

# Counting unlabelled topologies and transitive relations

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## Abstract

We enumerate isomorphism classes of several types of transitive relations (equivalently, finite topologies) up to 15 or 16 points.

## 1 Introduction

Pfeiffer [3] presented a classification of various types of order relations and determined their numbers up to 12 points. In this paper, we extend the counts to 15 or 16 points. We defer to [3] for historical survey, and only give enough background to precisely define the objects we are counting.

We consider only directed graphs (digraphs) that do not have multiple edges but may have up to one loop per point. A *transitive relation digraph (TRD)* is a digraph such that presence of edges  $(x, y)$  and  $(y, z)$  implies that  $(x, z)$  is also present (even if  $x, y, z$  are not distinct). (We cannot use the more natural term “transitive digraph” since its most common definition does not allow loops; this has the unfortunate consequence that transitive digraphs don’t correspond to transitive relations.)

A *strong component* of a digraph is a maximal set of points  $P$  such that there is a directed path within  $P$  from  $x$  to  $y$  for each pair  $x, y \in P$ . (This permits the zero-length path from  $x \in P$  to itself.)

If a strong component of a TRD has only one point, there may or may not be a loop on that point. However, larger strong components of a TRD have loops on every point and edges in both directions between each pair of points. If the strong components of a TRD are contracted to single points, the result is a TRD that has no directed cycles apart from loops.

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Two digraphs are *isomorphic* if there is a bijection between their point-sets that induces a bijection between their edge-sets. Thus we can refer to *isomorphism classes* of digraphs.

Of course a digraph can be viewed as representing some kind of order relation  $\leq$ , where  $x \leq y$  iff there is an edge  $(x, y)$ . In the language and notation of Pfeiffer, we have the following correspondences:

- A *transitive relation* corresponds to an arbitrary TRD. Let  $t(n)$  be the number of isomorphism classes of transitive relations.
- A *quasi-order* corresponds to a TRD such that every single-point strong component has a loop. Let  $q(n)$  be the number of isomorphism classes of quasi-orders.
- A *soft order* corresponds to a TRD whose strong components each have only one point (with or without loop). Let  $s(n)$  be the number of isomorphism classes of soft orders.
- A *partial order* corresponds to an TRD whose strong components are all single points with loops. Because we can remove and replace the loops in a unique fashion, the counts here are the same as for acyclic TRDs. Let  $p(n)$  be the number of isomorphism classes of partial orders.

There are also relationships with finite topologies: general topologies are in bijective correspondence with quasi-orders, and  $T_0$ -topologies are in bijective correspondence with partial orders. These bijections preserve isomorphism, so we are also counting isomorphism classes of topologies.

For enumerative purposes, we will define some specialized numbers.

- $q(n, m)$  is the number of isomorphism classes of TRDs with  $n$  points and  $m$  strong components, such that each single-point strong component has a loop.
- $t(n, m)$  is the number of isomorphism classes of TRDs with  $n$  points and  $m$  strong components (with single-point strong components having a loop or not).

In terms of these specialized numbers, we have

$$\begin{aligned} q(n) &= \sum_{m=1}^n q(n, m), & t(n) &= \sum_{m=1}^n t(n, m), \\ p(n) &= q(n, n), & s(n) &= t(n, n). \end{aligned}$$

## 2 Computing $t(n, m)$ and $q(n, m)$ for small $n$

Our main tool is a program which generates non-isomorphic partially ordered sets (*posets*). In our paper [1], we gave values of  $p(n)$  up to  $n = 16$  based on output from that program.

Given a poset  $P$ , we can make a TRD  $G$  by replacing each point by a directed clique (perhaps of a single point). Such directed cliques become the strong components of  $G$ . If points  $v, w$  of  $P$  become strong components  $c_v, c_w$  of  $G$ , then an edge  $(v, w)$  of  $P$  becomes

the set of all possible edges from  $c_v$  to  $c_w$  in  $G$ . Clearly non-isomorphic posets lead to non-isomorphic TRDs, since  $G$  uniquely determines  $P$ . The only remaining issue is that some of the TRDs made from  $P$  may be isomorphic due to symmetries (automorphisms) of  $P$ .

Given a poset  $P$ , we can represent its expansion to a TRD by assigning a positive integer *weight* to each point. This weight corresponds to the size of the directed clique that the point will be expanded to. We also consider *extended weights* where there are two types of weight with value 1 (corresponding to loop and non-loop).

