On the use of series in Hindu mathematics

INTRODUCTION.—The object of the present paper is to give an account of the use of series in Hindu Mathematics. For this purpose the Hindu literature, mathematical as well as non-mathematical, has been examined. Particular instances of the arithmetic and geometric series have been found to occur in Vedic literature as early as 2000 B.C. From Jaina literature it appears that the Hindus were in possession of the formulae for the sum of the arithmetic and the geometric series as early as the fourth century B.C., or earlier.

The earliest known Hindu work on mathematics available to us is a mutilated copy of what is known as the *Bakhshālī Manuscript* which contains a work on *Pātīgaṇita* composed about 300 A.D. (1) Amongst other works which contain an account of series may be mentioned the *Āryabhaṭīya* (499 A.D.), the *Brāhma-sphuṭa Siddhānta* (628 A.D.), the *Triśatikā* (750 A.D.), the *Gaṇita-sāra-saṁgraha* (850 A.D.), the *Mahāsiddhānta* (950 A.D.), the *Līlāvatī* (1150 A.D.), the *Gaṇita Kaumudī* (1350 A.D.), etc. In all these works series is treated as one of the fundamental operations *(śreḍhī vyavahāra)*, and a separate section is generally devoted to the formulae and problems relating to series. In Europe too series was looked upon as one of the fundamental operations (2), evidently due to Hindu influence through the Arabs. Besides the arithmetic and geometric series a number of other types of series, e.g., the series of sums, the series of the squares or cubes of the natural numbers, the arithmetico-geometric series, the series of polygonal or figurate numbers etc. occur in some

(1) The manuscript was edited and published by G. R. KAYE, who assigns a much later date to the work. KAYE's conclusions are unreliable. See B. DATTA, *Bull. Cal. Math. Soc.*, Vol. XXI (1929), pp. 1-60.

(2) See SMITH, *History of Mathematics*, Vol. II, p. 497.

of the works mentioned above. There is, however, no mention of the harmonic series.

Evidence of the use of infinite series in India is found in the fifteenth century. The infinite series appear to have been obtained in connection with the evaluation of the trigonometrical functions and the computation of π. Noteworthy achievements of the Hindu mathematicians in this theory are the expansions of Sin x, Cos x, and Tan ^{-1}x in powers of x, obtained long before they were known in Europe or anywhere else.

No account of the use of series in Hindu mathematics has been published by any previous writer and as such, I hope, the present discussion will be found useful by the historian of mathematics.

ORIGIN AND EARLY HISTORY.—Series of numbers developing according to certain laws have attracted the attention of people in all times and climes. The Egyptians are known to have used the arithmetic series about 1550 B.C. (3) Arithmetic as well as geometric series are found in the Vedic literature of the Hindus (*c.* 2000 B.C.). In the *Taittirīya Saṁhitā* (4) we find the following series :

(i) 1, 3, 5,... 19, 29,... 99.
(ii) 2, 4, 6,... 20.
(iii) 4, 8, 12,...
(iv) 10, 20, 30,...
(v) 1, 3, 5,... 33.

In the *Vājasaneyī Saṁhitā* (5), we find the *yugma* (even) and *ayugma* (odd) series :

(vi) 4, 8, 12, 16,... 48.
(vii) 1, 3, 5, 7,... 31.

The *Pañcaviṁśa Brāhmaṇa* (6) has the following geometric series :

(viii) 12, 24, 48, 96,... 196608, 393216.

Another geometric series occurs in the *Dīgha Nikāya* (7). It is :

(3) In the Ahmes Papyrus. *Cf.* PEET, *Rhind Papyrus*, p. 78; SMITH, *l.c.*

(4) See vii. 2. 12-17; iv. 3. 10.

(5) See xvii. 24. 25.

(6) xviii. 3. Compare also *Lāṭyāyana Śrauta sūtra*, viii. 10. 1 *et seq. Kātyāyana Śrauta sūtra*, xxii. 9. 1-6.

(7) T. W. RHYS DAVIDS, *Dialogues of the Buddha*, III, (1921), pp. 70-72.

