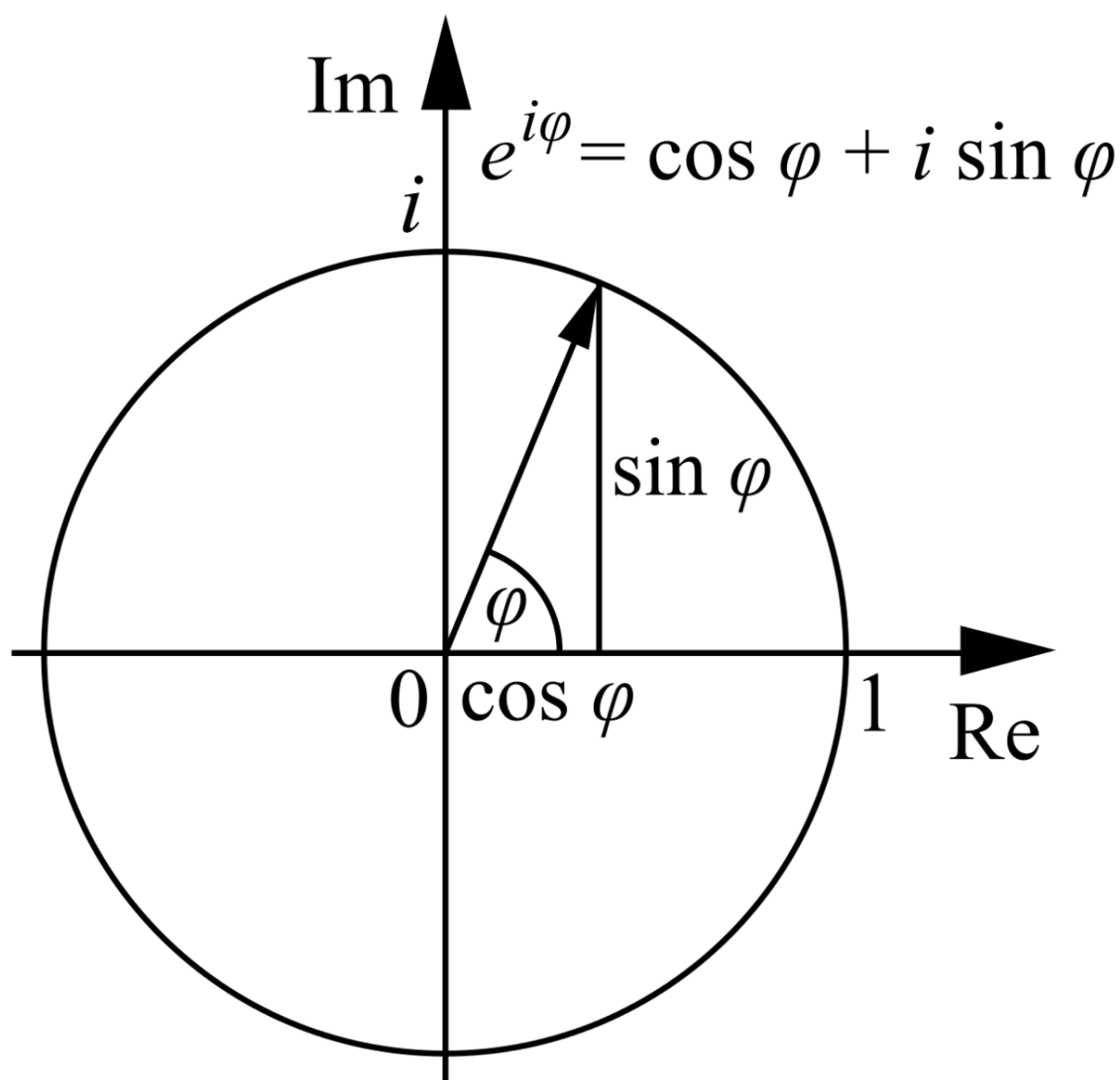


Figure 30 Euler's Formula

**Euler's formula** is a mathematical formula that establishes the deep relationship between the trigonometric functions and the complex exponential function. Euler's formula states that, for any real number  $x$ ,

$$e^{ix} = \cos x + i \sin x$$

where  $e$  is the base of the natural logarithm<sup>3</sup>,  $i$  is the square root of a negative one<sup>4</sup>, and  $\cos$  and  $\sin$  are the trig functions<sup>5</sup>, with the argument  $x$  given in radians<sup>6</sup>.

Euler's formula is ubiquitous in mathematics, physics, and engineering. The physicist Richard Feynman<sup>7</sup> called the equation "our jewel" and "one of the most remarkable, almost astounding, formulas in all of mathematics."

A more general version of Euler's equation is:

**[Euler's Equation]**

$$Me^{j(\omega t + \phi)} = M \cos(\omega t + \phi) + jM \sin(\omega t + \phi)$$

This equation allows us to view sinusoids as complex exponential functions. The circle in time of voltage, current or power turns into an arrow have length (magnitude  $\mathbb{C}$ ) and angle (phase  $\phi$ ) or a point having a real ( $X$ ) and imaginary ( $Y$ ) coordinates in the phasor domain/plane or complex domain/plane.

Generically, the phasor  $\mathbb{C}$ , (which could be voltage, current or power) can be written:

$$\mathbb{C} = X + jY \text{ (rectangular coordinates)}$$

$$\mathbb{C} = M_v \angle \phi \text{ (polar coordinates)}$$

We can graph the point ( $X$ ,  $Y$ ) on the complex plane and draw an arrow to it showing the relationship between  $X$ ,  $Y$ ,  $\mathbb{C}$  and  $\phi$ .

Using this fact, we can get the angle from the origin of the complex plane to our point ( $X$ ,  $Y$ ) with the function:

**[Angle equation]**

$$\theta_C = \arctan\left(\frac{Y}{X}\right)$$

And using the pythagorean theorem, we can find the magnitude of  $\mathbb{C}$  -- the distance from the origin to the point ( $X$ ,  $Y$ ) -- as:

**[Pythagorean Theorem]**

$$M_C = |\mathbb{C}| = \sqrt{X^2 + Y^2}$$

---

3 <http://en.wikibooks.org/wiki/%3Ae%20%28mathematical%20constant%29>

4 <http://en.wikibooks.org/wiki/%3Aa%3Aimaginary%20unit>

5 <http://en.wikibooks.org/wiki/%3Aa%3Atrigonometric%20functions>

6 <http://en.wikibooks.org/wiki/%3Aa%3Aradian>

7 <http://en.wikibooks.org/wiki/%3Aa%3ARichard%20Feynman>

## 19.5 Phasor Symbols

Phasors don't account for the frequency information, so make sure you write down the frequency some place safe.

Suppose in the time domain:

$$v(t) = M_v e^{j(\omega t + \phi)}$$

In the phasor domain, this voltage is expressed like this:

$$\mathbb{V} = M_v \angle \phi$$

The radial velocity  $\omega$  disappears from known functions (not the derivate and integral operations) and reappears in the time expression for the unknowns.

## 19.6 Not Vectors

Contrary to the statement made in this heading, phasors (phase vectors), are vectors. Phasors form a vector space with additional structure, hence they have some properties that are not common to all vector spaces; these additional properties exist because phasors form a field - thus you also get division.

For more details see [http://en.wikipedia.org/wiki/Phasor\\_\(electronics\)](http://en.wikipedia.org/wiki/Phasor_(electronics))

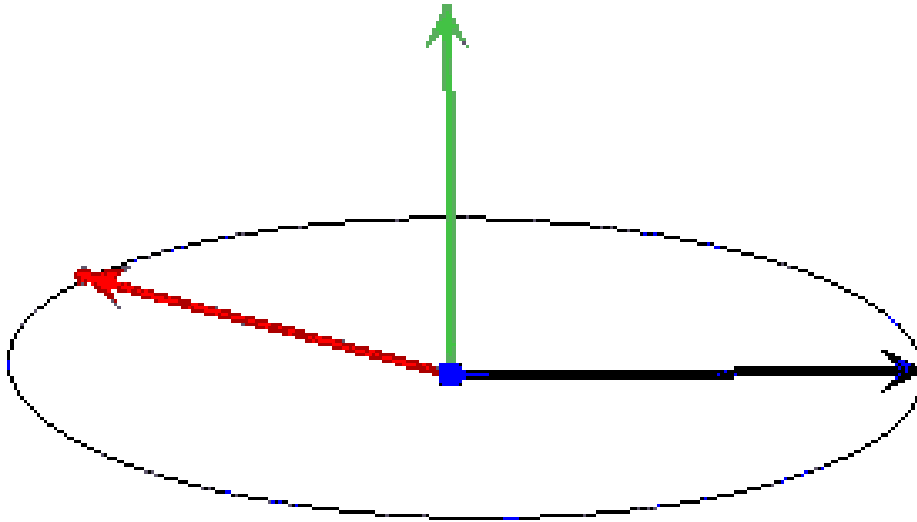
Like many kinds of vectors they have additional struct Phasors will always be written out either with a large bold letter (as above). They are not a vector. Vectors have two or more real axes that are not related by Euler, but are independent. They share some math in two dimensions, but this math diverges.

**Phasors can be divided, but vectors can not.**

Voltage can be divided by current (in the phasor domain), but East can not be divided by North (vectors can not be divided). Vectors move into three or more dimensions of linear algebra math that help build complicated structures in the real world such as space frames<sup>8</sup>. Phasors move into more complicated transforms related to differential equation math and electronics.

---

<sup>8</sup> [http://en.wikibooks.org/wiki/%3Aw%3ASpace\\_frame](http://en.wikibooks.org/wiki/%3Aw%3ASpace_frame)



**Figure 31** cross product visualization

The math of phasors is exactly the same as ordinary math, except with imaginary numbers. Vectors demand new mathematical operations such as dot product and cross product:

- The dot product of vectors finds the shadow of one vector on another.
- The cross product of vectors combines vectors into a third vector perpendicular to both.

## 19.7 Cosine Convention

In this book, all phasors correspond to a **cosine** function, not a **sine** function.

It is important to remember which trigonometric function your phasors are mapping to. Since a phasor only includes information on magnitude and phase angle, it is impossible to know whether a given phasor maps to a  $\sin(\ )$  function, or a  $\cos(\ )$  function instead. By convention, this wikibook and most electronic texts/documentation map to the cosine function.

If you end up with an answer that is sin, convert to cos by subtracting 90 degrees:

$$\sin(\omega t + \phi) = \cos(\omega t + \phi - \frac{\pi}{2})$$

If your simulator requires the source to be in sin form, but the starting point is cos, then convert to sin by adding 90 degrees:

$$\cos(\omega t + \phi) = \sin(\omega t + \phi + \frac{\pi}{2})$$

## 19.8 Phasor Concepts

Inside the phasor domain, concepts appear and are named. Inductors and capacitors can be coupled with their derivative operator transforms and appear as imaginary resistors called "reactance." The combination of resistance and reactance is called "impedance." Impedance can be treated algebraically as a phasor although technically it is not. Power concepts such as real, reactive, apparent and power factor appear in the phasor domain. Numeric math can be done in the phasor domain. Symbols can be manipulated in the phasor domain.

## 19.9 Phasor Math

There is more information about Phasors in  
**The Appendix<sup>a</sup>**

<sup>a</sup> Chapter 25 on page 125

Phasor math turns into the imaginary number math which is reviewed below.

Phasor A can be multiplied by phasor B:

**[Phasor Multiplication]**

$$\mathbb{A} \times \mathbb{B} = (M_a \times M_b) \angle (\phi_a + \phi_b)$$

The phase angles add because in the time domain they are exponents of two things multiplied together.

**[Phasor Division]**

$$\mathbb{A}/\mathbb{B} = (M_a/M_b)\angle(\phi_a - \phi_b)$$

Again the phase angles are treated like exponents ... so they subtract.

The magnitude and angle form of phasors can not be used for addition and subtraction. For this, we need to convert the phasors into rectangular notation:

$$\mathbb{C} = X + jY$$

Here is how to convert from polar form (magnitude and angle) to rectangular form (real and imaginary)

$$X = M \cos(\phi)$$

$$Y = M \sin(\phi)$$

Once in rectangular form:

- Real parts get add or subtract
- Imaginary parts add or subtract

**[Phasor Addition]**

$$\mathbb{C} = \mathbb{A} + \mathbb{B} = (X_A + X_B) + j(Y_A + Y_B) = X_C + jY_C$$

Here is how to convert from rectangular form to polar form:

$$\mathbb{C} = M_c \angle \phi_c = \sqrt{X^2 + Y^2} \angle \arctan\left(\frac{Y}{X}\right)$$

Once in polar phasor form, conversion back into the time domain is easy:

$$\text{Re}(Me^{j(\omega t + \phi)}) = M \cos(\omega t + \phi)$$

## 19.10 Function transformation Derivation

$g(t)$  represents either voltage, current or power.

$$g(t) = G_m \cos(\omega t + \phi) \text{ starting point}$$

$$g(t) = G_m \text{Re}(e^{j(\omega t + \phi)}) \text{ from Euler's Equation}$$

$$g(t) = G_m \text{Re}(e^{j\phi} e^{j\omega t}) \text{ law of exponents}$$

$g(t) = \text{Re}(G_m e^{j\phi} e^{j\omega t})$  ....  $G_m$  is a real number so it can be moved inside

$g(t) = \text{Re}(\mathbb{G} e^{j\omega t})$  ....  $\mathbb{G}$  is the definition of a phasor, here it is an expression substituting for  $G_m e^{j\phi}$

$g(t) \Leftrightarrow \mathbb{G}$  where  $\mathbb{G} = G_m e^{j\phi}$

What happens to  $e^{j\omega t}$  term? Long Answer<sup>9</sup>. It hangs around until it is time to transform back into the time domain. Because it is an exponent, and all the phasor math is algebra associated with exponents, the final phasor can be multiplied by it. Then the real part of the expression will be the time domain solution.

time domain	transformation	phasor domain
$A \cos(\omega t)$	$\Leftrightarrow$ proof <sup>10</sup>	$A$
$A \sin(\omega t)$	$\Leftrightarrow$ proof <sup>11</sup>	$-Aj$
$A \cos(\omega t) + B \sin(\omega t)$	$\Leftrightarrow$	$A - Bj$
$A \cos(\omega t) - B \sin(\omega t)$	$\Leftrightarrow$	$A + Bj$
$A \cos(\omega t + \phi)$	$\Leftrightarrow$ proof <sup>12</sup>	$A \cos(\phi) + A \sin(\phi)j$
$A \sin(\omega t + \phi)$	$\Leftrightarrow$ proof <sup>13</sup>	$A \sin(\phi) - A \cos(\phi)j$
$A \cos(\omega t - \phi)$	$\Leftrightarrow$ proof <sup>14</sup>	$A \cos(\phi) - A \sin(\phi)j$
$A \sin(\omega t - \phi)$	$\Leftrightarrow$ proof <sup>15</sup>	$-A \sin(\phi) - A \cos(\phi)j$

In all the cases above, remember that  $\phi$  is a constant, a known value in most cases. Thus the phasor is an complex number in most calculations.

There is another transform associated with a derivatives that is discussed in "phasor calculus."

## 19.11 Transforming calculus operators into phasors

When sinusoids are represented as phasors, differential equations become algebra. This result follows from the fact that the complex exponential is the eigenfunction<sup>16</sup> of the operation:

$$\frac{d}{dt}(e^{j\omega t}) = j\omega e^{j\omega t}$$

That is, only the complex amplitude is changed by the derivative operation. Taking the real part of both sides of the above equation gives the familiar result:

$$\frac{d}{dt} \cos \omega t = -\omega \sin \omega t$$

<sup>9</sup> <http://en.wikibooks.org/wiki/%2Fj%20omega%20disappears%2F>

<sup>10</sup> <http://en.wikibooks.org/wiki/%2Fproof1%2F>

<sup>11</sup> <http://en.wikibooks.org/wiki/%2Fproof2%2F>

<sup>12</sup> <http://en.wikibooks.org/wiki/%2Fproof5%2F>

<sup>13</sup> <http://en.wikibooks.org/wiki/%2Fproof6%2F>

<sup>14</sup> <http://en.wikibooks.org/wiki/%2Fproof7%2F>

<sup>15</sup> <http://en.wikibooks.org/wiki/%2Fproof8%2F>

<sup>16</sup> <http://en.wikibooks.org/wiki/%3Aw%3AEigenfunction>

Thus, a time derivative of a sinusoid becomes, when transformed into the phasor domain, algebra:

$$\frac{d}{dt}i(t) \rightarrow j\omega \mathbb{I} \text{ } j \text{ is the square root of } -1 \text{ or an imaginary number}$$

In a similar way the time integral, when transformed into the phasor domain is:

$$\int V(t)dt \rightarrow \frac{\mathbb{V}}{j\omega}$$

The above is true of voltage, current, and power.

The question is why does this work? Where is the proof? Lets do this three times: once for a resistor, then inductor, then capacitor. The symbols for the current and voltage going through the terminals are:  $V_m \cos(\omega t + \phi_v)$  and  $I_m \cos(\omega t + \phi_I)$

### 19.11.1 Resistor Terminal Equation

$V = RI$  . terminal relationship

$V_m \cos(\omega t + \phi_V) = RI_m \cos(\omega t + \phi_I)$  .. substituting example functions

$V_m e^{j\omega t + j\phi_V} = RI_m e^{j\omega t + j\phi_I}$  .. Euler's version of the terminal relationship

$V_m e^{j\omega t} e^{j\phi_V} = RI_m e^{j\omega t} e^{j\phi_I}$  .. law of exponents

$V_m e^{j\omega t} e^{j\phi_V} = RI_m e^{j\omega t} e^{j\phi_I}$  .. do same thing go both sides of equal sign

$V_m e^{j\phi_V} = RI_m e^{j\phi_I}$  .. time domain result

$\mathbb{V} = R\mathbb{I}$  .. phasor expression

Just put the voltage and current in phasor form and substitute to migrate equation into the phasor domain.