For any permutation  $\gamma$ , define

$$C_\gamma(x) = \prod_{i=1}^k (1 - x^{a_i})^{-1}$$

$$C'_\gamma(x) = \prod_{i=1}^k (1 + (1 - x^{a_i})^{-1}),$$

where  $a_1, a_2, \dots, a_k$  are the cycle lengths of  $\gamma$ .

**Theorem 1.** *Let  $1 \leq m \leq n$ . Then*

$$q(n, m) = [x^{n-m}] \sum_P \left( |\text{Aut}(P)|^{-1} \sum_{\gamma \in \text{Aut}(P)} C_\gamma(x) \right)$$

$$t(n, m) = [x^{n-m}] \sum_P \left( |\text{Aut}(P)|^{-1} \sum_{\gamma \in \text{Aut}(P)} C'_\gamma(x) \right),$$

where the first sum in each case is over isomorphism class representatives  $P$  of posets on  $m$  points,  $\text{Aut}(P)$  is the automorphism group of  $P$ , and  $[x^{n-m}]$  denotes extraction of the coefficient of  $x^{n-m}$ .

*Proof.* We prove the formula for  $q(n, m)$ ; the other is similar. Let  $P$  be a poset with  $m$  points. By the Frobenius-Burnside Lemma, the number of equivalence classes under  $\text{Aut}(P)$  of weight assignments with total weight  $n$  is the average over  $\gamma \in \text{Aut}(P)$  of the number  $w_n(\gamma)$  of such weight assignments fixed by  $\gamma$ .

Consider one such element  $\gamma \in \text{Aut}(P)$  with cycles of length  $a_1, a_2, \dots, a_k$ . A weight assignment is fixed by  $\gamma$  iff it is constant on each cycle of  $\gamma$ , so  $w_n(\gamma)$  is the coefficient of  $x^n$  in

$$\prod_{i=1}^k (x^{a_i} + x^{2a_i} + x^{3a_i} + \dots) = x^m C_\gamma(x).$$

The result now follows by averaging over  $\gamma$ . □

As mentioned earlier,  $q(n, n) = p(n)$ . We can also identify  $q(n, n-1)$  as the total number of orbits of  $\text{Aut}(P)$  over all posets on  $n-1$  points. Using our previously-computed value of  $p(16)$ , this enabled us to determine  $q(n, m)$  for  $n \leq 16$  and  $t(n, m)$  for  $n \leq 15$  by computing the automorphism groups of all the posets up to 15 points. Except in

some simple (but common) situations where the poset generator had already determined  $\text{Aut}(P)$ , this was computed using `nauty` [2].

The resulting values are given in Tables 1–3. The programs were run twice in case of machine errors. We also successfully recovered the numbers given by Pfeiffer [3] and the values of  $p(n)$  up to  $n = 15$  given by Brinkmann and McKay [1].

| $n$ | $q(n)$           | $s(n)$              | $t(n)$              |
|-----|------------------|---------------------|---------------------|
| 1   | 1                | 2                   | 2                   |
| 2   | 3                | 7                   | 8                   |
| 3   | 9                | 32                  | 39                  |
| 4   | 33               | 192                 | 242                 |
| 5   | 139              | 1490                | 1895                |
| 6   | 718              | 15067               | 19051               |
| 7   | 4535             | 198296              | 246895              |
| 8   | 35979            | 3398105             | 4145108             |
| 9   | 363083           | 75734592            | 90325655            |
| 10  | 4717687          | 2191591226          | 2555630036          |
| 11  | 79501654         | 82178300654         | 93810648902         |
| 12  | 1744252509       | 3984499220967       | 4461086120602       |
| 13  | 49872339897      | 249298391641352     | 274339212258846     |
| 14  | 1856792610995    | 20089200308020179   | 21775814889230580   |
| 15  | 89847422244493   | 2081351202770089728 | 2226876304576948549 |
| 16  | 5637294117525695 |                     |                     |

Table 1: Quasi-orders, soft orders, and transitive relations

## References

- [1] G. Brinkmann and B. D. McKay, Posets on up to 16 points, *Order*, **19** (2002) 147–179.
- [2] B. McKay, `nauty` – a program for graph isomorphism and automorphism, available at <http://cs.anu.edu.au/~bdm/nauty/>.
- [3] G. Pfeiffer, Counting transitive relations, *J. Integer Seq.*, **7** (2004) 11pp.