### 19.11.2 Inductor Terminal Equation

$V = L \frac{d}{dt} I$  ... terminal relationship

$V_m \cos(\omega t + \phi_V) = L \frac{d}{dt} (I_m \cos(\omega t + \phi_I))$  .. substitution of a generic sinusoidal

$V_m \cos(\omega t + \phi_V) = -\omega L I_m \sin(\omega t + \phi_I)$  .. taking the derivative

$-\sin(\omega t + \phi_I) = \cos(\omega t + \phi_I + \frac{\pi}{2})$  .. trig

$V_m \cos(\omega t + \phi_V) = \omega L I_m \cos(\omega t + \phi_I + \frac{\pi}{2})$  .. substitution

$V_m \operatorname{Re}(e^{j(\omega t + \phi_V)}) = \omega L I_m \operatorname{Re}(e^{j(\omega t + \phi_I + \frac{\pi}{2})})$  from Euler's Equation

$V_m \operatorname{Re}(e^{j\omega t} e^{j\phi_V}) = \omega L I_m \operatorname{Re}(e^{j\omega t} e^{j\phi_I} e^{j\frac{\pi}{2}})$  law of exponents

$\operatorname{Re}(V_m e^{j\phi_V} e^{j\omega t}) = \operatorname{Re}(e^{j\frac{\pi}{2}} \omega L I_m e^{j\phi_I} e^{j\omega t})$  .... real numbers can be moved inside

$e^{j\frac{\pi}{2}} = \cos(\frac{\pi}{2}) + j \sin(\frac{\pi}{2}) = j$  ... substitute in above



$\mathbb{I} = I_m e^{j\phi_L}$  and  $\mathbb{V} = V_m e^{j\phi_V}$  .. substitute in above

cancel out the  $e^{j\omega t}$  terms on both sides

$\text{Re}(\mathbb{V} e^{j\omega t}) = \text{Re}(j\omega L \mathbb{I} e^{j\omega t})$  .... definition of phasors

$\mathbb{V} = j\omega L \mathbb{I}$  .... equation transformed into phasor domain

Conclusion, put the voltage and current in phasor form, replace  $\frac{d}{dt}$  with  $j\omega$  to translate the equation to the phasor domain.

### 19.11.3 Capacitor Terminal Equation

A capacitor is basically the same form, V and I switch sides, C is substituted for L.

$I = C \frac{d}{dt} V$  ... terminal relationship

$I_m \cos(\omega t + \phi_I) = C \frac{d}{dt} (V_m \cos(\omega t + \phi_V))$  .. substitution of a generic sinusoidal

$I_m \cos(\omega t + \phi_I) = -\omega C V_m \sin(\omega t + \phi_V)$  .. taking the derivative

$-\sin(\omega t + \phi_V) = \cos(\omega t + \phi_V + \frac{\pi}{2})$  .. trig

$I_m \cos(\omega t + \phi_I) = \omega C V_m \cos(\omega t + \phi_V + \frac{\pi}{2})$  .. substitution

$I_m \text{Re}(e^{j(\omega t + \phi_I)}) = \omega C V_m \text{Re}(e^{j(\omega t + \phi_V + \frac{\pi}{2})})$  from Euler's Equation

$I_m \text{Re}(e^{j\omega t} e^{j\phi_I}) = \omega C V_m \text{Re}(e^{j\omega t} e^{j\phi_V} e^{j\frac{\pi}{2}})$  law of exponents

$\text{Re}(I_m e^{j\phi_I} e^{j\omega t}) = \text{Re}(e^{j\frac{\pi}{2}} \omega C V_m e^{j\phi_V} e^{j\omega t})$  .... real numbers can be moved inside

$e^{j\frac{\pi}{2}} = \cos(\frac{\pi}{2}) + j \sin(\frac{\pi}{2}) = j$  ... substitute in above equation

$\mathbb{V} = V_m e^{j\phi_V}$  and  $\mathbb{I} = I_m e^{j\phi_I}$  .. substitute in above

cancel out the  $e^{j\omega t}$  terms on both sides

$\text{Re}(\mathbb{I} e^{j\omega t}) = \text{Re}(j\omega C \mathbb{V} e^{j\omega t})$  .... definition of phasors

$\mathbb{I} = j\omega C \mathbb{V}$  .... equation transformed into phasor domain

Conclusion, put the voltage and current in phasor form, replace  $\frac{d}{dt}$  with  $j\omega$  to translate the equation to the phasor domain.

In summary, all the terminal relations have  $e^{j\omega t}$  terms that cancel:

$$V_m e^{j\phi} e^{j\omega t} = I_m e^{j\phi} e^{j\omega t} * R$$

$$\mathbb{V} = \mathbb{I} R$$

$$V_m e^{j\phi} e^{j\omega t} = I_m e^{j\phi} e^{j\omega t} * j\omega * L$$

$$\mathbb{V} = \mathbb{I} j\omega L$$

$$I_m e^{j\phi} e^{j\omega t} = V_m e^{j\phi} e^{j\omega t} * j\omega * C$$

$$\mathbb{I} = \mathbb{V} j\omega C$$

What is interesting about this path of inquiry/logic/thought is a new concept emerges:

Device	$\mathbb{V}$	$\mathbb{I}$
Resistor	$R$	$\frac{\mathbb{V}}{R}$
Capacitor	$\frac{1}{j\omega C}$	$j\omega C$
Inductor	$j\omega L$	$\frac{1}{j\omega L}$

The  $j\omega$  terms that don't cancel out come from the derivative terms in the terminal relations. These derivative terms are associated with the capacitors and inductors themselves, not the sources. Although the derivative is applied to a source, the independent device the derivative originates from (a capacitor or inductor) is left with it's feature after the transform! So if we leave the driving forces as  $\frac{\text{output}}{\text{input}}$  ratios on one side of the equal sign, we can consider separately the other side of the equal sign as a function! These functions have a name ... Transfer Functions. When we analyze the voltage/current ratios's in terms of R, L an C, we can sweep  $\omega$  through a variety of driving source frequencies, or keep the frequency constant and sweep through a variety of inductor values .. . we can analyze the circuit response!

Note: Transfer Functions are an entire section of this course. They come up in mechanical engineering control system classes also. There are similarities. Driving over a bump is like a surge or spike. Driving over a curb is like turning on a circuit. And when mechanical engineers study vibrations, they deal with sinusoidal driving functions, but they are dealing with a three dimensional object rather than a one dimensional object like we are in this course.

## 19.12 Phasor Domain to Time Domain

Getting back into the time domain is just about as simple. After working through the equations in the phasor domain and finding  $\mathbb{V}$  and  $\mathbb{I}$ , the goal is to convert them to  $V$  and  $I$ .

The phasor solutions will have the form  $\mathbb{G} = A + Bj = G_m e^{j\phi}$  you should be able now to convert between the two forms of the solution. Then:

$$G = \text{Re}(\mathbb{G} e^{j\omega t}) = \text{Re}(G_m e^{j\phi} e^{j\omega t}) = \text{Re}(G_m e^{j(\omega t + \phi)}) = G_m \cos(\omega t + \phi)$$

## 19.13 What is not covered

There is another way of thinking about circuits where inductors and capacitors are complex resistances. The idea is:

$$\text{impedance} = \text{resistance} + j * \text{reactance}$$

Or symbolically

$$Z = R + j * X$$

Here<sup>17</sup> the derivative is attached to the inductance and capacitance, rather than to the terminal equation as we have done. This spreads the math of solving circuit problems into smaller pieces that is more easily checked, but it makes symbolic solutions more complex and can cause numeric solution errors to accumulate because of intermediate calculations.

The phasor concept is found everywhere. Some day it will be necessary to study this if you get involved in microwave projects that involve "stubs"<sup>18</sup> or antenna projects that involve a "loading coil"<sup>19</sup> ... the list is huge.

The goal here is to avoid the concepts of conductance, reactance, impedance, susceptance, and admittance<sup>20</sup> ... and avoid the confusion of relating these concepts while trying to compare phasor math with calculus and Laplace transforms.

## 19.14 Phasor Notation

Remember, a phasor represents a single value that can be displayed in multiple ways.

$\mathbb{C} = M \angle \phi$  "Polar Notation"

$C = M e^{j(\omega t + \phi)}$  "Exponential Notation"

$\mathbb{C} = A + jB$  "Rectangular Notation"

$C = M \cos(\omega t + \phi) + jM \sin(\omega t + \phi)$  "time domain notation"

These 4 notations are all just different ways of writing the same exact thing.

## 19.15 Phasor symbols

When writing on a board or on paper, use hats  $\hat{V}$  to denote phasors. Expect variations in books and online:

- $\mathbb{V}$  (the large bold block-letters we use in this wikibook)
- $\bar{V}$  ("bar" notation, used by Wikipedia)
- $\vec{V}$  (bad ... save for vectors ... vector arrow notation)
- $\tilde{V}$  (some text books)
- $\hat{V}$  (some text books)

<sup>17</sup> <http://en.wikibooks.org/wiki/Electronics%2FImpedance>

<sup>18</sup> [http://en.wikibooks.org/wiki/Communication\\_Systems%2FMicrowave\\_Systems%23Microwave\\_Components](http://en.wikibooks.org/wiki/Communication_Systems%2FMicrowave_Systems%23Microwave_Components)

<sup>19</sup> [http://en.wikibooks.org/wiki/%3Aw%3ALoading\\_coil%23Radio\\_antenna](http://en.wikibooks.org/wiki/%3Aw%3ALoading_coil%23Radio_antenna)

<sup>20</sup> <http://en.wikibooks.org/wiki/%3Aw%3AAdmittance>



## 20 Differential Equations

### 20.1 Phasors Generate the Particular Solution

Phasors can replace calculus, they can replace Laplace transforms, they can replace trig. But there is one thing they can not do: initial conditions/integration constants. When doing problems with both phasors and Laplace, or phasors and calculus, the difference in the answers is going to be an integration constant.

Differential equations are solved in this course in three steps:

- finding the particular solution ... particular to the driving function ... particular to the voltage or current source
- finding the homogenous solution ... the solution that is the same no matter what the driving function is ... the solution that explores how an initial energy imbalance in the circuit is balanced
- determining the coefficients, the constants of integration from initial conditions

### 20.2 Phasors Don't Generate Integration Constants

The integration constant doesn't appear in phasor solutions. But they will appear in the Laplace and Calculus alternatives to phasor solutions. If the full differential equation is going to be solved, it is absolutely necessary to see where the phasors fail to create a symbol for the unknown integration constant ... that is calculated in the third step.

Phasors are the technique used to find the particular AC solution. Integration constants document the initial DC bias or energy difference in the circuit. Finding these constants requires first finding the homogeneous solution which deals with the fact that capacitors may or may not be charged when a circuit is first turned on. Phasors don't completely replace the steps of Differential Equations. Phasors just replace the first step: finding the particular solution.

### 20.3 Differential Equations Review

The goal is to solve Ordinary Differential Equations (ODE) of the first and second order with both phasors, calculus, and Laplace transforms. This way the phasor solution can be compared with content of pre-requisite or co-requisite math courses. The goal is to do these problems with numeric and symbolic tools such as matLab and mupad/mathematica/wolframalpha. If you have already had the differential equations course, this is a quick review.

The most important thing to understand is the nature of a function. Trig, Calculus, and Laplace transforms and phasors are all associated with functions, not algebra. If you don't understand the difference between algebra and a function, maybe this /student professor/<sup>1</sup> dialogue will help.

We start with equations from terminal definitions, loops and junctions. Each of the symbols in these algebraic equations is a function. We are not transforming the equations. We are transforming the functions in these equations. All sorts of operators appear in these equations including  $+$   $-$   $*$   $/$  and  $\frac{d}{dt}$ . The first table focuses on transforming these operators. The second focuses on transforming the functions themselves.

The real power of the Laplace tranform is that it eliminates the integral and differential operators. Then the functions themselves can be transformed. Then unknowns can be found with just algebra. Then the functions can be transformed back into time domain functions.

Here are some of the Properties and Theorems<sup>2</sup> needed to transform the typical sinusoidal voltages, powers and currents in this class.

### 20.3.1 Laplace Operator Transforms

Properties of the unilateral Laplace transform	Time domain	's' domain	Comment
<b>Time scaling</b>	$f(at)$	$\frac{1}{ a }F\left(\frac{s}{a}\right)$	for figuring out how $\omega$ affects the equation
<b>Time shifting</b>	$f(t-a)u(t-a)$	$e^{-as}F(s)$	$u(t)$ is the unit step function .. for figuring out the $\phi$ phase angle
<b>Linearity</b>	$af(t) + bg(t)$	$aF(s) + bG(s)$	Can be proved using basic rules of integration.
<b>Differentiation</b>	$f'(t)$	$sF(s) - f(0)$	$f$ is assumed to be a differentiable function, and its derivative is assumed to be of exponential type. This can then be obtained by integration by parts
<b>Integration</b>	$\int_0^t f(\tau) d\tau = (u * f)(t)$	$\frac{1}{s}F(s)$	a constant pops out at the end of this too

---

<sup>1</sup> <http://en.wikibooks.org/wiki/%2Fstudent%20professor%2F>

<sup>2</sup> [http://en.wikibooks.org/wiki/%3Aa%3ALaplace\\_transform%23Properties\\_and\\_theorems](http://en.wikibooks.org/wiki/%3Aa%3ALaplace_transform%23Properties_and_theorems)

### 20.3.2 Laplace Function Transform

Here are some of the transforms<sup>3</sup> needed in this course:

Function	Time domain $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Laplace s-domain $F(s) = \mathcal{L}\{f(t)\}$	Region of convergence	Reference
exponential decay	$e^{-\alpha t} \cdot u(t)$	$\frac{1}{s+\alpha}$	$\text{Re}(s) > -\alpha$	Frequency shift of unit step
exponential approach	$(1 - e^{-\alpha t}) \cdot u(t)$	$\frac{\alpha}{s(s+\alpha)}$	$\text{Re}(s) > 0$	Unit step minus exponential decay
sine	$\sin(\omega t) \cdot u(t)$	$\frac{\omega}{s^2 + \omega^2}$	$\text{Re}(s) > 0$	
cosine	$\cos(\omega t) \cdot u(t)$	$\frac{s}{s^2 + \omega^2}$	$\text{Re}(s) > 0$	
exponentially decaying sine wave	$e^{-\alpha t} \sin(\omega t) \cdot u(t)$	$\frac{\omega}{(s+\alpha)^2 + \omega^2}$	$\text{Re}(s) > -\alpha$	
exponentially decaying cosine wave	$e^{-\alpha t} \cos(\omega t) \cdot u(t)$	$\frac{s+\alpha}{(s+\alpha)^2 + \omega^2}$	$\text{Re}(s) > -\alpha$	

<sup>3</sup> [http://en.wikibooks.org/wiki/%3Aw%3ALaplace\\_transform%23Table\\_of\\_selected\\_Laplace\\_transforms](http://en.wikibooks.org/wiki/%3Aw%3ALaplace_transform%23Table_of_selected_Laplace_transforms)





# 21 Phasor Circuit Analysis

## 21.1 Phasor Analysis

The mathematical representations of individual circuit elements can be converted into phasor notation, and then the circuit can be solved using phasors.

## 21.2 Resistance, Impedance and Admittance

In phasor notation, resistance, capacitance, and inductance can all be lumped together into a single term called "impedance". The phasor used for impedance is  $\mathbb{Z}$ . The inverse of Impedance is called "Admittance" and is denoted with a  $\mathbb{Y}$ .  $\mathbb{V}$  is Voltage and  $\mathbb{I}$  is current.

$$\mathbb{Z} = \frac{1}{\mathbb{Y}}$$

And the Ohm's law for phasors becomes:

$$\mathbb{V} = \mathbb{Z}\mathbb{I} = \frac{\mathbb{I}}{\mathbb{Y}}$$

It is important to note at this point that *Ohm's Law still holds true* even when we switch from the time domain to the phasor domain. This is made all the more amazing by the fact that the new term, impedance, is no longer a property only of resistors, but now encompasses all load elements on a circuit (capacitors and inductors too!).

Impedance is still measured in units of Ohms, and admittance (like Conductance, its DC-counterpart) is still measured in units of Siemens.

Let's take a closer look at this equation:

**[Ohm's Law with Phasors]**

$$\mathbb{V} = \mathbb{Z}\mathbb{I}$$

If we break this up into polar notation, we get the following result:

$$M_V \angle \phi_V = (M_Z \times M_I) \angle (\phi_Z + \phi_I)$$

w:Electrical impedance<sup>1</sup>

---

<sup>1</sup> <http://en.wikipedia.org/wiki/Electrical%20impedance>

This is important, because it shows that not only are the magnitude values of voltage and current related to each other, but also the phase angle of their respective waves are also related. Different circuit elements will have different effects on both the magnitude and the phase angle of the voltage given a certain current. We will explore those relationships below.

## 21.3 Resistors

Resistors do not affect the phase of the voltage or current, only the magnitude. Therefore, the impedance of a resistor with resistance  $R$  is:

**[Resistor Impedance]**

$$\mathbb{Z} = R\angle 0$$

Through a resistor, the phase difference between current and voltage will not change. This is important to remember when analyzing circuits.

## 21.4 Capacitors

A capacitor with a capacitance of  $C$  has a phasor value:

**[Capacitor Impedance]**

$$\mathbb{Z} = C\angle\left(-\frac{\pi}{2}\right)$$

To write this in terms of degrees, we can say:

$$\mathbb{Z} = C\angle(-90^\circ)$$

We can accept this for now as being axiomatic. If we consider the fact that phasors can be graphed on the imaginary plane, we can easily see that the angle of  $-\pi/2$  points directly downward, along the negative imaginary axis. We then come to an important conclusion: The impedance of a capacitor is *imaginary*, in a sense. Since the angle follows directly along the imaginary axis, there is no real part to the phasor at all. Because there is no real part to the impedance, we can see that capacitors have no resistance (because resistance is a real value, as stated above).

### 21.4.1 Reactance

A capacitor with a capacitance of  $C$  in an AC circuit with an angular velocity  $\omega$  has a reactance given by

$$\mathbb{X} = \frac{1}{\omega C}\angle(-90^\circ)$$

Reactance is the impedance specific to an AC circuit with angular velocity  $\omega$ .

## 21.5 Inductors

Inductors have a phasor value:

[Inductor Impedance]

$$\mathbb{Z} = L \angle \left( \frac{\pi}{2} \right)$$

Where L is the inductance of the inductor. We can also write this using degrees:

$$\mathbb{Z} = L \angle (90^\circ)$$

Like capacitors, we can see that the phasor for inductor shows that the value of the impedance is located directly on the imaginary axis. However, the phasor value for inductance points in exactly the opposite direction from the capacitance phasor. We notice here also that inductors have no resistance, because the resistance is a real value, and inductors have only an imaginary value.

### 21.5.1 Reactance

In an AC circuit with a source angular velocity of  $\omega$ , and inductor with inductance L.

$$\mathbb{X} = \omega L \angle (90^\circ)$$

## 21.6 Impedances Connected in Series

If there are several impedances connected in series, the equivalent impedance is simply a sum of the impedance values:

$$\text{----}[ Z_1 ]\text{----}[ Z_2 ]\text{---} \dots \text{---}[ Z_n ]\text{---} \implies \text{---}[ Z_{\text{series}} ]\text{---}$$

[Impedances in Series]

$$\sum_{\text{series}} \mathbb{Z}_n = \mathbb{Z}_{\text{series}}$$

Notice how much easier this is than having to differentiate between the formulas for combining capacitors, resistors, and inductors in series. Notice also that resistors, capacitors, and inductors can all be mixed without caring which type of element they are. This is valuable, because we can now combine different elements into a single impedance value, as opposed to different values of inductance, capacitance, and resistance.

Keep in mind however, that phasors need to be converted to rectangular coordinates before they can be added together. If you know the formulas, you can write a small computer

program, or even a small application on a programmable calculator to make the conversion for you.

## 21.7 Impedances in Parallel

Impedances connected in parallel can be combined in a slightly more complicated process:

**[Impedances in Parallel]**

$$Z_{parallel} = \frac{\prod_N Z_n}{\sum_N Z_n}$$

Where N is the total number of impedances connected in parallel with each other. Impedances may be multiplied in the polar representation, but they must be converted to rectangular coordinates for the summation. This calculation can be a little bit time consuming, but when you consider the alternative (having to deal with each type of element separately), we can see that this is much easier.

## 21.8 Steps For Solving a Circuit With Phasors

There are a few general steps for solving a circuit with phasors:

1. Convert all elements to phasor notation
2. Combine impedances, if possible
3. Combine Sources, if possible
4. Use Ohm's Law, and Kirchoff's laws to solve the circuit
5. Convert back into time-domain representation

Unfortunately, phasors can only be used with sinusoidal input functions. We cannot employ phasors when examining a DC circuit, nor can we employ phasors when our input function is any non-sinusoidal periodic function. To handle these cases, we will look at more general methods in later chapters

## 21.9 Network Function

The network function is a phasor,  $\mathbb{H}$  that is a ratio of the circuit's input to its output. This is important, because if we can solve a circuit down to find the network function, we can find the response to *any* sinusoidal input, by simply multiplying by the network function. With time-domain analysis, we would have to solve the circuit for every new input, and this would be very time consuming indeed.

Network functions are defined in the following way:

**[Network Function]**

$$\mathbb{H} = \frac{\mathbb{Y}}{\mathbb{X}}$$

Where  $\mathbb{Y}$  is the phasor representation of the circuit's output, and  $\mathbb{X}$  is the representation of the circuit's input. In the time domain, to find the output, we would need to convolute the input with the impulse response. With the network function, however, it becomes a simple matter of multiplying the input phasor with the network function, to get the output phasor. Using this method, we have converted an entire circuit to become a simple function that changes magnitude and phase angle.

## 21.10 Gain

Gain is the amount by which the magnitude of the sinusoid is amplified or attenuated by the circuit. Gain can be computed from the Network function as such:

[Gain]

$$Gain = |\mathbb{H}(\omega)| = \frac{|\mathbb{Y}(\omega)|}{|\mathbb{X}(\omega)|}$$

Where the bars around the phasors are the "magnitude" of the phasor, and not the "absolute value" as they are in other math texts. Again, gain may be a measure of the magnitude change in either current or voltage. Most frequently, however, it is used to describe voltage.

## 21.11 Phase Shift

The phase shift of a function is the amount of phase change between the input signal and the output signal. This can be calculated from the network function as such:

[Phase Shift]

$$\angle \mathbb{H}(\omega) = \angle \mathbb{Y}(\omega) - \angle \mathbb{X}(\omega)$$

Where the  $\angle$  denotes the phase of the phasor.

Again, the phase change may represent current or voltage.



## 22 Phasor Theorems

### 22.1 Circuit Theorems

Phasors would be absolutely useless if they didn't make the analysis of a circuit easier. Luckily for us, all our old circuit analysis tools work with values in the phasor domain. Here is a quick list of tools that we have already discussed, that continue to work with phasors:

- Ohm's Law
- Kirchhoff's Laws
- Superposition
- Thevenin and Norton Sources
- Maximum Power Transfer

This page will describe how to use some of the tools we discussed for DC circuits in an AC circuit using phasors.

### 22.2 Ohm's Law

Ohm's law, as we have already seen, becomes the following equation when in the phasor domain:

$$\mathbb{V} = \mathbb{Z}\mathbb{I}$$

Separating this out, we get:

$$M_V \angle \phi_V = (M_Z \times M_I) \angle (\phi_Z + \phi_I)$$

Where we can clearly see the magnitude and phase relationships between the current, the impedance, and the voltage phasors.

### 22.3 Kirchhoff's Laws

Kirchhoff's laws still hold true in phasors, with no alterations.

### 22.3.1 Kirchoff's Current Law

Kirchoff's current law states that the amount of current entering a particular node must equal the amount of current leaving that node. Notice that KCL never specifies what form the current must be in: any type of current works, and KCL always holds true.

[KCL With Phasors]

$$\sum_n \mathbb{I}_n = 0$$

### 22.3.2 Kirchoff's Voltage Law

KVL states: The sum of the voltages around a closed loop must always equal zero. Again, the form of the voltage forcing function is never considered: KVL holds true for any input function.

[KVL With Phasors]

$$\sum_n \mathbb{V}_n = 0$$

## 22.4 Superposition

Superposition may be applied to a circuit if all the sources have the same frequency. However, superposition *must* be used as the only possible method to solve a circuit with sources that have different frequencies. The important part to remember is that impedance values in a circuit are based on the frequency. Different reactive elements react to different frequencies differently. Therefore, the circuit must be solved once for every source frequency. This can be a long process, but it is the only good method to solve these circuits.

## 22.5 Thevenin and Norton Circuits

Thevenin Circuits and Norton Circuits can be manipulated in a similar manner to their DC counterparts: Using the phasor-domain implementation of Ohm's Law.

$$\mathbb{V} = \mathbb{Z}\mathbb{I}$$

It is important to remember that the  $\mathbb{Z}$  does not change in the calculations, although the phase and the magnitude of both the current and the voltage sources might change as a result of the calculation.



## 22.6 Maximum Power Transfer

The maximum power transfer theorem in phasors is slightly different than the theorem for DC circuits. To obtain maximum power transfer from a thevenin source to a load, the internal thevenin impedance ( $Z_t$ ) must be the complex conjugate of the load impedance ( $Z_l$ ):

**[Maximum Power Transfer, with Phasors]**

$$Z_l = R_t - jX_t$$



## 23 Complex Power

### 23.1 Laplace Transform

The **Laplace Transform** is a powerful tool that is very useful in Electrical Engineering. The transform allows equations in the "time domain" to be transformed into an equivalent equation in the **Complex S Domain**. The laplace transform is an integral transform, although the reader does not need to have a knowledge of integral calculus because all results will be provided. This page will discuss the Laplace transform as being simply a tool for solving and manipulating ordinary differential equations.

Laplace transformations of circuit elements are similar to phasor representations, but they are not the same. Laplace transformations are more general than phasors, and can be easier to use in some instances. Also, do not confuse the term "Complex S Domain" with the complex power ideas that we have been talking about earlier. Complex power uses the variable  $\mathbb{S}$ , while the Laplace transform uses the variable  $s$ . The Laplace variable  $s$  has nothing to do with power.

The transform is named after the mathematician **Pierre Simon Laplace** (1749-1827). The transform itself did not become popular until Oliver Heaviside, a famous electrical engineer, began using a variation of it to solve electrical circuits.

### 23.2 Laplace Domain

The **Laplace domain**, or the "Complex  $s$  Domain" is the domain into which the Laplace transform transforms a time-domain equation.  $s$  is a complex variable, composed of real and imaginary parts:

$$s = \sigma + j\omega$$

The Laplace domain graphs the real part ( $\sigma$ ) as the horizontal axis, and the imaginary part ( $\omega$ ) as the vertical axis. The real and imaginary parts of  $s$  can be considered as independent quantities.

The similarity of this notation with the notation used in Fourier transform theory is no coincidence; for  $\sigma = 0$ , the Laplace transform is the same as the Fourier transform if the signal is causal.

## 23.3 The Transform

The mathematical definition of the Laplace transform is as follows:

[The Laplace Transform]

$$F(s) = \mathcal{L}\{f(t)\} = \int_{0^-}^{\infty} e^{-st} f(t) dt$$

**Note:**

The letter **s** has no special significance, and is used with the Laplace Transform as a matter of common convention.

The transform, by virtue of the definite integral, removes all **t** from the resulting equation, leaving instead the new variable **s**, a complex number that is normally written as  $s = \sigma + j\omega$ . In essence, this transform takes the function  $f(t)$ , and "transforms it" into a function in terms of **s**,  $F(s)$ . As a general rule the transform of a function  $f(t)$  is written as  $F(s)$ . Time-domain functions are written in lower-case, and the resultant **s**-domain functions are written in upper-case.

There is a table of Laplace Transform pairs in  
**the Appendix<sup>a</sup>**

---

<sup>a</sup> Chapter 27 on page 133

we will use the following notation to show the transform of a function:

$$f(t) \Leftrightarrow F(s)$$

We use this notation, because we can convert  $F(s)$  back into  $f(t)$  using the **inverse Laplace transform**.

## 23.4 The Inverse Transform

The **inverse laplace transform** converts a function in the complex **S**-domain to its counterpart in the time-domain. Its mathematical definition is as follows:

[Inverse Laplace Transform]

$$\mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi} \int_{c-i\infty}^{c+i\infty} e^{ft} F(s) ds = f(t)$$

where  $c$  is a real constant such that all of the poles  $s_1, s_2, \dots, s_n$  of  $F(s)$  fall in the region  $\Re\{s_i\} < c$ . In other words,  $c$  is chosen so that all of the poles of  $F(s)$  are to the left of the vertical line intersecting the real axis at  $s = c$ .

The inverse transform is more difficult mathematically than the transform itself is. However, luckily for us, extensive tables of laplace transforms and their inverses have been computed, and are available for easy browsing.

## 23.5 Transform Properties

There is a table of Laplace Transform properties in  
**The Appendix<sup>a</sup>**

<sup>a</sup> Chapter 27 on page 133

The most important property of the Laplace Transform (for now) is as follows:

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$

Likewise, we can express higher-order derivatives in a similar manner:

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$

Or for an arbitrary derivative:

$$\mathcal{L}\{f^{(n)}(t)\} = s^nF(s) - \sum_{i=0}^{n-1} s^{(n-1-i)} f^{(i)}(0)$$

where the notation  $f^{(n)}(t)$  means the  $n^{\text{th}}$  derivative of the function  $f$  at the point  $t$ , and  $f^{(0)}(t)$  means  $f(t)$ .

In plain English, the laplace transform converts differentiation into polynomials. The only important thing to remember is that we must add in the initial conditions of the time domain function, but for most circuits, the initial condition is 0, leaving us with nothing to add.

For integrals, we get the following:

$$\mathcal{L}\left\{\int_0^t f(t) dt\right\} = \frac{1}{s}F(s)$$

## 23.6 Initial Value Theorem

The **Initial Value Theorem** of the laplace transform states as follows:

**[Initial Value Theorem]**

$$f(0) \Leftrightarrow \lim_{s \rightarrow \infty} sF(s)$$

This is useful for finding the initial conditions of a function needed when we perform the transform of a differentiation operation (see above).

### 23.7 Final Value Theorem

Similar to the Initial Value Theorem, the **Final Value Theorem** states that we can find the value of a function  $f$ , as  $t$  approaches infinity, in the laplace domain, as such:

[**Final Value Theorem**]

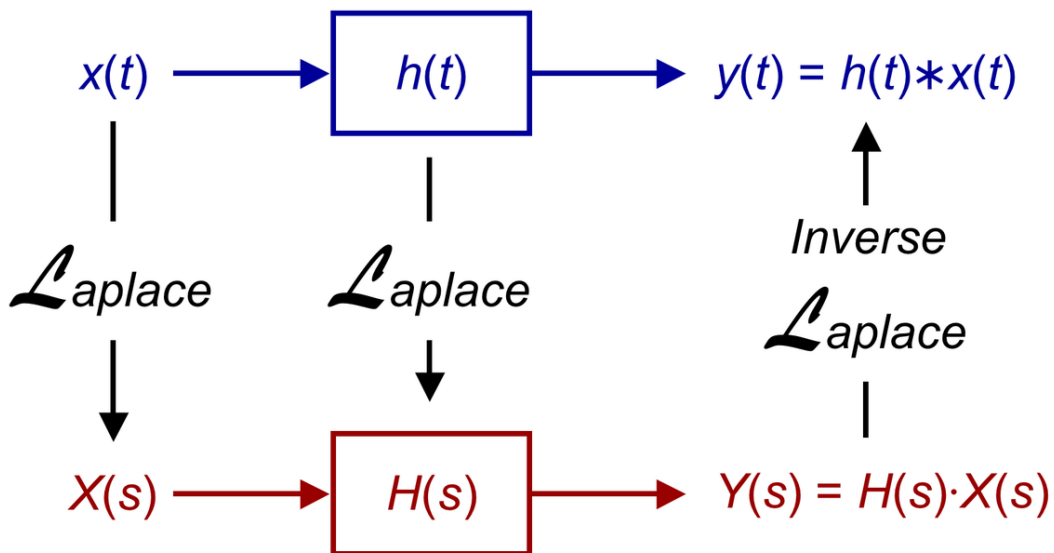
$$\lim_{t \rightarrow \infty} f(t) \Leftrightarrow \lim_{s \rightarrow 0} sF(s)$$

This is useful for finding the steady state response of a circuit. The final value theorem may only be applied to stable systems.

### 23.8 Transfer Function

If we have a circuit with impulse-response  $h(t)$  in the time domain, with input  $x(t)$  and output  $y(t)$ , we can find the **Transfer Function** of the circuit, in the laplace domain, by transforming all three elements:

Time domain



Frequency domain

Figure 32

In this situation,  $H(s)$  is known as the "Transfer Function" of the circuit. It can be defined as both the transform of the impulse response, or the ratio of the circuit output to its input in the Laplace domain:

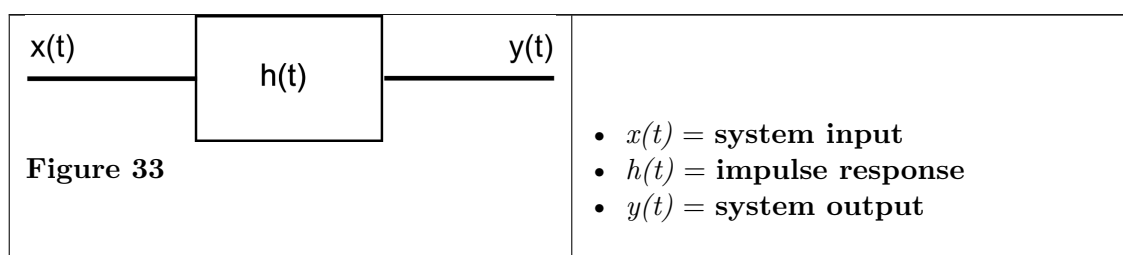
[Transfer Function]

$$H(s) = \mathcal{L}\{h(t)\} = \frac{Y(s)}{X(s)}$$

Transfer functions are powerful tools for analyzing circuits. If we know the transfer function of a circuit, we have all the information we need to understand the circuit, and we have it in a form that is easy to work with. When we have obtained the transfer function, we can say that the circuit has been "solved" completely.

## 23.9 Convolution Theorem

Earlier it was mentioned that we could compute the output of a system from the input and the impulse response by using the convolution operation. As a reminder, given the following system:



We can calculate the output using the convolution operation, as such:

$$y(t) = x(t) * h(t)$$

Where the asterisk denotes convolution, not multiplication. However, in the S domain, this operation becomes much easier, because of a property of the laplace transform:

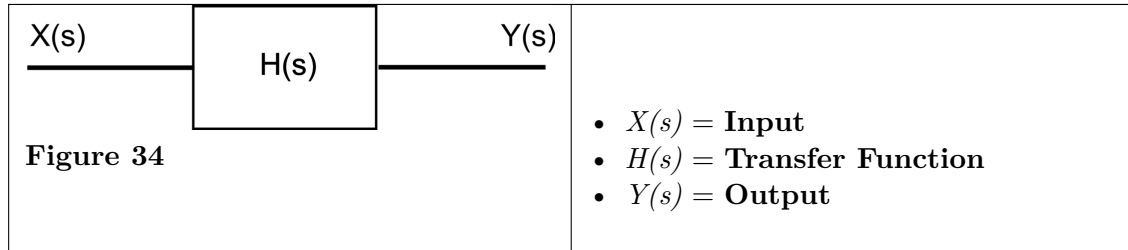
[Convolution Theorem]

$$\mathcal{L}\{a(t) * b(t)\} = A(s)B(s)$$

Where the asterisk operator denotes the convolution operation. This leads us to an English statement of the convolution theorem:

Convolution in the time domain becomes multiplication in the S domain, and convolution in the S domain becomes multiplication in the time domain.<sup>1</sup>

Now, if we have a system in the Laplace S domain:



We can compute the output  $Y(s)$  from the input  $X(s)$  and the Transfer Function  $H(s)$ :

$$Y(s) = X(s)H(s)$$

Notice that this property is very similar to phasors, where the output can be determined by multiplying the input by the network function. The network function and the transfer function then, are very similar quantities.

## 23.10 Resistors

The laplace transform can be used independently on different circuit elements, and then the circuit can be solved entirely in the S Domain (Which is much easier). Let's take a look at some of the circuit elements:

Resistors are time and frequency invariant. Therefore, the transform of a resistor is the same as the resistance of the resistor:

**[Transform of Resistors]**

$$R(s) = r$$

Compare this result to the phasor impedance value for a resistance  $r$ :

$$Z_r = r \angle 0$$

You can see very quickly that resistance values are very similar between phasors and laplace transforms.