|   |   |        |    |    |             |    |    |                  |
|---|---|--------|----|----|-------------|----|----|------------------|
| 1 | 1 | 1      | 10 | 1  | 1           | 14 | 1  | 1                |
| 2 | 1 | 1      | 10 | 2  | 14          | 14 | 2  | 20               |
| 2 | 2 | 2      | 10 | 3  | 120         | 14 | 3  | 256              |
| 3 | 1 | 1      | 10 | 4  | 849         | 14 | 4  | 2790             |
| 3 | 2 | 3      | 10 | 5  | 5123        | 14 | 5  | 27637            |
| 3 | 3 | 5      | 10 | 6  | 27439       | 14 | 6  | 260840           |
| 4 | 1 | 1      | 10 | 7  | 127965      | 14 | 7  | 2385741          |
| 4 | 2 | 5      | 10 | 8  | 501591      | 14 | 8  | 21304106         |
| 4 | 3 | 11     | 10 | 9  | 1487301     | 14 | 9  | 184860968        |
| 4 | 4 | 16     | 10 | 10 | 2567284     | 14 | 10 | 1535230287       |
| 5 | 1 | 1      | 11 | 1  | 1           | 14 | 11 | 11832383054      |
| 5 | 2 | 6      | 11 | 2  | 15          | 14 | 12 | 80018898562      |
| 5 | 3 | 22     | 11 | 3  | 150         | 14 | 13 | 425004096962     |
| 5 | 4 | 47     | 11 | 4  | 1193        | 14 | 14 | 1338193159771    |
| 5 | 5 | 63     | 11 | 5  | 8406        | 15 | 1  | 1                |
| 6 | 1 | 1      | 11 | 6  | 53443       | 15 | 2  | 21               |
| 6 | 2 | 8      | 11 | 7  | 309719      | 15 | 3  | 299              |
| 6 | 3 | 35     | 11 | 8  | 1599822     | 15 | 4  | 3525             |
| 6 | 4 | 113    | 11 | 9  | 7040921     | 15 | 5  | 38420            |
| 6 | 5 | 243    | 11 | 10 | 23738557    | 15 | 6  | 401602           |
| 6 | 6 | 318    | 11 | 11 | 46749427    | 15 | 7  | 4124565          |
| 7 | 1 | 1      | 12 | 1  | 1           | 15 | 8  | 41997196         |
| 7 | 2 | 9      | 12 | 2  | 17          | 15 | 9  | 424381067        |
| 7 | 3 | 52     | 12 | 3  | 182         | 15 | 10 | 4221560826       |
| 7 | 4 | 213    | 12 | 4  | 1632        | 15 | 11 | 40603719604      |
| 7 | 5 | 682    | 12 | 5  | 13011       | 15 | 12 | 365485856203     |
| 7 | 6 | 1533   | 12 | 6  | 96226       | 15 | 13 | 2906408331804    |
| 7 | 7 | 2045   | 12 | 7  | 664467      | 15 | 14 | 18255153928204   |
| 8 | 1 | 1      | 12 | 8  | 4268404     | 15 | 15 | 68275077901156   |
| 8 | 2 | 11     | 12 | 9  | 24858756    | 16 | 1  | 1                |
| 8 | 3 | 71     | 12 | 10 | 124784466   | 16 | 2  | 23               |
| 8 | 4 | 367    | 12 | 11 | 484673601   | 16 | 3  | 343              |
| 8 | 5 | 1503   | 12 | 12 | 1104891746  | 16 | 4  | 4396             |
| 8 | 6 | 4989   | 13 | 1  | 1           | 16 | 5  | 52033            |
| 8 | 7 | 12038  | 13 | 2  | 18          | 16 | 6  | 597502           |
| 8 | 8 | 16999  | 13 | 3  | 218         | 16 | 7  | 6804011          |
| 9 | 1 | 1      | 13 | 4  | 2154        | 16 | 8  | 77823441         |
| 9 | 2 | 12     | 13 | 5  | 19320       | 16 | 9  | 897440095        |
| 9 | 3 | 95     | 13 | 6  | 162404      | 16 | 10 | 10402896209      |
| 9 | 4 | 570    | 13 | 7  | 1304373     | 16 | 11 | 119938485210     |
| 9 | 5 | 2923   | 13 | 8  | 10009358    | 16 | 12 | 1348204100877    |
| 9 | 6 | 12591  | 13 | 9  | 72589838    | 16 | 13 | 14281079724622   |
| 9 | 7 | 44842  | 13 | 10 | 483531684   | 16 | 14 | 134410089884839  |
| 9 | 8 | 118818 | 13 | 11 | 2803234294  | 16 | 15 | 1003992754517006 |
| 9 | 9 | 183231 | 13 | 12 | 12677658783 | 16 | 16 | 4483130665195087 |
|   |   |        | 13 | 13 | 33823827452 |    |    |                  |