## 23.11 Ohm's Law

If we transform Ohm's law, we get the following equation:

**[Transform of Ohm's Law]**

$$V(s) = I(s)R$$



Now, following ohms law, the resistance of the circuit element is a ratio of the voltage to the current. So, we will solve for the quantity  $\frac{V(s)}{I(s)}$ , and the result will be the resistance of our circuit element:

$$R = \frac{V(s)}{I(s)}$$

This ratio, the input/output ratio of our resistor is an important quantity, and we will find this quantity for all of our circuit elements. We can say that the transform of a resistor with resistance  $r$  is given by:

**[Transform of Resistor]**

$$\mathcal{L}\{resistor\} = R = r$$

## 23.12 Capacitors

Let us look at the relationship between voltage, current, and capacitance, in the time domain:

$$i(t) = C \frac{dv(t)}{dt}$$

Solving for voltage, we get the following integral:

$$v(t) = \frac{1}{C} \int_{t_0}^{\infty} i(t) dt$$

Then, transforming this equation into the laplace domain, we get the following:

$$V(s) = \frac{1}{C} \frac{1}{s} I(s)$$

Again, if we solve for the ratio  $\frac{V(s)}{I(s)}$ , we get the following:

$$\frac{V(s)}{I(s)} = \frac{1}{sC}$$

Therefore, the transform for a capacitor with capacitance  $C$  is given by:

**[Transform of Capacitor]**

$$\mathcal{L}\{capacitor\} = \frac{1}{sC}$$

## 23.13 Inductors

Let us look at our equation for inductance:

$$v(t) = L \frac{di(t)}{dt}$$

putting this into the laplace domain, we get the formula:

$$V(s) = sLI(s)$$

And solving for our ratio  $\frac{V(s)}{I(s)}$ , we get the following:

$$\frac{V(s)}{I(s)} = sL$$

Therefore, the transform of an inductor with inductance  $L$  is given by:

[Transform of Inductor]

$$\mathcal{L}\{Inductor\} = sL$$

## 23.14 Impedance

Since all the load elements can be combined into a single format dependent on  $s$ , we call the effect of all load elements **impedance**, the same as we call it in phasor representation. We denote impedance values with a capital  $Z$  (but not a phasor  $\mathbb{Z}$ ).

## 23.15 References

## 23.16 Laplace Circuit Solution

One of the most important uses of the Laplace transform is to solve linear differential equations, just like the type of equations that represent our first- and second-order circuits. This page will discuss the use of the Laplace Transform to find the complete response of a circuit.

## 23.17 Steps

Here are the general steps for solving a circuit using the Laplace Transform:

1. Determine the differential equation for the circuit.
2. Use the Laplace Transform on the differential equation.

3. Solve for the unknown variable in the laplace domain.
4. Use the inverse laplace transform to find the time domain solution.

Another method that we can use is:

1. Transform the individual circuit components into impedance values using the Laplace Transform.
2. Find the Transfer function that describes the circuit
3. Solve for the unknown variable in the laplace domain.
4. Use the inverse laplace transform to find the time domain solution.



**Figure 35** Joseph Fourier<sup>a</sup>, after whom the Fourier Transform is named, was a famous mathematician who worked for Napoleon.

<sup>a</sup> <http://en.wikipedia.org/wiki/Joseph%20Fourier>

## 23.18 Fourier Transform

The **Fourier Transform** is a specific case of the Laplace transform. If we separate  $s$  into its real and imaginary parts:

$$s = \sigma + j\omega$$

Where  $s$  is the complex laplace variable,  $\sigma$  is the real part of  $s$ , and  $\omega$  is the imaginary part of  $s$ . Remember, in Electrical Engineering,  $j$  is the imaginary number, not  $i$ .

Now, if we set  $\sigma \rightarrow 0$ , we can get the following:

$$s = j\omega$$

Plugging into the Laplace transform, we get the following formula:

[Fourier Transform]

$$F(j\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt$$

The variable  $\omega$  is known as the "radial frequency" of the circuit. This term refers to the frequency of the circuit. The Fourier transform, in the respect that it accounts only for the response of the circuit to a given frequency is very similar to phasor notation. However the Fourier Transform produces an equation that can be used to analyze the circuit *for all frequencies*, not just a single frequency like phasors are limited to.

There is a table of Fourier Transform pairs in  
**the Appendix<sup>a</sup>**

---

<sup>a</sup> Chapter 27 on page 133

As with the Laplace transform, there is an **inverse Fourier transform**:

[Inverse Fourier Transform]

$$\mathcal{F}^{-1}\{F(j\omega)\} = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega)e^{j\omega t} d\omega$$

However, there are extensive tables of Fourier transforms and their inverses available, so we need not waste time computing individual transforms.

## 23.19 Radial Frequency

In the Fourier transform, the value  $\omega$  is known as the **Radial Frequency**, and has units of radians/second (rad/s). People might be more familiar with the variable  $f$ , which is called the "Frequency", and is measured in units called Hertz (Hz). The conversion is done as such:

**[Radial Frequency]**

$$\omega = 2\pi f$$

**Radial Frequency** is measured in radians, **frequency** is measured in hertz. Both describe the same quantity.

For instance, if a given AC source has a frequency of 60Hz, the resultant radial frequency is:

$$\omega = 2\pi f = 2\pi(60) = 120\pi$$

## 23.20 Fourier Domain

The Laplace transform converts functions from the time domain to the complex  $s$  domain.  $s$  has real and imaginary parts, and these parts form the axes of the  $s$  domain: the real part is the horizontal axis, and the imaginary part is the vertical axis. However, in the Fourier transform, we have the relationship:

$$s \rightarrow j\omega$$

And therefore we don't have a real part of  $s$ . The Fourier domain then is broken up into two distinct parts: the **magnitude graph**, and the **phase graph**. The magnitude graph has  $j\omega$  as the horizontal axis, and the magnitude of the transform as the vertical axis. Remember, we can compute the magnitude of a complex value  $C$  as:

$$C = A + jB$$

$$|C| = \sqrt{A^2 + B^2}$$

The Phase graph has  $j\omega$  as the horizontal axis, and the phase value of the transform as the vertical axis. Remember, we can compute the phase of a complex value as such:

$$C = A + jB$$

$$\angle C = \tan^{-1} \left( \frac{B}{A} \right)$$

The phase and magnitude values of the Fourier transform can be considered independent values, although some abstract relationships do apply. Every fourier transform must include a phase value and a magnitude value, or it cannot be uniquely transformed back into the time domain.

The combination of graphs of the magnitude and phase responses of a circuit, along with some special types of formatting and interpretation are called Bode Plots<sup>2</sup>, and are discussed in more detail in the next chapter.

## 23.21 Impedance and Reactance

In the Fourier domain, the concepts of capacitance, inductance, and resistance can be generalized into a single complex term called "Impedance." Impedance in this sense is exactly the same as the impedance quantities from the Laplace domain and the phasor domain. In the fourier domain however, the impedance of a circuit element is defined in terms of the voltage frequency across that element, as such:

**Remember:**

Reactance is a combination of inductance and capacitance.

$$Z(j\omega) = R(j\omega) + jX(j\omega)$$

Where R is the fourier transform of resistance, and X is the transform of reactance, that we discussed earlier.

## 23.22 Frequency Domain Analysis

Individual circuit elements can be transformed into the Fourier frequency domain according to a few simple rules. These transformed circuit elements can then be used to find the Frequency Response of the circuit.

### 23.22.1 Resistors

Resistors are not reactive elements, and their resistance is not a function of time. Therefore, when transformed, the fourier impedance value of a resistor is given as such:

[Transform of Resistor]

$$Z_{resistor}(j\omega) = r$$

Resistors act equally on all frequencies of input.

---

<sup>2</sup> Chapter 23.23 on page 113

### 23.22.2 Capacitors

Capacitors are reactive elements, and therefore they have reactance, but no resistance, as such:

[Transform of Capacitor]

$$Z_{\text{capacitor}}(j\omega) = \frac{1}{j\omega C} = \frac{-j}{\omega C}$$

### 23.22.3 Inductors

Inductors are also reactive elements, and have the following fourier transform:

[Transform of Inductor]

$$Z_{\text{inductor}}(j\omega) = j\omega L$$

### 23.22.4 Current and Voltage Sources

The frequency representation of a source is simply the transform of that source's input function.

## 23.23 Frequency Response

If we set  $s \rightarrow j\omega$ , and plug this value into our transfer function:

[Frequency Response]

$$H(s)|_{s \rightarrow j\omega} = H(j\omega)$$

The function  $H(j\omega)$  is called the "Frequency Response". The frequency response can be used to find the output of a circuit from the input, in exactly the same way that the Transfer function can be:

$$Y(j\omega) = X(j\omega)H(j\omega)$$

In addition, the Convolution Theorem holds for the  $\omega$  domain the same way as it works for the  $S$  domain:

Convolution in the time domain is multiplication in the frequency domain. Multiplication in the time domain is Convolution in the Frequency domain.

## 23.24 Decibel

More information about **Decibels** can be found in:  
**The Appendix<sup>a</sup>**

---

<sup>a</sup> Chapter 26 on page 129

Decibels (abbreviated "dB") are not units per se. Instead, a Decibel is simply a logarithmic ratio of the input to the output of a circuit. Decibels represent a power ratio unless they are clearly tagged differently. Most db tags are NOT indicative of the ratio units, but rather, they indicate a comparison of a variable with an accepted standard. For example, dbm means 10 times the log of a power variable in ratio to 1 milliwatt. This is very popular in dealing with telephone lines. Decibels were invented by power weenies and belong to them, so it's power unless someone indicates otherwise.

### 23.24.1 Voltage

[Decibel relation]

$$dB = 20 \log \frac{V_{out}}{V_{in}}$$

This is the power ratio ( out / in ) that exists for a voltage ratio of V\_(out) to V\_(in). Power goes as the square of the voltage, therefore the 10 log becomes 20 log. Bels, the precursors to decibels, were invented to describe power ratios; therefore, it is customary to describe a voltage ratio in terms of the power ratio that voltage ratio represents. This is used even when the input and output impedance differ. This makes little sense, but it is common practice. This "voltage-ratio-yields-power-ratio" comparison only really works when the impedance is the same for In and Out. This is true in such cases as telephone and video systems. The terms "Vin" and "Vout" can either be the time-domain values of the voltage input and voltage output, or they can be the magnitudes of the respective phasors. The original definition was bels = log (Power-out / Power-in), where log is the common (base 10) log, but the bel is a very large unit, hence the birth of the decibel, one-tenth of a bel. You can take a log ratio of any two quantities, but they must be the same unit. Oranges over oranges = unit-less.

### 23.24.2 Power Gain

$$dB = 10 \log \frac{P_{out}}{P_{in}}$$

This is used to compare the output power of a system to the input power of a system. Notice that since power is being compared to power, the log prefix is 10. For voltage comparisons the prefix was 20 (due to power being proportional to voltage squared).



## 23.25 Notes on Decibels

### Remember:

**"Decibels are not numbers, they are ratios"**

Decibels are plotted on a "log10" graph, where each hash-mark on the axis is a successive power of 10. Also, the values on the X-axis are plotted on a log-10 scale as well. Each successive power-of-10 on the frequency axis is known as a "decade". It is important to mention that decibels are simply a convenient way to represent a scaling factor, and that *decibels are not numbers: they are ratios*. Any quantity ratio can be expressed in decibels. The decibel is really just a common (base ten) log expressed in 0.1 log points; a log with enhanced resolution. It is customary to use a letter to warn the reader that a non-typical quantity is being compared by the log of a ratio. For voltage, it is usually dbv. This is a voltage comparison, NOT a power-due-to-the-voltage comparison. So 100 volts out for 1 volt in is 20 dbv. Impedance is not relevant for dbv. But this usage is fraught with peril, because dbv is also used to indicate that a variable is being ratioed to 1 volt. Decibels has been converted to an absolute. Beware and make yourself clear. Early telephony usage converted decibels to an absolute power level by setting the denominator to a fixed agreed power value: 6 milliwatts at 500 or 600 ohms impedance. The ratio of a variable to this fixed quantity was called dbm. Modern practice is to use 1 milliwatt as the reference for dbm.

## 23.26 Bode Plots

Bode plots can be broken down into 2 separate graphs: the magnitude graph, and the phase graph. Both graphs represent the circuit response in each category to sinusoids of different frequencies.

### 23.26.1 Magnitude Graph

The Bode Magnitude Graph is a graph where the radial frequency is plotted along the X-axis, and the gain of the circuit *at that frequency* is plotted (in Decibels) on the Y-axis. The bode magnitude graph most frequently plots the power gain against the frequency, although they may also be used to graph the voltage gain against the frequency. Also, the frequency axis may be in terms of hertz or radians, so the person drawing a bode plot should make sure to label their axes correctly.

### 23.26.2 Phase Graph

The Bode Phase Plot is a graph where the radial frequency is plotted along the X axis, and phase shift of the circuit *at that frequency* is plotted on the Y-axis. The phase change is almost always represented in terms of radians, although it is not unheard of to express them in terms of degrees. Likewise, the frequency axis may be in units of hertz or radians per second, so the axes need to be labeled correctly.

## 23.27 Bode Plots by Different Methods

Bode plots can be used both with Phasors (Network Functions), and with the Fourier Transform (Frequency Response). However, there are slightly different methods to doing it each way, and those methods will be examined in the following chapters. The Laplace Transform can be used to construct a bode plot by transposing from the s-domain to the fourier-domain. However, this is rarely done in practice and the Laplace Transform is instead used with other graphical methods that are unfortunately, outside the scope of this wikibook.

## 23.28 Bode Analysis

The pages in this section will talk about how to analyze a bode plot of a given circuit, and draw conclusions from that plot.

## 23.29 Log Magnitude Graph

Using a network function (remember phasors?), we can find the log magnitude and the phase bode plots of a circuit. This page will discuss how to find the bode plot from the network function of a circuit.

## 23.30 Phase Graph

## 23.31 Poles and Zeros

The topic of "Poles and Zeros" are discussed in excruciating detail in advanced texts in Electrical Engineering. We will introduce the concepts of what a pole and a zero are in this chapter.

### 23.31.1 Transfer Polynomials

Let's say that we have a given frequency response:

$$H(j\omega) = \frac{Z(j\omega)}{P(j\omega)}$$

Where both Z and P are polynomials. We then set each of these equations to zero, and solve:

$$Z(j\omega) = 0$$

$$P(j\omega) = 0$$

The solutions to the equation  $Z = 0$  are called the "Zeros" of  $H$ . The solutions to the equation  $P = 0$  are called the "Poles" of  $H$ .

### 23.31.2 Properties of poles and zeros

Let's say that we have a frequency response that has a zero at  $N$ , and a pole at  $M$ . We then plug in these values to our frequency response:

$$H(j\omega)|_{\omega \rightarrow N} = \frac{Z(j\omega)}{P(j\omega)} = \frac{0}{P(j\omega)} = 0$$

And:

$$H(j\omega)|_{\omega \rightarrow M} = \frac{Z(j\omega)}{P(j\omega)} = \frac{Z(j\omega)}{0} = \infty$$

Now, some of the purists will immediately say "but you aren't allowed to divide by zero", and to those people I say: you can write in a limit, if you really want to.

## 23.32 Bode Equation Format

let us say that we have a generic transfer function with poles and zeros:

$$H(j\omega) = \frac{(\omega_A + j\omega)(\omega_B + j\omega)}{(\omega_C + j\omega)(\omega_D + j\omega)}$$

Each term, on top and bottom of the equation, is of the form  $(\omega_N + j\omega)$ . However, we can rearrange our numbers to look like the following:

$$\omega_N \left(1 + \frac{j\omega}{\omega_N}\right)$$

Now, if we do this for every term in the equation, we get the following:

$$H_{bode}(j\omega) = \frac{\omega_A \omega_B}{\omega_C \omega_D} \frac{\left(1 + \frac{j\omega}{\omega_A}\right) \left(1 + \frac{j\omega}{\omega_B}\right)}{\left(1 + \frac{j\omega}{\omega_C}\right) \left(1 + \frac{j\omega}{\omega_D}\right)}$$

This is the format that we are calling "Bode Equations", although they are simply another way of writing an ordinary frequency response equation.

## 23.33 DC Gain

The constant term out front:

$$\frac{\omega_A \omega_B}{\omega_C \omega_D}$$

is called the "DC Gain" of the function. If we set  $\omega \rightarrow 0$ , we can see that everything in the equation cancels out, and the value of H is simply our DC gain. DC then is simply the input with a frequency of zero.

## 23.34 Break Frequencies

in each term:

$$(1 + \frac{j\omega}{\omega_N})$$

the quantity  $\omega_N$  is called the "Break Frequency". When the radial frequency of the circuit equals a break frequency, that term becomes  $(1 + 1) = 2$ . When the radial frequency is much higher than the break frequency, the term becomes much greater than 1. When the radial Frequency is much smaller than the break frequency, the value of that term becomes approximately 1.

### 23.34.1 Much Greater and Much Less

We use the term "much" as a synonym for the term "At least 10 times". So "Much Greater" becomes "At least 10 times greater" and "Much less" becomes "At least 10 times less". We also use the symbol "<<" to mean "is much less than" and ">>" to mean "Is much greater than". Here are some examples:

- $1 \ll 10$
- $10 \ll 1000$
- $2 \ll 20$  Right!
- $2 \ll 10$  WRONG!

### 23.34.2 Engineering Approximations

For a number of reasons, Electrical Engineers find it appropriate to approximate and round some values very heavily. For instance, manufacturing technology will never create electrical circuits that perfectly conform to mathematical calculations. When we combine this with the << and >> operators, we can come to some important conclusions that help us to simplify our work:

If  $A \ll B$ :

- $A + B = B$

- $A - B = -B$
- $A / B = 0$

All other mathematical operations need to be performed, but these 3 forms can be approximated away. This point will come important for later work on bode plots.

Using our knowledge of the Bode Equation form, the DC gain value, Decibels, and the "much greater, much less" inequalities, we can come up with a fast way to approximate a bode magnitude plot. Also, it is important to remember that these gain values are not constants, but rely instead on changing frequency values. Therefore, the gains that we find are all *slopes* of the bode plot. Our slope values all have units of "decibel per decade", or "db/decade", for short.

### 23.35 At Zero Radial Frequency

At zero radial frequency, the value of the bode plot is simply the DC gain value *in decibels*. Remember, bode plots have a log-10 magnitude Y-axis, so we need to convert our gain to decibels:

$$Magnitude = 20\log_{10}(DCGain)$$

### 23.36 At a Break Point

We can notice that each given term changes it's effect as the radial frequency goes from below the break point, to above the break point. Let's show an example:

$$(1 + \frac{j\omega}{5})$$

Our breakpoint occurs at 5 radians per second. When our radial frequency is *much less* than the break point, we have the following:

$$Gain = (1 + 0) = 1$$

$$Magnitude = 20\log_{10}(1) = 0db/decade$$

When our radial frequency is equal to our break point we have the following:

$$Gain = |(1 + j)| = \sqrt{2}$$

$$Magnitude = 20\log_{10}(\sqrt{2}) = 3db/decade$$

And when our radial frequency is much higher (10 times) our break point we get:

$$Gain = |(1 + 10j)| \approx 10$$

$$Magnitude = 20\log_{10}(10) = 20\text{db/decade}$$

However, we need to remember that some of our terms are "Poles" and some of them are "Zeros".

### 23.36.1 Zeros

Zeros have a positive effect on the magnitude plot. The contributions of a zero are all positive:

#### **Radial Frequency << Break Point**

0db/decade gain.

#### **Radial Frequency = Break Point**

3db/decade gain.

#### **Radial Frequency >> Break Point**

20db/decade gain.

### 23.36.2 Poles

Poles have a negative effect on the magnitude plot. The contributions of the poles are as follows:

#### **Radial Frequency << Break Point**

0db/decade gain.

#### **Radial Frequency = Break Point**

-3db/decade gain.

#### **Radial Frequency >> Break Point**

-20db/decade gain.

## 23.37 Conclusions

To draw a bode plot effectively, follow these simple steps:

1. Put the frequency response equation into bode equation form.
2. identify the DC gain value, and mark this as a horizontal line coming in from the far left (where the radial frequency conceptually is zero).
3. At every "zero" break point, increase the slope of the line upwards by 20db/decade.
4. At every "pole" break point, decrease the slope of the line downwards by 20db/decade.

5. at every breakpoint, note that the "actual value" is 3db off from the value graphed.  
And then you are done!

## 23.38 See Also

- Bode Plots on ControlTheoryPro.com<sup>3</sup>

## 23.39 Impedance

Let's recap: In the transform domain, the quantities of resistance, capacitance, and inductance can all be combined into a single complex value known as "Impedance". Impedance is denoted with the letter  $Z$ , and can be a function of  $s$  or  $j\omega$ , depending on the transform used (Laplace or Fourier). This impedance is very similar to the phasor concept of impedance, except that we are in the complex domain (laplace or fourier), and not the phasor domain.

Impedance is a complex quantity, and is therefore comprised of two components: The real component (resistance), and the complex component (reactance). Resistors, because they do not vary with time or frequency, have real values. Capacitors and inductors however, have imaginary values of impedance. The resistance is denoted (as always) with a capital  $R$ , and the reactance is denoted with an  $X$  (this is common, although it is confusing because  $X$  is also the most common input designator). We have therefore, the following relationship between resistance, reactance, and impedance:

[Complex Laplace Impedance]

$$Z = R + jX$$

## 23.40 Susceptance and Admittance

The inverse of resistance is a quantity called "Conductance". Similarly, the inverse of reactance is called "Susceptance". The inverse of impedance is called "Admittance". Conductance, Susceptance, and Admittance are all denoted by the variables  $Y$  or  $G$ , and are given the units **Siemens**. This book will not use any of these terms again, and they are just included here for completeness.

## 23.41 Parallel Components

Once in the transform domain, all circuit components act like basic resistors. Components in parallel are related as follows:

---

3 [http://wikis.controltheorypro.com/index.php?title=Bode\\_Plot](http://wikis.controltheorypro.com/index.php?title=Bode_Plot)

$$Z_1 || Z_2 = \frac{Z_1 Z_2}{Z_1 + Z_2}$$

### 23.42 Series Components

Series components in the transform domain all act like resistors in the time domain as well. If we have two impedances in series with each other, we can combine them as follows:

$$Z_1 \text{ in series with } Z_2 = Z_1 + Z_2$$

### 23.43 Solving Circuits



## 24 Circuit Functions

### 24.1 Circuit Functions

This appendix page will list the various values of the variable  $H$  that have been used throughout the circuit theory textbooks. These values of  $H$  are all equivalent, but are represented in different domains. All of the  $H$  functions are a ratio of the circuit input over the circuit output.

### 24.2 The "Impulse Response"

The **impulse response** is the time-domain relationship between the circuit input and the circuit output, denoted with the following notation:

$$h(t)$$

The impulse response is, strictly speaking, the output that the circuit will produce when an ideal impulse function is the input. The impulse response can be used to determine the output from the input through the convolution operation:

$$y(t) = h(t) * x(t)$$

### 24.3 The "Network Function"

The **network function** is the phasor-domain representation of the impulse response. The network function is denoted as such:

$$\mathbb{H}(\omega)$$

The network function is related to the input and output of the circuit through the following relationships:

$$\mathbb{Y}(\omega) = \mathbb{H}(\omega)\mathbb{X}(\omega)$$

Similarly, the network function can be received by dividing the output by the input, in the phasor domain.

## 24.4 The "Transfer Function"

The **transfer function** is the laplace-transformed representation of the impulse response. It is denoted with the following notation:

$$H(s)$$

The transfer function can be obtained by one of two methods:

1. Transform the impulse response.
2. Transform the circuit, and solve.

The Transfer function is related to the input and output as follows:

$$Y(s) = H(s)X(s)$$

## 24.5 The "Frequency Response"

The **Frequency Response** is the fourier-domain representation of the impulse response. It is denoted as such:

$$H(j\omega)$$

The frequency response can be obtained in one of three ways:

1. Transform the impulse response
2. Transform the circuit and solve
3. Substitute  $s = j\omega$  into the transfer function

The frequency response has the following relationship to the circuit input and output:

$$Y(j\omega) = H(j\omega)X(j\omega)$$

The frequency response is particularly useful when discussing a sinusoidal input, or when constructing a bode diagram.

# 25 Phasor Arithmetic

## 25.1 Phasor Arithmetic

This page will review phasors and phasor arithmetic topics.

## 25.2 Forms

Phasors have two components, the magnitude (M) and the phase angle ( $\phi$ ). Phasors are related to sinusoids through our cosine convention:

$$\mathbb{C} = M\angle\phi = M\cos(t\omega + \phi)$$

Remember, there are **3 forms to phasors**:

- $\mathbb{C} = M\angle\phi$  "Polar Form"
- $\mathbb{C} = A + jB$  "Rectangular Form"
- $\mathbb{C} = Me^{j\phi}$  "Exponential Form"

Phasor and Exponential forms are identical and are also referred to as polar form.

## 25.3 Converting between Forms

When working with phasors it is often necessary to convert between rectangular and polar form. To convert from rectangular form to polar form:

$$M = \sqrt{A^2 + B^2}$$

$$\phi = \arctan\left(\frac{B}{A}\right)$$

To convert from polar to rectangular form:

A is the part of the phasor along the real axis

$$A = M\cos(\phi)$$

B is the part of the phasor along the imaginary axis

$$B = M \sin(\phi)$$

## 25.4 Addition

To add two phasors together, we must convert them into rectangular form:

$$\mathbb{C}_1 = A_1 + jB_1$$

$$\mathbb{C}_2 = A_2 + jB_2$$

$$\mathbb{C}_1 + \mathbb{C}_2 = (A_1 + A_2) + j(B_1 + B_2)$$

This is a well-known property of complex arithmetic.

## 25.5 Subtraction

Subtraction is similar to addition, except now we subtract

$$\mathbb{C}_1 = A_1 + jB_1$$

$$\mathbb{C}_2 = A_2 + jB_2$$

$$\mathbb{C}_1 - \mathbb{C}_2 = (A_1 - A_2) + j(B_1 - B_2)$$

## 25.6 Multiplication

To multiply two phasors, we should first convert them to polar form to make things simpler. The product in polar form is simply the product of their magnitudes, and the phase is the sum of their phases.

$$\mathbb{C}_1 = M_1 \angle \phi_1$$

$$\mathbb{C}_2 = M_2 \angle \phi_2$$

$$\mathbb{C}_1 \times \mathbb{C}_2 = M_1 \times M_2 \angle \phi_1 + \phi_2$$

Keep in mind that in polar form, phasors are exponential quantities with a magnitude ( $M$ ), and an argument ( $\varphi$ ). Multiplying two exponentials together forces us to multiply the magnitudes, and add the exponents.

## 25.7 Division

Division is similar to multiplication, except now we divide the magnitudes, and subtract the phases

$$\mathbb{C}_1 = M_1 \angle \phi_1$$

$$\mathbb{C}_2 = M_2 \angle \phi_2$$

$$\frac{\mathbb{C}_1}{\mathbb{C}_2} = \frac{M_1}{M_2} \angle \phi_1 - \phi_2$$

## 25.8 Inversion

An important relationship that is worth understanding is the **inversion property** of phasors:

$$\mathbb{C} = M_C \angle 0 = -M_C \angle \pi$$

Or, in degrees,

$$\mathbb{C} = M_C \angle 0^\circ = -M_C \angle 180^\circ$$

On the normal cartesian plane, for instance, the negative X axis is 180 degrees around from the positive X axis. By using that fact on an imaginary axis, we can see that the Negative Real axis is facing in the exact opposite direction from the Positive Real axis, and therefore is 180 degrees apart.

## 25.9 Complex Conjugation

Similar to the inversion property is the **complex conjugation property** of phasors. Complex conjugation is denoted with an asterisk above the phasor to be conjugated. Since phasors can be graphed on the Real-Imaginary plane, a 90 degree phasor is a purely imaginary number, and a -90 degree phasor is its complex conjugate:

$$\mathbb{C} = M \angle 90^\circ$$

$$\mathbb{C}^* = M \angle -90^\circ = M \angle 270^\circ$$

Essentially, this holds true for phasors with all angles: the sign of the angle is reversed to produce the complex conjugate of the phasor in polar notation. In general, for polar notation, we have:

$$\mathbb{C} = M\angle\phi$$

$$\mathbb{C}^* = M\angle -\phi$$

In rectangular form, we can express complex conjugation as:

$$\mathbb{C} = A + jB$$

$$\mathbb{C}^* = A - jB$$

Notice the only difference in the complex conjugate of  $\mathbb{C}$  is the sign change of the imaginary part.

## 26 Decibels

This appendix page is going to take a deeper look at the units of decibels, it will describe some of the properties of decibels, and will demonstrate how to use them in calculations.

### 26.1 Definition

Decibels are, first and foremost, a power calculation. With that in mind, we will state the definition of a decibel:

$$dB = 10 \log \frac{P_{out}}{P_{in}}$$

The letters "dB" are used as the units for the result of this calculation. dB ratios are always in terms of watts, unless otherwise noted.

### 26.2 Voltage Calculation

now, another formula has been demonstrated that allows a decibel calculation to be made using voltages, instead of power measurements. We will derive that equation here:

First, we will use the power calculation and Ohm's law to produce a common identity:

$$P = VI = \frac{V^2}{R}$$

Now, if we plug that result into the definition of a decibel, we can create a complicated equation:

$$dB = 10 \log \left[ \frac{\frac{V_{out}^2}{R}}{\frac{V_{in}^2}{R}} \right]$$

Now, we can cancel out the resistance values (R) from the top and bottom of the fraction, and rearrange the exponent as such:

$$dB = 10 \log \left[ \left( \frac{V_{out}}{V_{in}} \right)^2 \right]$$

If we remember the properties of logarithms, we will remember that if we have an exponent inside a logarithm, we can move the exponent outside, as a coefficient. This rule gives us our desired result:

$$dB = 20 \log \left[ \frac{V_{out}}{V_{in}} \right]$$

## 26.3 Inverse Calculation

It is a simple matter of arithmetic to find the inverse of the decibel calculation, so it will not be derived here, but stated simply:

$$P = 10^{dB/10}$$

## 26.4 Reference Units

Now, this decibel calculation has proven to be so useful, that occasionally they are applied to other units of measurement, instead of just watts. Specifically, the units "dBm" are used when the power unit being converted was in terms of milliwatts, not just watts. Let's say we have a value of 10dBm, we can go through the inverse calculation:

$$P = 10^{10dBm/10} = 10mW$$

Likewise, let's say we want to apply the decibel calculation to a completely unrelated unit: hertz. If we have 100Hz, we can apply the decibel calculation:

$$dB = 10 \log 100Hz = 20dBHz$$

If no letters follow the "dB" label, the decibels are referenced to watts.

## 26.5 Decibel Arithmetic

Decibels are ratios, and *are not real numbers*. Therefore, specific care should be taken not to use decibel values in equations that call for gains, unless decibels are specifically called for (which they usually aren't). However, since decibels are calculated using logarithms, a few principles of logarithms can be used to make decibels usable in calculations.

### 26.5.1 Multiplication

Let's say that we have three values, **a**, **b** and **c**, with their respective decibel equivalents denoted by the upper-case letters **A**, **B** and **C**. We can show that for the following equation:



$$a = b \cdot c$$

That we can change all the quantities to decibels, and convert the multiplication operations to addition:

$$A = B + C$$

### 26.5.2 Division

Let's say that we have three values, **a**, **b** and **c**, with their respective decibel equivalents denoted by the upper-case letters **A**, **B** and **C**. We can show that for the following equation:

$$a = b / c$$

Then we can show through the principals of logarithms that we can convert all the values to decibels, and we can then convert the division operation to subtraction:

$$A = B - C$$



## 27 Transform Tables



## 28 Resources

### 28.1 Further Reading

Pages listed here are sources of further information on the topic of electric circuits, or are additional subjects that may be of interest for a reader of this book. Many of the resources listed here are sources of information, and this may be treated as a bibliography for this wikibook.

#### 28.1.1 Wikibooks

- Wikibooks: Electronics<sup>1</sup>
- Wikibooks: Signals and Systems<sup>2</sup>
- Wikibooks: Digital Circuits<sup>3</sup>
- Wikibooks: Circuit Idea<sup>4</sup>

The following Wikibooks list Circuit Theory<sup>5</sup> as a prerequisite: <dynamicpagelist>

```
category=Circuit Theory/Prerequisite
namespace=Main
suppresserrors=true
```

</dynamicpagelist>

#### 28.1.2 Other Resources

- Horowitz and Hill, *The Art of Electronics*, Second Edition, Cambridge University Press, 1989. ISBN 0521370957
- US Navy, *Basic Electrity*, Dover, 1970. ISBN 0486209733
- US Navy, *Basic Electronics*, Dover, 1973. ISBN 0486210766
- Comer and Comer, *Fundamentals of Electronic Circuit Design*, John Wiley & Sons, 2003. <http://www.wiley.com/college/comer/> ISBN 0471410160
- Dorf and Svoboda, *Introduction to Electric Circuits*, Sixth Edition, John Wiley & Sons, 2004. ISBN 0471447951

---

1 <http://en.wikibooks.org/wiki/Electronics>  
2 <http://en.wikibooks.org/wiki/Signals%20and%20Systems>  
3 <http://en.wikibooks.org/wiki/Digital%20Circuits>  
4 <http://en.wikibooks.org/wiki/Circuit%20Idea>  
5 <http://en.wikibooks.org/wiki/Circuit%20Theory>



## 29 Contributors

Edits	User
308	1sfoerster <sup>1</sup>
1	A.schwarz <sup>2</sup>
4	Adicarlo <sup>3</sup>
5	Adrignola <sup>4</sup>
10	Avicennasis <sup>5</sup>
2	Az1568 <sup>6</sup>
3	Candamil <sup>7</sup>
3	CommonsDelinker <sup>8</sup>
1	Danduddy <sup>9</sup>
14	Darklama <sup>10</sup>
3	DavidCary <sup>11</sup>
1	Derbeth <sup>12</sup>
4	Dirk Hünninger <sup>13</sup>
1	Eaglesfein <sup>14</sup>
7	Fale <sup>15</sup>
1	Ferengi <sup>16</sup>
2	Frohro <sup>17</sup>
1	Gautamraj <sup>18</sup>
2	Hanspi <sup>19</sup>
1	Hgmjr <sup>20</sup>
1	Ifuwant2know <sup>21</sup>