Table 2: Values of  $q(n, m)$  for various  $n, m$

|   |   |         |    |    |               |    |    |                     |
|---|---|---------|----|----|---------------|----|----|---------------------|
| 1 | 1 | 2       | 9  | 1  | 1             | 13 | 1  | 1                   |
| 2 | 1 | 1       | 9  | 2  | 15            | 13 | 2  | 21                  |
| 2 | 2 | 7       | 9  | 3  | 174           | 13 | 3  | 335                 |
| 3 | 1 | 1       | 9  | 4  | 1769          | 13 | 4  | 4852                |
| 3 | 2 | 6       | 9  | 5  | 17694         | 13 | 5  | 70797               |
| 3 | 3 | 32      | 9  | 6  | 170391        | 13 | 6  | 1066041             |
| 4 | 1 | 1       | 9  | 7  | 1577763       | 13 | 7  | 16906476            |
| 4 | 2 | 8       | 9  | 8  | 12823256      | 13 | 8  | 282183725           |
| 4 | 3 | 41      | 9  | 9  | 75734592      | 13 | 9  | 4922404711          |
| 4 | 4 | 192     | 10 | 1  | 1             | 13 | 10 | 87597193530         |
| 5 | 1 | 1       | 10 | 2  | 17            | 13 | 11 | 1521294651297       |
| 5 | 2 | 9       | 10 | 3  | 207           | 13 | 12 | 23426706135708      |
| 5 | 3 | 63      | 10 | 4  | 2380          | 13 | 13 | 249298391641352     |
| 5 | 4 | 332     | 10 | 5  | 26352         | 14 | 1  | 1                   |
| 5 | 5 | 1490    | 10 | 6  | 294156        | 14 | 2  | 23                  |
| 6 | 1 | 1       | 10 | 7  | 3243880       | 14 | 3  | 381                 |
| 6 | 2 | 11      | 10 | 8  | 34592661      | 14 | 4  | 5964                |
| 6 | 3 | 84      | 10 | 9  | 325879156     | 14 | 5  | 93159               |
| 6 | 4 | 583     | 10 | 10 | 2191591226    | 14 | 6  | 1521101             |
| 6 | 5 | 3305    | 11 | 1  | 1             | 14 | 7  | 26315265            |
| 6 | 6 | 15067   | 11 | 2  | 18            | 14 | 8  | 486434324           |
| 7 | 1 | 1       | 11 | 3  | 248           | 14 | 9  | 9568317752          |
| 7 | 2 | 12      | 11 | 4  | 3068          | 14 | 10 | 197959166598        |
| 7 | 3 | 112     | 11 | 5  | 37919         | 14 | 11 | 4196507844123       |
| 7 | 4 | 883     | 11 | 6  | 472519        | 14 | 12 | 86930341478767      |
| 7 | 5 | 6537    | 11 | 7  | 6031290       | 14 | 13 | 1595279690032943    |
| 7 | 6 | 41054   | 11 | 8  | 77251333      | 14 | 14 | 20089200308020179   |
| 7 | 7 | 198296  | 11 | 9  | 960789368     | 15 | 1  | 1                   |
| 8 | 1 | 1       | 11 | 10 | 10587762484   | 15 | 2  | 24                  |
| 8 | 2 | 14      | 11 | 11 | 82178300654   | 15 | 3  | 435                 |
| 8 | 3 | 139     | 12 | 1  | 1             | 15 | 4  | 7190                |
| 8 | 4 | 1294    | 12 | 2  | 20            | 15 | 5  | 120514              |
| 8 | 5 | 11096   | 12 | 3  | 288           | 15 | 6  | 2110489             |
| 8 | 6 | 90758   | 12 | 4  | 3911          | 15 | 7  | 39534004            |
| 8 | 7 | 643701  | 12 | 5  | 52415         | 15 | 8  | 798048843           |
| 8 | 8 | 3398105 | 12 | 6  | 724866        | 15 | 9  | 17393487215         |
|   |   |         | 12 | 7  | 10377573      | 15 | 10 | 406531057397        |
|   |   |         | 12 | 8  | 153952401     | 15 | 11 | 10041016241522      |
|   |   |         | 12 | 9  | 2314756589    | 15 | 12 | 254873608116034     |
|   |   |         | 12 | 10 | 33895893064   | 15 | 13 | 6326335208572503    |
|   |   |         | 12 | 11 | 440211138507  | 15 | 14 | 138933427209562650  |
|   |   |         | 12 | 12 | 3984499220967 | 15 | 15 | 2081351202770089728 |

Table 3: Values of  $t(n, m)$  for various  $n, m$