---

1	<a href="http://en.wikibooks.org/w/index.php?title=User:1sfoerster">http://en.wikibooks.org/w/index.php?title=User:1sfoerster</a>
2	<a href="http://en.wikibooks.org/w/index.php?title=User:A.schwarz">http://en.wikibooks.org/w/index.php?title=User:A.schwarz</a>
3	<a href="http://en.wikibooks.org/w/index.php?title=User:Adicarlo">http://en.wikibooks.org/w/index.php?title=User:Adicarlo</a>
4	<a href="http://en.wikibooks.org/w/index.php?title=User:Adrignola">http://en.wikibooks.org/w/index.php?title=User:Adrignola</a>
5	<a href="http://en.wikibooks.org/w/index.php?title=User:Avicennasis">http://en.wikibooks.org/w/index.php?title=User:Avicennasis</a>
6	<a href="http://en.wikibooks.org/w/index.php?title=User:Az1568">http://en.wikibooks.org/w/index.php?title=User:Az1568</a>
7	<a href="http://en.wikibooks.org/w/index.php?title=User:Candamil">http://en.wikibooks.org/w/index.php?title=User:Candamil</a>
8	<a href="http://en.wikibooks.org/w/index.php?title=User:CommonsDelinker">http://en.wikibooks.org/w/index.php?title=User:CommonsDelinker</a>
9	<a href="http://en.wikibooks.org/w/index.php?title=User:Danduddy">http://en.wikibooks.org/w/index.php?title=User:Danduddy</a>
10	<a href="http://en.wikibooks.org/w/index.php?title=User:Darklama">http://en.wikibooks.org/w/index.php?title=User:Darklama</a>
11	<a href="http://en.wikibooks.org/w/index.php?title=User:DavidCary">http://en.wikibooks.org/w/index.php?title=User:DavidCary</a>
12	<a href="http://en.wikibooks.org/w/index.php?title=User:Derbeth">http://en.wikibooks.org/w/index.php?title=User:Derbeth</a>
13	<a href="http://en.wikibooks.org/w/index.php?title=User:Dirk_H%C3%BCnniger">http://en.wikibooks.org/w/index.php?title=User:Dirk_H%C3%BCnniger</a>
14	<a href="http://en.wikibooks.org/w/index.php?title=User:Eaglesfein">http://en.wikibooks.org/w/index.php?title=User:Eaglesfein</a>
15	<a href="http://en.wikibooks.org/w/index.php?title=User:Fale">http://en.wikibooks.org/w/index.php?title=User:Fale</a>
16	<a href="http://en.wikibooks.org/w/index.php?title=User:Ferengi">http://en.wikibooks.org/w/index.php?title=User:Ferengi</a>
17	<a href="http://en.wikibooks.org/w/index.php?title=User:Frohro">http://en.wikibooks.org/w/index.php?title=User:Frohro</a>
18	<a href="http://en.wikibooks.org/w/index.php?title=User:Gautamraj">http://en.wikibooks.org/w/index.php?title=User:Gautamraj</a>
19	<a href="http://en.wikibooks.org/w/index.php?title=User:Hanspi">http://en.wikibooks.org/w/index.php?title=User:Hanspi</a>
20	<a href="http://en.wikibooks.org/w/index.php?title=User:Hgmjr">http://en.wikibooks.org/w/index.php?title=User:Hgmjr</a>
21	<a href="http://en.wikibooks.org/w/index.php?title=User:Ifuwant2know">http://en.wikibooks.org/w/index.php?title=User:Ifuwant2know</a>

2 Inductiveload<sup>22</sup>  
1 Int21h<sup>23</sup>  
1 JenVan<sup>24</sup>  
1 Jfrederickson<sup>25</sup>  
14 Jguk<sup>26</sup>  
9 Jomegat<sup>27</sup>  
1 JonLS<sup>28</sup>  
1 Kenji.ohi<sup>29</sup>  
1 Klules<sup>30</sup>  
1 Luminous2269<sup>31</sup>  
3 Macaddict<sup>32</sup>  
1 Mattb112885<sup>33</sup>  
3 Mattpanico<sup>34</sup>  
1 Mikelee<sup>35</sup>  
3 Mroberto<sup>36</sup>  
1 Ndriscoll<sup>37</sup>  
1 Ntlhui<sup>38</sup>  
4 Outopiate<sup>39</sup>  
3 Panic2k4<sup>40</sup>  
1 Pdiehr<sup>41</sup>  
1 Petello12000<sup>42</sup>  
1 Pnavarrc<sup>43</sup>  
10 QuiteUnusual<sup>44</sup>  
11 Rakarlin<sup>45</sup>  
11 Recent\_Runes<sup>46</sup>

---

22 <http://en.wikibooks.org/w/index.php?title=User:Inductiveload>  
23 <http://en.wikibooks.org/w/index.php?title=User:Int21h>  
24 <http://en.wikibooks.org/w/index.php?title=User:JenVan>  
25 <http://en.wikibooks.org/w/index.php?title=User:Jfrederickson>  
26 <http://en.wikibooks.org/w/index.php?title=User:Jguk>  
27 <http://en.wikibooks.org/w/index.php?title=User:Jomegat>  
28 <http://en.wikibooks.org/w/index.php?title=User:JonLS>  
29 <http://en.wikibooks.org/w/index.php?title=User:Kenji.ohi>  
30 <http://en.wikibooks.org/w/index.php?title=User:Klules>  
31 <http://en.wikibooks.org/w/index.php?title=User:Luminous2269>  
32 <http://en.wikibooks.org/w/index.php?title=User:Macaddict>  
33 <http://en.wikibooks.org/w/index.php?title=User:Mattb112885>  
34 <http://en.wikibooks.org/w/index.php?title=User:Mattpanico>  
35 <http://en.wikibooks.org/w/index.php?title=User:Mikelee>  
36 <http://en.wikibooks.org/w/index.php?title=User:Mroberto>  
37 <http://en.wikibooks.org/w/index.php?title=User:Ndriscoll>  
38 <http://en.wikibooks.org/w/index.php?title=User:Ntlhui>  
39 <http://en.wikibooks.org/w/index.php?title=User:Outopiate>  
40 <http://en.wikibooks.org/w/index.php?title=User:Panic2k4>  
41 <http://en.wikibooks.org/w/index.php?title=User:Pdiehr>  
42 <http://en.wikibooks.org/w/index.php?title=User:Petello12000>  
43 <http://en.wikibooks.org/w/index.php?title=User:Pnavarrc>  
44 <http://en.wikibooks.org/w/index.php?title=User:QuiteUnusual>  
45 <http://en.wikibooks.org/w/index.php?title=User:Rakarlin>  
46 [http://en.wikibooks.org/w/index.php?title=User:Recent\\_Runes](http://en.wikibooks.org/w/index.php?title=User:Recent_Runes)







# List of Figures

- GFDL: Gnu Free Documentation License. <http://www.gnu.org/licenses/fdl.html>
- cc-by-sa-3.0: Creative Commons Attribution ShareAlike 3.0 License. <http://creativecommons.org/licenses/by-sa/3.0/>
- cc-by-sa-2.5: Creative Commons Attribution ShareAlike 2.5 License. <http://creativecommons.org/licenses/by-sa/2.5/>
- cc-by-sa-2.0: Creative Commons Attribution ShareAlike 2.0 License. <http://creativecommons.org/licenses/by-sa/2.0/>
- cc-by-sa-1.0: Creative Commons Attribution ShareAlike 1.0 License. <http://creativecommons.org/licenses/by-sa/1.0/>
- cc-by-2.0: Creative Commons Attribution 2.0 License. <http://creativecommons.org/licenses/by/2.0/>
- cc-by-2.0: Creative Commons Attribution 2.0 License. <http://creativecommons.org/licenses/by/2.0/deed.en>
- cc-by-2.5: Creative Commons Attribution 2.5 License. <http://creativecommons.org/licenses/by/2.5/deed.en>
- cc-by-3.0: Creative Commons Attribution 3.0 License. <http://creativecommons.org/licenses/by/3.0/deed.en>
- GPL: GNU General Public License. <http://www.gnu.org/licenses/gpl-2.0.txt>
- LGPL: GNU Lesser General Public License. <http://www.gnu.org/licenses/lgpl.html>
- PD: This image is in the public domain.
- ATTR: The copyright holder of this file allows anyone to use it for any purpose, provided that the copyright holder is properly attributed. Redistribution, derivative work, commercial use, and all other use is permitted.
- EURO: This is the common (reverse) face of a euro coin. The copyright on the design of the common face of the euro coins belongs to the European Commission. Authorised reproduction in a format without relief (drawings, paintings, films) provided they are not detrimental to the image of the euro.
- LFK: Lizenz Freie Kunst. <http://artlibre.org/licence/lal/de>
- CFR: Copyright free use.

- EPL: Eclipse Public License. <http://www.eclipse.org/org/documents/epl-v10.php>

Copies of the GPL, the LGPL as well as a GFDL are included in chapter Licenses<sup>64</sup>. Please note that images in the public domain do not require attribution. You may click on the image numbers in the following table to open the webpage of the images in your webbrowser.

---

<sup>64</sup> Chapter 30 on page 145

1	lsfoerster <sup>65</sup>	cc-by-sa-3.0
2		GFDL
3		
4	Inductiveload <sup>66</sup>	PD
5	Inductiveload <sup>67</sup>	PD
6	Inductiveload <sup>68</sup>	PD
7	Inductiveload <sup>69</sup>	PD
8	Inductiveload <sup>70</sup>	PD
9	Inductiveload <sup>71</sup>	PD
10	Inductiveload <sup>72</sup>	PD
11	Inductiveload <sup>73</sup>	PD
12	Inductiveload <sup>74</sup>	PD
13	Inductiveload <sup>75</sup>	PD
14		GFDL
15		GFDL
16		GFDL
17		GFDL
18		
19		
20		
21		GFDL
22		GFDL
23		
24		GFDL
25	inductiveload	PD
26	Traced by User:Stannered <sup>76</sup>	GFDL
27	Dirk-Lüder Kreie <sup>77</sup>	GFDL
28		PD
29		
30		GFDL
31	Nicostella <sup>78</sup>	PD
32		PD
33		GFDL
34		GFDL
35		PD

- <sup>65</sup> <http://en.wikibooks.org/wiki/User%3A1sfoerster>  
<sup>66</sup> <http://en.wikibooks.org/wiki/User%3AInductiveload>  
<sup>67</sup> <http://en.wikibooks.org/wiki/User%3AInductiveload>  
<sup>68</sup> <http://en.wikibooks.org/wiki/User%3AInductiveload>  
<sup>69</sup> <http://en.wikibooks.org/wiki/User%3AInductiveload>  
<sup>70</sup> <http://en.wikibooks.org/wiki/User%3AInductiveload>  
<sup>71</sup> <http://en.wikibooks.org/wiki/User%3AInductiveload>  
<sup>72</sup> <http://en.wikibooks.org/wiki/User%3AInductiveload>  
<sup>73</sup> <http://en.wikibooks.org/wiki/User%3AInductiveload>  
<sup>74</sup> <http://en.wikibooks.org/wiki/User%3AInductiveload>  
<sup>75</sup> <http://en.wikibooks.org/wiki/User%3AInductiveload>  
<sup>76</sup> <http://en.wikibooks.org/wiki/User%3AStannered>  
<sup>77</sup> <http://de.wikipedia.org/wiki/Benutzer%3ADeelkar>  
<sup>78</sup> <http://en.wikibooks.org/wiki/User%3ANicostella>



# 30 Licenses

## 30.1 GNU GENERAL PUBLIC LICENSE

Version 3, 29 June 2007

Copyright © 2007 Free Software Foundation, Inc.  
<<http://fsf.org/>>

Everyone is permitted to copy and distribute verbatim copies of this license document, but changing it is not allowed. Preamble

The GNU General Public License is a free, copyleft license for software and other kinds of works.

The licenses for most software and other practical works are designed to take away your freedom to share and change the works. By contrast, the GNU General Public License is intended to guarantee your freedom to share and change all versions of a program—to make sure it remains free software for all its users. We, the Free Software Foundation, use the GNU General Public License for most of our software; it applies also to any other work released this way by its authors. You can apply it to your programs, too.

When we speak of free software, we are referring to freedom, not price. Our General Public Licenses are designed to make sure that you have the freedom to distribute copies of free software (and charge for them if you wish), that you receive source code or can get it if you want it, that you can change the software or use pieces of it in new free programs, and that you know you can do these things.

To protect your rights, we need to prevent others from denying you these rights or asking you to surrender the rights. Therefore, you have certain responsibilities if you distribute copies of the software, or if you modify it: responsibilities to respect the freedom of others.

For example, if you distribute copies of such a program, whether gratis or for a fee, you must pass on to the recipients the same freedoms that you received. You must make sure that they, too, receive or can get the source code. And you must show them these terms so they know their rights.

Developers that use the GNU GPL protect your rights with two steps: (1) assert copyright on the software, and (2) offer you this License giving you legal permission to copy, distribute and/or modify it.

For the developers' and authors' protection, the GPL clearly explains that there is no warranty for this free software. For both users' and authors' sake, the GPL requires that modified versions be marked as changed, so that their problems will not be attributed erroneously to authors of previous versions.

Some devices are designed to deny users access to install or run modified versions of the software inside them, although the manufacturer can do so. This is fundamentally incompatible with the aim of protecting users' freedom to change the software. The systematic pattern of such abuse occurs in the area of products for individuals to use, which is precisely where it is most unacceptable. Therefore, we have designed this version of the GPL to prohibit the practice for those products. If such problems arise substantially in other domains, we stand ready to extend this provision to those domains in future versions of the GPL, as needed to protect the freedom of users.

Finally, every program is threatened constantly by software patents. States should not allow patents to restrict development and use of software on general-purpose computers, but in those that do, we wish to avoid the special danger that patents applied to a free program could make it effectively proprietary. To prevent this, the GPL assures that patents cannot be used to render the program non-free.

The precise terms and conditions for copying, distribution and modification follow. TERMS AND CONDITIONS 0. Definitions.

"This License" refers to version 3 of the GNU General Public License.

"Copyright" also means copyright-like laws that apply to other kinds of works, such as semiconductor masks.

"The Program" refers to any copyrightable work licensed under this License. Each licensee is addressed as "you". "Licensees" and "recipients" may be individuals or organizations.

To "modify" a work means to copy from or adapt all or part of the work in a fashion requiring copyright permission, other than the making of an exact copy. The resulting work is called a "modified version" of the earlier work or a work "based on" the earlier work.

A "covered work" means either the unmodified Program or a work based on the Program.

To "propagate" a work means to do anything with it that, without permission, would make you directly or secondarily liable for infringement under applicable copyright law, except executing it on a computer or modifying a private copy. Propagation includes copying, distribution (with or without modification), making available to the public, and in some countries other activities as well.

To "convey" a work means any kind of propagation that enables other parties to make or receive copies. Mere interaction with a user through a computer

network, with no transfer of a copy, is not conveying.

An interactive user interface displays "Appropriate Legal Notices" to the extent that it includes a convenient and prominently visible feature that (1) displays an appropriate copyright notice, and (2) tells the user that there is no warranty for the work (except to the extent that warranties are provided); that licensees may convey the work under this License, and how to view a copy of this License. If the interface presents a list of user commands or options, such as a menu, a prominent item in the list meets this criterion. 1. Source Code.

The "source code" for a work means the preferred form of the work for making modifications to it. "Object code" means any non-source form of a work.

A "Standard Interface" means an interface that either is an official standard defined by a recognized standards body, or, in the case of interfaces specified for a particular programming language, one that is widely used among developers working in that language.

The "System Libraries" of an executable work include anything, other than the work as a whole, that (a) is included in the normal form of packaging a Major Component, but which is not part of that Major Component, and (b) serves only to enable use of the work with that Major Component, or to implement a Standard Interface for which an implementation is available to the public in source code form. A "Major Component", in this context, means a major essential component (kernel, window system, and so on) of the specific operating system (if any) on which the executable work runs, or a compiler used to produce the work, or an object code interpreter used to run it.

The "Corresponding Source" for a work in object code form means all the source code needed to generate, install, and (for an executable work) run the object code and to modify the work, including scripts to control those activities. However, it does not include the work's System Libraries, or general-purpose tools or generally available free programs which are used unmodified in performing those activities but which are not part of the work. For example, Corresponding Source includes interface definition files associated with source files for the work, and the source code for shared libraries and dynamically linked subprograms that the work is specifically designed to require, such as by intimate data communication or control flow between those subprograms and other parts of the work.

The Corresponding Source need not include anything that users can regenerate automatically from other parts of the Corresponding Source.

The Corresponding Source for a work in source code form is that same work. 2. Basic Permissions.

All rights granted under this License are granted for the term of copyright on the Program, and are irrevocable provided the stated conditions are met. This License explicitly affirms your unlimited permission to run the unmodified Program. The output from running a covered work is covered by this License only if the output, given its content, constitutes a covered work. This License acknowledges your rights of fair use or other equivalent, as provided by copyright law.

You may make, run and propagate covered works that you do not convey, without conditions so long as your license otherwise remains in force. You may convey covered works to others for the sole purpose of having them make modifications exclusively for you, or provide you with facilities for running those works, provided that you comply with the terms of this License in conveying all material for which you do not control copyright. Those thus making or running the covered works for you must do so exclusively on your behalf, under your direction and control, on terms that prohibit them from making any copies of your copyrighted material outside their relationship with you.

Conveying under any other circumstances is permitted solely under the conditions stated below. Sub-licensing is not allowed; section 10 makes it unnecessary. 3. Protecting Users' Legal Rights From Anti-Circumvention Law.

No covered work shall be deemed part of an effective technological measure under any applicable law fulfilling obligations under article 11 of the WIPO copyright treaty adopted on 20 December 1996, or similar laws prohibiting or restricting circumvention of such measures.

When you convey a covered work, you waive any legal power to forbid circumvention of technological measures to the extent such circumvention is effected by exercising rights under this License with respect to the covered work, and you disclaim any intention to limit operation or modification of the work as a means of enforcing, against the work's users, your or third parties' legal rights to forbid circumvention of technological measures. 4. Conveying Verbatim Copies.

You may convey verbatim copies of the Program's source code as you receive it, in any medium, provided that you conspicuously and appropriately publish on each copy an appropriate copyright notice; keep intact all notices stating that this License and any non-permissive terms added in accord with section 7 apply to the code; keep intact all notices of the absence of any warranty; and give all recipients a copy of this License along with the Program.

You may charge any price or no price for each copy that you convey, and you may offer support or warranty protection for a fee. 5. Conveying Modified Source Versions.

You may convey a work based on the Program, or the modifications to produce it from the Program, in the form of source code under the terms of section 4, provided that you also meet all of these conditions:

\* a) The work must carry prominent notices stating that you modified it, and giving a relevant date. \* b) The work must carry prominent notices stating that it is released under this License and any conditions added under section 7. This requirement modifies the requirement in section 4 to "keep intact all notices". \* c) You must license the entire work, as a whole, under this License to anyone who comes into possession of a copy. This License will therefore apply, along with any applicable section 7 additional terms, to the whole of the work, and all its parts, regardless of how they are packaged. This License gives no permission to license the work in any other way, but it does not invalidate such permission if you have separately received it. \* d) If the work has interactive user interfaces, each must display Appropriate Legal Notices; however, if the Program has interactive interfaces that do not display Appropriate Legal Notices, your work need not make them do so.

A compilation of a covered work with other separate and independent works, which are not by their nature extensions of the covered work, and which are not combined with it such as to form a larger program, in or on a volume of a storage or distribution medium, is called an "aggregate" if the compilation and its resulting copyright are not used to limit the access or legal rights of the compilation's users beyond what the individual works permit. Inclusion of a covered work in an aggregate does not cause this License to apply to the other parts of the aggregate. 6. Conveying Non-Source Forms.

You may convey a covered work in object code form under the terms of sections 4 and 5, provided that you also convey the machine-readable Corresponding Source under the terms of this License, in one of these ways:

\* a) Convey the object code in, or embodied in, a physical product (including a physical distribution medium), accompanied by the Corresponding Source fixed on a durable physical medium customarily used for software interchange. \* b) Convey the object code in, or embodied in, a physical product (including a physical distribution medium), accompanied by a written offer, valid for at least three years and valid for as long as you offer spare parts or customer support for that product model, to give anyone who possesses the object code either (1) a copy of the Corresponding Source for all the software in the product that is covered by this License, on a durable physical medium customarily used for software interchange, for a price no more than your reasonable cost of physically performing this conveying of source, or (2) access to copy the Corresponding Source from a network server at no charge. \* c) Convey individual copies of the object code with a copy of the written offer to provide the Corresponding Source. This alternative is allowed only occasionally and noncommercially, and only if you received the object code with such an offer, in accord with subsection 6b. \* d) Convey the object code by offering access from a designated place (gratis or for a charge), and offer equivalent access to the Corresponding Source in the same way through the same place at no further charge. You need not require recipients to copy the Corresponding Source along with the object code. If the place to copy the object code is a network server, the Corresponding Source may be on a different server (operated by you or a third party) that supports equivalent copying facilities, provided you maintain clear directions next to the object code saying where to find the Corresponding Source. Regardless of what server hosts the Corresponding Source, you remain obligated to ensure that it is available for as long as needed to satisfy these requirements. \* e) Convey the object code using peer-to-peer transmission, provided you inform other peers where the object code and Corresponding Source of the work are being offered to the general public at no charge under subsection 6d.

A separable portion of the object code, whose source code is excluded from the Corresponding Source as a System Library, need not be included in conveying the object code work.

A "User Product" is either (1) a "consumer product", which means any tangible personal property which is normally used for personal, family, or household purposes, or (2) anything designed or sold for incorporation into a dwelling. In determining whether a product is a consumer product, doubtful cases shall be resolved in favor of coverage. For a particular product received by a particular user, "normally used" refers to a typical or common use of that class of product, regardless of the status of the particular user or of the way in which the particular user actually uses, or expects or is expected to use, the product. A product is a consumer product regardless of whether the producer has substantial commercial, industrial, or non-consumer uses, unless such uses represent the only significant mode of use of the product.

"Installation Information" for a User Product means any methods, procedures, authorization keys, or other information required to install and execute modified versions of a covered work in that User Product from a modified version of its Corresponding Source. The information must suffice to ensure that the continued functioning of the modified object code is in no case prevented or interfered with solely because modification has been made.

If you convey an object code work under this section in, or with, or specifically for use in, a User Product, and the conveying occurs as part of a transaction in which the right of possession and use of the User Product is transferred to the recipient in perpetuity or for a fixed term (regardless of how the transaction is characterized), the Corresponding Source conveyed under this section must be accompanied by the Installation Information. But this requirement does not apply if neither you nor any third party retains the ability to install modified object code on the User Product (for example, the work has been installed in ROM).

The requirement to provide Installation Information does not include a requirement to continue to provide support service, warranty, or updates for a work that has been modified or installed by the recipient, or for the User Product in which it has been modified or installed. Access to a network may be denied when the modification itself materially and adversely affects the operation of the network or violates the rules and protocols for communication across the network.

Corresponding Source conveyed, and Installation Information provided, in accord with this section must be in a format that is publicly documented (and with an implementation available to the public in source code form), and must require no special password or key for unpacking, reading or copying. 7. Additional Terms.

"Additional permissions" are terms that supplement the terms of this License by making exceptions from one or more of its conditions. Additional permissions that are applicable to the entire Program shall be treated as though they were included in this License, to the extent that they are valid under applicable law. If additional permissions apply only to part of the Program, that part may be used separately under those permissions, but the entire Program remains governed by this License without regard to the additional permissions.

When you convey a copy of a covered work, you may at your option remove any additional permissions from that copy, or from any part of it. (Additional permissions may be written to require their own removal in certain cases when you modify the work.) You may place additional permissions on material, added by you to a covered work, for which you have or can give appropriate copyright permission.

Notwithstanding any other provision of this License, for material you add to a covered work, you may (if authorized by the copyright holders of that material) supplement the terms of this License with terms:

\* a) Disclaiming warranty or limiting liability differently from the terms of sections 15 and 16 of this License; or \* b) Requiring preservation of specified reasonable legal notices or author attributions in that material or in the Appropriate Legal Notices displayed by works containing it; or \* c) Prohibiting misrepresentation of the origin of that material, or requiring that modified versions of such material be marked in reasonable ways as different from the original version; or \* d) Limiting the use for publicity purposes of names of licensors or authors of the material; or \* e) Declining to grant rights under trademark law for use of some trade names, trademarks, or service marks; or \* f) Requiring indemnification of licensors and authors of that material by anyone who conveys the material (or modified versions of it) with contractual assumptions of liability to the recipient, for any liability that these contractual assumptions directly impose on those licensors and authors.

All other non-permissive additional terms are considered "further restrictions" within the meaning of section 10. If the Program as you received it, or any part of it, contains a notice stating that it is governed by this License along with a term that is a further restriction, you may remove that term. If a license document contains a further restriction but permits relicensing or conveying under this License, you may add to a covered work material governed by the terms of that license document, provided that the further restriction does not survive such relicensing or conveying.

If you add terms to a covered work in accord with this section, you must place, in the relevant source files, a statement of the additional terms that apply to those files, or a notice indicating where to find the applicable terms.

Additional terms, permissive or non-permissive, may be stated in the form of a separately written license, or stated as exceptions; the above requirements apply either way. 8. Termination.

You may not propagate or modify a covered work except as expressly provided under this License. Any attempt otherwise to propagate or modify it is void, and will automatically terminate your rights under this License (including any patent licenses granted under the third paragraph of section 11).

However, if you cease all violation of this License, then your license from a particular copyright holder is reinstated (a) provisionally, unless and until the copyright holder explicitly and finally terminates your license, and (b) permanently, if the copyright holder fails to notify you of the violation by some reasonable means prior to 60 days after the cessation.

Moreover, your license from a particular copyright holder is reinstated permanently if the copyright holder notifies you of the violation by some reasonable means, this is the first time you have received notice of violation of this License (for any work)

from that copyright holder, and you cure the violation prior to 30 days after your receipt of the notice.

Termination of your rights under this section does not terminate the licenses of parties who have received copies or rights from you under this License. If your rights have been terminated and not permanently reinstated, you do not qualify to receive new licenses for the same material under section 10. 9. Acceptance Not Required for Having Copies.

You are not required to accept this License in order to receive or run a copy of the Program. Ancillary propagation of a covered work occurring solely as a consequence of using peer-to-peer transmission to receive a copy likewise does not require acceptance. However, nothing other than this License grants you permission to propagate or modify any covered work. These actions infringe copyright if you do not accept this License. Therefore, by modifying or propagating a covered work, you indicate your acceptance of this License to do so. 10. Automatic Licensing of Downstream Recipients.

Each time you convey a covered work, the recipient automatically receives a license from the original licensors, to run, modify and propagate that work, subject to this License. You are not responsible for enforcing compliance by third parties with this License.

An "entity transaction" is a transaction transferring control of an organization, or substantially all assets of one, or subdividing an organization, or merging organizations. If propagation of a covered work results from an entity transaction, each party to that transaction who receives a copy of the work also receives whatever licenses to the work the party's predecessor in interest had or could give under the previous paragraph, plus a right to possession of the Corresponding Source of the work from the predecessor in interest, if the predecessor has it or can get it with reasonable efforts.

You may not impose any further restrictions on the exercise of the rights granted or affirmed under this License. For example, you may not impose a license fee, royalty, or other charge for exercise of rights granted under this License, and you may not initiate litigation (including a cross-claim or counterclaim in a lawsuit) alleging that any patent claim is infringed by making, using, selling, offering for sale, or importing the Program or any portion of it. 11. Patents.

A "contributor" is a copyright holder who authorizes use under this License of the Program or a work on which the Program is based. The work thus licensed is called the contributor's "contributor version".

A contributor's "essential patent claims" are all patent claims owned or controlled by the contributor, whether already acquired or hereafter acquired, that would be infringed by some manner, permitted by this License, of making, using, or selling its contributor version, but do not include claims that would be infringed only as a consequence of further modification of the contributor version. For purposes of this definition, "control" includes the right to grant patent sublicenses in a manner consistent with the requirements of this License.

Each contributor grants you a non-exclusive, worldwide, royalty-free patent license under the contributor's essential patent claims, to make, use, sell, offer for sale, import and otherwise run, modify and propagate the contents of its contributor version.

## 30.2 GNU Free Documentation License

Version 1.3, 3 November 2008

Copyright © 2000, 2001, 2002, 2007, 2008 Free Software Foundation, Inc. <<http://fsf.org/>>

Everyone is permitted to copy and distribute verbatim copies of this license document, but changing it is not allowed. 0. PREAMBLE

The purpose of this License is to make a manual, textbook, or other functional and useful document "free" in the sense of freedom: to assure everyone the effective freedom to copy and redistribute it, with or without modifying it, either commercially or noncommercially. Secondly, this License preserves for the author and publisher a way to get credit for their work, while not being considered responsible for modifications made by others.

This License is a kind of "copyleft", which means that derivative works of the document must themselves be free in the same sense. It implements the GNU General Public License, which is a copyleft license designed for free software.

We have designed this License in order to use it for manuals for free software, because free software needs free documentation: a free program should come with manuals providing the same freedoms that the software does. But this License is not limited to software manuals; it can be used for any textual work, regardless of subject matter or whether it is published as a printed book. We recommend this License principally for works whose purpose is instruction or reference. 1. APPLICABILITY AND DEFINITIONS

This License applies to any manual or other work, in any medium, that contains a notice placed by the copyright holder saying it can be distributed under the terms of this License. Such a notice grants a world-wide, royalty-free license, unlimited in duration, to use that work under the conditions stated herein. The "Document", below, refers to any such manual or work. Any member of the public is a licensee, and is addressed as "you". You accept the license if you copy, modify or distribute the work in a way requiring permission under copyright law.

A "Modified Version" of the Document means any work containing the Document or a portion of it, either copied verbatim, or with modifications and/or translated into another language.

A "Secondary Section" is a named appendix or a front-matter section of the Document that deals exclusively with the relationship of the publishers or

In the following three paragraphs, a "patent license" is any express agreement or commitment, however denominated, not to enforce a patent (such as an express permission to practice a patent or covenant not to sue for patent infringement). To "grant" such a patent license to a party means to make such an agreement or commitment not to enforce a patent against the party.

If you convey a covered work, knowingly relying on a patent license, and the Corresponding Source of the work is not available for anyone to copy, free of charge and under the terms of this License, through a publicly available network server or other readily accessible means, then you must either (1) cause the Corresponding Source to be so available, or (2) arrange to deprive yourself of the benefit of the patent license for this particular work, or (3) arrange, in a manner consistent with the requirements of this License, to extend the patent license to downstream recipients. "Knowingly relying" means you have actual knowledge that, but for the patent license, your conveying the covered work in a country, or your recipient's use of the covered work in a country, would infringe one or more identifiable patents in that country that you have reason to believe are valid.

If, pursuant to or in connection with a single transaction or arrangement, you convey, or propagate by procuring conveyance of, a covered work, and grant a patent license to some of the parties receiving the covered work authorizing them to use, propagate, modify or convey a specific copy of the covered work, then the patent license you grant is automatically extended to all recipients of the covered work and works based on it.

A patent license is "discriminatory" if it does not include within the scope of its coverage, prohibits the exercise of, or is conditioned on the exercise of, one or more of the rights that are specifically granted under this License. You may not convey a covered work if you are a party to an arrangement with a third party that is in the business of distributing software, under which you make payment to the third party based on the extent of your activity of conveying the work, and under which the third party grants, to any of the parties who may receive the covered work from you, a discriminatory patent license (a) in connection with copies of the covered work conveyed by you (or copies made from those copies), or (b) primarily for and in connection with specific products or compilations that contain the covered work, unless you entered into that arrangement, or that patent license was granted, prior to 28 March 2007.

Nothing in this License shall be construed as excluding or limiting any implied license or other defenses to infringement that may otherwise be available to you under applicable patent law. 12. No Surrender of Others' Freedom.

If conditions are imposed on you (whether by court order, agreement or otherwise) that contradict the conditions of this License, they do not excuse you from the conditions of this License. If you cannot convey a covered work so as to satisfy simultaneously your obligations under this License and any other pertinent obligations, then as a consequence you may not convey it at all. For example, if you agree to terms that obligate you to collect a royalty for further conveying from those to whom you convey the Program, the only way you could satisfy both those terms and this License would be to refrain entirely from conveying the Program. 13. Use with the GNU Affero General Public License.

authors of the Document to the Document's overall subject (or to related matters) and contains nothing that could fall directly within that overall subject. (Thus, if the Document is in part a textbook of mathematics, a Secondary Section may not explain any mathematics.) The relationship could be a matter of historical connection with the subject or with related matters, or of legal, commercial, philosophical, ethical or political position regarding them.

The "Invariant Sections" are certain Secondary Sections whose titles are designated, as being those of Invariant Sections, in the notice that says that the Document is released under this License. If a section does not fit the above definition of Secondary then it is not allowed to be designated as Invariant. The Document may contain zero or more Invariant Sections. If the Document does not identify any Invariant Sections then there are none.

The "Cover Texts" are certain short passages of text that are listed, as Front-Cover Texts or Back-Cover Texts, in the notice that says that the Document is released under this License. A Front-Cover Text may be at most 5 words, and a Back-Cover Text may be at most 25 words.

A "Transparent" copy of the Document means a machine-readable copy, represented in a format whose specification is available to the general public, that is suitable for revising the document straightforwardly with generic text editors or (for images composed of pixels) generic paint programs or (for drawings created with vector drawing editors) and that is suitable for input to text formatters or for automatic translation to a variety of formats suitable for input to text formatters. A copy made in an otherwise Transparent file format whose markup, or absence of markup, has been arranged to thwart or discourage subsequent modification by readers is not Transparent. An image format is not Transparent if used for any substantial amount of text. A copy that is not "Transparent" is called "Opaque".

Examples of suitable formats for Transparent copies include plain ASCII without markup, Texinfo input format, LaTeX input format, SGML or XML using a publicly available DTD, and standard-conforming simple HTML, PostScript or PDF designed for human modification. Examples of transparent image formats include PNG, XCF and JPG. Opaque formats include proprietary formats that can be read and edited only by proprietary word processors, SGML or XML for which the DTD and/or processing tools are not generally available, and the machine-generated HTML, PostScript or

Notwithstanding any other provision of this License, you have permission to link or combine any covered work with a work licensed under version 3 of the GNU Affero General Public License into a single combined work, and to convey the resulting work. The terms of this License will continue to apply to the part which is the covered work, but the special requirements of the GNU Affero General Public License, section 13, concerning interaction through a network will apply to the combination as such. 14. Revised Versions of this License.

The Free Software Foundation may publish revised and/or new versions of the GNU General Public License from time to time. Such new versions will be similar in spirit to the present version, but may differ in detail to address new problems or concerns.

Each version is given a distinguishing version number. If the Program specifies that a certain numbered version of the GNU General Public License "or any later version" applies to it, you have the option of following the terms and conditions either of that numbered version or of any later version published by the Free Software Foundation. If the Program does not specify a version number of the GNU General Public License, you may choose any version ever published by the Free Software Foundation.

If the Program specifies that a proxy can decide which future versions of the GNU General Public License can be used, that proxy's public statement of acceptance of a version permanently authorizes you to choose that version for the Program.

Later license versions may give you additional or different permissions. However, no additional obligations are imposed on any author or copyright holder as a result of your choosing to follow a later version. 15. Disclaimer of Warranty.

THERE IS NO WARRANTY FOR THE PROGRAM, TO THE EXTENT PERMITTED BY APPLICABLE LAW. EXCEPT WHEN OTHERWISE STATED IN WRITING THE COPYRIGHT HOLDERS AND/OR OTHER PARTIES PROVIDE THE PROGRAM "AS IS" WITHOUT WARRANTY OF ANY KIND, EITHER EXPRESSED OR IMPLIED, INCLUDING, BUT NOT LIMITED TO, THE IMPLIED WARRANTIES OF MERCHANTABILITY AND FITNESS FOR A PARTICULAR PURPOSE. THE ENTIRE RISK AS TO THE QUALITY AND PERFORMANCE OF THE PROGRAM IS WITH YOU. SHOULD THE PROGRAM PROVE DEFECTIVE, YOU ASSUME THE COST OF ALL NECESSARY SERVICING, REPAIR OR CORRECTION. 16. Limitation of Liability.

IN NO EVENT UNLESS REQUIRED BY APPLICABLE LAW OR AGREED TO IN WRITING WILL ANY COPYRIGHT HOLDER, OR ANY OTHER PARTY WHO MODIFIES AND/OR CONVEYS THE PROGRAM AS PERMITTED ABOVE, BE LIABLE TO YOU FOR DAMAGES, INCLUDING ANY GENERAL, SPECIAL, INCIDENTAL OR CONSEQUENTIAL DAMAGES ARISING OUT OF THE USE OR INABILITY TO USE THE PROGRAM (INCLUDING BUT NOT LIMITED TO LOSS OF DATA OR DATA BEING RENDERED INACCURATE OR LOSSES SUSTAINED BY YOU OR THIRD PARTIES OR A FAILURE OF THE PROGRAM TO OPERATE WITH ANY OTHER PROGRAMS), EVEN IF SUCH HOLDER OR OTHER PARTY HAS BEEN ADVISED OF THE POSSIBILITY OF SUCH DAMAGES. 17. Interpretation of Sections 15 and 16.

If the disclaimer of warranty and limitation of liability provided above cannot be given local legal ef-

fect according to their terms, reviewing courts shall apply local law that most closely approximates an absolute waiver of all civil liability in connection with the Program, unless a warranty or assumption of liability accompanies a copy of the Program in return for a fee.

END OF TERMS AND CONDITIONS How to Apply These Terms to Your New Programs

If you develop a new program, and you want it to be of the greatest possible use to the public, the best way to achieve this is to make it free software which everyone can redistribute and change under these terms.

To do so, attach the following notices to the program. It is safest to attach them to the start of each source file to most effectively state the exclusion of warranty; and each file should have at least the "copyright" line and a pointer to where the full notice is found.

<one line to give the program's name and a brief idea of what it does.> Copyright (C) <year>  
<name of author>

This program is free software; you can redistribute it and/or modify it under the terms of the GNU General Public License as published by the Free Software Foundation, either version 3 of the License, or (at your option) any later version.

This program is distributed in the hope that it will be useful, but WITHOUT ANY WARRANTY; without even the implied warranty of MERCHANTABILITY or FITNESS FOR A PARTICULAR PURPOSE. See the GNU General Public License for more details.

You should have received a copy of the GNU General Public License along with this program. If not, see <<http://www.gnu.org/licenses/>>.

Also add information on how to contact you by electronic and paper mail.

If the program does terminal interaction, make it output a short notice like this when it starts in an interactive mode:

<program> Copyright (C) <year> <name of author>  
This program comes with ABSOLUTELY NO WARRANTY; for details type 'show w'. This is free software, and you are welcome to redistribute it under certain conditions; type 'show c' for details.

The hypothetical commands 'show w' and 'show c' should show the appropriate parts of the General Public License. Of course, your program's commands might be different; for a GUI interface, you would use an "about box".

You should also get your employer (if you work as a programmer) or school, if any, to sign a "copyright disclaimer" for the program, if necessary. For more information on this, and how to apply and follow the GNU GPL, see <<http://www.gnu.org/licenses/>>.

The GNU General Public License does not permit incorporating your program into proprietary programs. If your program is a subroutine library, you may consider it more useful to permit linking proprietary applications with the library. If this is what you want to do, use the GNU Lesser General Public License instead of this License. But first, please read <<http://www.gnu.org/philosophy/why-not-lgpl.html>>.

PDF produced by some word processors for output purposes only.

The "Title Page" means, for a printed book, the title page itself, plus such following pages as are needed to hold, legibly, the material this License requires to appear in the title page. For works in formats which do not have any title page as such, "Title Page" means the text near the most prominent appearance of the work's title, preceding the beginning of the body of the text.

The "publisher" means any person or entity that distributes copies of the Document to the public.

A section "Entitled XYZ" means a named subunit of the Document whose title either is precisely XYZ or contains XYZ in parentheses following text that translates XYZ in another language. (Here XYZ stands for a specific section name mentioned below, such as "Acknowledgements", "Dedications", "Endorsements", or "History".) To "Preserve the Title" of such a section when you modify the Document means that it remains a section "Entitled XYZ" according to this definition.

The Document may include Warranty Disclaimers next to the notice which states that this License applies to the Document. These Warranty Disclaimers are considered to be included by reference in this License, but only as regards disclaiming warranties; any other implication that these Warranty Disclaimers may have is void and has no effect on the meaning of this License. 2. VERBATIM COPYING

You may copy and distribute the Document in any medium, either commercially or noncommercially, provided that this License, the copyright notices, and the license notice saying this License applies to the Document are reproduced in all copies, and that you add no other conditions whatsoever to those of this License. You may not use technical measures to obstruct or control the reading or further copying of the copies you make or distribute. However, you may accept compensation in exchange for copies. If you distribute a large enough number of copies you must also follow the conditions in section 3.

You may also lend copies, under the same conditions stated above, and you may publicly display copies. 3. COPYING IN QUANTITY

If you publish printed copies (or copies in media that commonly have printed covers) of the Document, numbering more than 100, and the Document's license notice requires Cover Texts, you

\* A. Use in the Title Page (and on the covers, if any) a title distinct from that of the Document, and from those of previous versions (which should, if there were any, be listed in the History section of the Document). You may use the same title as a previous version if the original publisher of that version gives permission. \* B. List on the Title



Page, as authors, one or more persons or entities responsible for authorship of the modifications in the Modified Version, together with at least five of the principal authors of the Document (all of its principal authors, if it has fewer than five), unless they release you from this requirement. \* C. State on the Title page the name of the publisher of the Modified Version, as the publisher. \* D. Preserve all the copyright notices of the Document. \* E. Add an appropriate copyright notice for your modifications adjacent to the other copyright notices. \* F. Include, immediately after the copyright notices, a license notice giving the public permission to use the Modified Version under the terms of this License, in the form shown in the Addendum below. \* G. Preserve in that license notice the full lists of Invariant Sections and required Cover Texts given in the Document's license notice. \* H. Include an unaltered copy of this License. \* I. Preserve the section Entitled "History", Preserve its Title, and add to it an item stating at least the title, year, new authors, and publisher of the Modified Version as given on the Title Page. If there is no section Entitled "History" in the Document, create one stating the title, year, authors, and publisher of the Document as given on its Title Page, then add an item describing the Modified Version as stated in the previous sentence. \* J. Preserve the network location, if any, given in the Document for public access to a Transparent copy of the Document, and likewise the network locations given in the Document for previous versions it was based on. These may be placed in the "History" section. You may omit a network location for a work that was published at least four years before the Document itself, or if the original publisher of the version it refers to gives permission. \* K. For any section Entitled "Acknowledgements" or "Dedications", Preserve the Title of the section, and preserve in the section all the substance and tone of each of the contributor acknowledgements and/or dedications given therein. \* L. Preserve all the Invariant Sections of the Document, unaltered in their text and in their titles. Section numbers or the equivalent are not considered part of the section titles. \* M. Delete any section Entitled "Endorsements". Such a section may not be included in the Modified Version. \* N. Do not retile any existing section to be Entitled "Endorsements" or to conflict in title with any Invariant Section. \* O. Preserve any Warranty Disclaimers.

If the Modified Version includes new front-matter sections or appendices that qualify as Secondary Sections and contain no material copied from the Document, you may at your option designate some or all of these sections as invariant. To do this, add their titles to the list of Invariant Sections in the Modified Version's license notice. These titles must be distinct from any other section titles.

You may add a section Entitled "Endorsements", provided it contains nothing but endorsements of your Modified Version by various parties—for example, statements of peer review or that the text has been approved by an organization as the authoritative definition of a standard.

You may add a passage of up to five words as a Front-Cover Text, and a passage of up to 25 words as a Back-Cover Text, to the end of the list of Cover Texts in the Modified Version. Only one passage of Front-Cover Text and one of Back-Cover Text may be added by (or through arrangements made by) any one entity. If the Document already includes a cover text for the same cover, previously added by you or by arrangement made by the same entity you are acting on behalf of, you may not add an-

other; but you may replace the old one, on explicit permission from the previous publisher that added the old one.

The author(s) and publisher(s) of the Document do not by this License give permission to use their names for publicity for or to assert or imply endorsement of any Modified Version. 5. COMBINING DOCUMENTS

You may combine the Document with other documents released under this License, under the terms defined in section 4 above for modified versions, provided that you include in the combination all of the Invariant Sections of all of the original documents, unmodified, and list them all as Invariant Sections of your combined work in its license notice, and that you preserve all their Warranty Disclaimers.

The combined work need only contain one copy of this License, and multiple identical Invariant Sections may be replaced with a single copy. If there are multiple Invariant Sections with the same name but different contents, make the title of each such section unique by adding at the end of it, in parentheses, the name of the original author or publisher of that section if known, or else a unique number. Make the same adjustment to the section titles in the list of Invariant Sections in the license notice of the combined work.

In the combination, you must combine any sections Entitled "History" in the various original documents, forming one section Entitled "History"; likewise combine any sections Entitled "Acknowledgements", and any sections Entitled "Dedications". You must delete all sections Entitled "Endorsements". 6. COLLECTIONS OF DOCUMENTS

You may make a collection consisting of the Document and other documents released under this License, and replace the individual copies of this License in the various documents with a single copy that is included in the collection, provided that you follow the rules of this License for verbatim copying of each of the documents in all other respects.

You may extract a single document from such a collection, and distribute it individually under this License, provided you insert a copy of this License into the extracted document, and follow this License in all other respects regarding verbatim copying of that document. 7. AGGREGATION WITH INDEPENDENT WORKS

A compilation of the Document or its derivatives with other separate and independent documents or works, in or on a volume of a storage or distribution medium, is called an "aggregate" if the copyright resulting from the compilation is not used to limit the legal rights of the compilation's users beyond what the individual works permit. When the Document is included in an aggregate, this License does not apply to the other works in the aggregate which are not themselves derivative works of the Document.

If the Cover Text requirement of section 3 is applicable to these copies of the Document, then if the Document is less than one half of the entire aggregate, the Document's Cover Texts may be placed on covers that bracket the Document within the aggregate, or the electronic equivalent of covers if the Document is in electronic form. Otherwise they must appear on printed covers that bracket the whole aggregate. 8. TRANSLATION

The "Corresponding Application Code" for a Combined Work means the object code and/or source code for the Application, including any data and utility programs needed for reproducing the Combined Work from the Application, but excluding the System Libraries of the Combined Work. 1. Exception to Section 3 of the GNU GPL.

You may convey a covered work under sections 3 and 4 of this License without being bound by section 3 of the GNU GPL. 2. Conveying Modified Versions.

If you modify a copy of the Library, and, in your modifications, a facility refers to a function or data to be supplied by an Application that uses the facility (other than as an argument passed when the facility is invoked), then you may convey a copy of the modified version:

\* a) under this License, provided that you make a good faith effort to ensure that, in the event an Application does not supply the function or data, the facility still operates, and performs whatever part of its purpose remains meaningful, or \* b) under the GNU GPL, with none of the additional permissions of this License applicable to that copy.

3. Object Code Incorporating Material from Library Header Files.

The object code form of an Application may incorporate material from a header file that is part of the Library. You may convey such object code under terms of your choice, provided that, if the incorporated material is not limited to numerical parameters, data structure layouts and accessors, or small macros, inline functions and templates (ten or fewer lines in length), you do both of the following:

\* a) Give prominent notice with each copy of the object code that the Library is used in it and that the Library and its use are covered by this License. \* b) Accompany the object code with a copy of the GNU GPL and this license document.

Translation is considered a kind of modification, so you may distribute translations of the Document under the terms of section 4. Replacing Invariant Sections with translations requires special permission from their copyright holders, but you may include translations of some or all Invariant Sections in addition to the original versions of these Invariant Sections. You may include a translation of this License, and all the license notices in the Document, and any Warranty Disclaimers, provided that you also include the original English version of this License and the original versions of those notices and disclaimers. In case of a disagreement between the translation and the original version of this License or a notice or disclaimer, the original version will prevail.

If a section in the Document is Entitled "Acknowledgements", "Dedications", or "History", the requirement (section 4) to Preserve its Title (section 1) will typically require changing the actual title. 9. TERMINATION

You may not copy, modify, sublicense, or distribute the Document except as expressly provided under this License. Any attempt otherwise to copy, modify, sublicense, or distribute it is void, and will automatically terminate your rights under this License.

However, if you cease all violation of this License, then your license from a particular copyright holder is reinstated (a) provisionally, unless and until the copyright holder explicitly and finally terminates your license, and (b) permanently, if the copyright holder fails to notify you of the violation by some reasonable means prior to 60 days after the cessation.

Moreover, your license from a particular copyright holder is reinstated permanently if the copyright holder notifies you of the violation by some reasonable means, this is the first time you have received notice of violation of this License (for any work) from that copyright holder, and you cure the violation prior to 30 days after your receipt of the notice.

Termination of your rights under this section does not terminate the licenses of parties who have received copies or rights from you under this License. If your rights have been terminated and not permanently reinstated, receipt of a copy of some or all of the same material does not give you any rights to use it. 10. FUTURE REVISIONS OF THIS LICENSE

The Free Software Foundation may publish new, revised versions of the GNU Free Documentation License from time to time. Such new versions will be similar in spirit to the present version, but may differ in detail to address new problems or concerns. See <http://www.gnu.org/copyleft/>.

Each version of the License is given a distinguishing version number. If the Document specifies that a particular numbered version of this License "or any later version" applies to it, you have the option of following the terms and conditions either of that specified version or of any later version that has been published (not as a draft) by the Free Software Foundation. If the Document does not specify a version number of this License, you may choose any version ever published (not as a draft) by the Free Software Foundation. If the Document specifies that a proxy can decide which future versions of

4. Combined Works.

You may convey a Combined Work under terms of your choice that, taken together, effectively do not restrict modification of the portions of the Library contained in the Combined Work and reverse engineering for debugging such modifications, if you also do each of the following:

\* a) Give prominent notice with each copy of the Combined Work that the Library is used in it and that the Library and its use are covered by this License. \* b) Accompany the Combined Work with a copy of the GNU GPL and this license document. \* c) For a Combined Work that displays copyright notices during execution, include the copyright notice for the Library among these notices, as well as a reference directing the user to the copies of the GNU GPL and this license document. \* d) Do one of the following: o 0) Convey the Minimal Corresponding Source under the terms of this License, and the Corresponding Application Code in a form suitable for, and under terms that permit, the user to recombine or relink the Application with a modified version of the Linked Version to produce a modified Combined Work, in the manner specified by section 6 of the GNU GPL for conveying Corresponding Source. o 1) Use a suitable shared library mechanism for linking with the Library. A suitable mechanism is one that (a) uses at run time a copy of the Library already present on the user's computer system, and (b) will operate properly with a modified version of the Library that is interface-compatible with the Linked Version. \* e) Provide Installation Information, but only if you would otherwise be required to provide such information under section 6 of the GNU GPL, and only to the extent that such information is necessary to install and execute a modified version of the Combined Work produced by recombining or relinking the Application with a modified version of the Linked Version. (If you use option 4d0, the Installation Information must accompany the Minimal Corresponding Source and Corresponding Application Code. If you use option 4d1, you must provide the Installation Information in the manner specified by section 6 of the GNU GPL for conveying Corresponding Source.)

this License can be used, that proxy's public statement of acceptance of a version permanently authorizes you to choose that version for the Document. 11. RELICENSING

"Massive Multiauthor Collaboration Site" (or "MMC Site") means any World Wide Web server that publishes copyrightable works and also provides prominent facilities for anybody to edit those works. A public wiki that anybody can edit is an example of such a server. A "Massive Multiauthor Collaboration" (or "MMC") contained in the site means any set of copyrightable works thus published on the MMC site.

"CC-BY-SA" means the Creative Commons Attribution-Share Alike 3.0 license published by Creative Commons Corporation, a not-for-profit corporation with a principal place of business in San Francisco, California, as well as future copyleft versions of that license published by that same organization.

"Incorporate" means to publish or republish a Document, in whole or in part, as part of another Document.

An MMC is "eligible for relicensing" if it is licensed under this License, and if all works that were first published under this License somewhere other than this MMC, and subsequently incorporated in whole or in part into the MMC, (1) had no cover texts or invariant sections, and (2) were thus incorporated prior to November 1, 2008.

The operator of an MMC Site may republish an MMC contained in the site under CC-BY-SA on the same site at any time before August 1, 2009, provided the MMC is eligible for relicensing. ADDENDUM: How to use this License for your documents

To use this License in a document you have written, include a copy of the License in the document and put the following copyright and license notices just after the title page:

Copyright (C) YEAR YOUR NAME. Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.3 or any later version published by the Free Software Foundation; with no Invariant Sections, no Front-Cover Texts, and no Back-Cover Texts. A copy of the license is included in the section entitled "GNU Free Documentation License".

If you have Invariant Sections, Front-Cover Texts and Back-Cover Texts, replace the "with ... Texts." line with this:

with the Invariant Sections being LIST THEIR TITLES, with the Front-Cover Texts being LIST, and with the Back-Cover Texts being LIST.

If you have Invariant Sections without Cover Texts, or some other combination of the three, merge those two alternatives to suit the situation.

If your document contains nontrivial examples of program code, we recommend releasing these examples in parallel under your choice of free software license, such as the GNU General Public License, to permit their use in free software.

5. Combined Libraries.

You may place library facilities that are a work based on the Library side by side in a single library together with other library facilities that are not Applications and are not covered by this License, and convey such a combined library under terms of your choice, if you do both of the following:

\* a) Accompany the combined library with a copy of the same work based on the Library, uncombined with any other library facilities, conveyed under the terms of this License. \* b) Give prominent notice with the combined library that part of it is a work based on the Library, and explaining where to find the accompanying uncombined form of the same work.

6. Revised Versions of the GNU Lesser General Public License.

The Free Software Foundation may publish revised and/or new versions of the GNU Lesser General Public License from time to time. Such new versions will be similar in spirit to the present version, but may differ in detail to address new problems or concerns.

Each version is given a distinguishing version number. If the Library as you received it specifies that a certain numbered version of the GNU Lesser General Public License "or any later version" applies to it, you have the option of following the terms and conditions either of that published version or of any later version published by the Free Software Foundation. If the Library as you received it does not specify a version number of the GNU Lesser General Public License, you may choose any version of the GNU Lesser General Public License ever published by the Free Software Foundation.

If the Library as you received it specifies that a proxy can decide whether future versions of the GNU Lesser General Public License shall apply, that proxy's public statement of acceptance of any version is permanent authorization for you to choose that version for the Library.

## 30.3 GNU Lesser General Public License

GNU LESSER GENERAL PUBLIC LICENSE

Version 3, 29 June 2007

Copyright © 2007 Free Software Foundation, Inc. <<http://fsf.org/>>

Everyone is permitted to copy and distribute verbatim copies of this license document, but changing it is not allowed.

This version of the GNU Lesser General Public License incorporates the terms and conditions of version 3 of the GNU General Public License, supplemented by the additional permissions listed below. 0. Additional Definitions.

As used herein, "this License" refers to version 3 of the GNU Lesser General Public License, and the "GNU GPL" refers to version 3 of the GNU General Public License.

"The Library" refers to a covered work governed by this license, other than an Application or a Combined Work as defined below.

An "Application" is any work that makes use of an interface provided by the Library, but which is not otherwise based on the Library. Defining a subclass of a class defined by the Library is deemed a mode of using an interface provided by the Library.

A "Combined Work" is a work produced by combining or linking an Application with the Library. The particular version of the Library with which the Combined Work was made is also called the "Linked Version".

The "Minimal Corresponding Source" for a Combined Work means the Corresponding Source for the Combined Work, excluding any source code for portions of the Combined Work that, considered in isolation, are based on the Application, and not on the Linked Version.