(ix) 10, 20, 40,... 80000.

The Hindus must have obtained the formula for the sum of an arithmetic series at a very early date, but when exactly they did so can not be said with certainty. It is, however, definite that in the fifth century B.C. they were in possession of the formula for the sum of the series of natural numbers; for in the *Bṛhaddevatā* (8) (500-400 B.C.) we have the result

$$2 + 3 + 4 + \ldots + 1000 = 500499.$$

In the *Kalpa-sūtra* of BHADRABĀHU (*c.* 350 B.C.) we have the sum of the following geometric series

$$1 + 2 + 4 + \ldots + 8192$$

given correctly as 16383, showing that the Hindus possessed some method of finding the sum of the geometric series in the fourth century B.C.

KINDS OF SERIES.—It thus appears that the Hindus studied the arithmetic and the geometric series at a very early date. ĀRYABHAṬA I (499), BRAHMAGUPTA (628) and other posterior writers considered also the cases of the sum of the sums, the squares and the cubes of the natural numbers. MAHĀVĪRA (850) gave a rule for the summation of an interesting arithmetico-geometric series, *viz.*,

$$\sum_{1}^{n} t_m$$

where $t_1 = a$ and $t_m = rt_{m-1} \pm b$. $(m \geqslant 2)$; and NĀRĀYAṆA (1356) considered the summation of the figurate numbers of higher orders.

TECHNICAL TERMS.—The Sanskrit term for a series is *śreḍhī* meaning literally " progression," " any set or succession of distinct things," or śreṇī (or śreṇi), literally " line," " row," " series " or " succession "; hence in relation to mathematics it implies " a series or progression of numbers." Thus it is clear that the modern terms progression and series are analogous to the Hindu terms, and they seem to have been adopted in the West under Hindu influence, in preference to the Greek term ἔκθεσις (ekthesis) which literally means a setting forth. The Sanskrit name

(8) *Bṛhaddevatā* edited in original Sanskrit with English translation by MACDONELL, Harvard, (1904).

for a term of a series is *dhana* (9) (literally, " any valued object "). The first term is called *ādi-dhana* (" first term "), and any other term *iṣṭa-dhana* (" desired term "). When the series is finite, its last term is called *antya-dhana* (" last term ") and the middle term *madhya-dhana* (" middle term "). Often, for the sake of abridgement, the second words of these compound names are deleted, so that we have the terms *ādi*, *iṣṭa*, *madhya* and *antya* in their places. The first term is also called *prabhava* (" initial term ") or *mukha* (" face ") and its synonyms. The technical names for the common difference in an arithmetic series are *caya* or *pracaya* (from the root *caya* " to go," hence meaning " that by which the term goes," that is, " increment "), *uttara* (" difference," " excess "), *vṛddhi* (" increment ") etc. The common ratio in a geometric series is technically called *guṇa* or *guṇaka* (" multiplier ") and so this series is distinguished from the arithmetic series by the specific name *guṇa-śreḍhī*. The number of terms in a series is known as *pada* (" step," meaning " the number of steps in the sequence ") or *gaccha* (" period "). The sum is called *sarva-dhana* (" total of terms "), *śreḍhī-phala* (" result of progression ") *śreḍhī gaṇita* (or simply *gaṇita*, because the sum of the series is obtained by computation), *śreḍhī-saṁkalita* (or in short *saṁkalita*, " sum of the series ").

The above mentioned technical terms occur commonly in almost all the known Hindu treatises on arithmetic, from the so called *Bakhshālī* treatise (*c.* 300) onwards. But in the latter, series has also been designated by *varga*, meaning " group." Occasionally we meet with the terms *paṅkti* (10) and *dhārā* (11) which signify " continuous line or series." NĀRĀYAṆA has used the special term *āya* (literally, " income ") for a sum of natural numbers.

SUM OF AN A. P.—Problems on the summation of an arithmetic series are met with in the earliest available work on Hindu mathematics, the *Bakhshālī Manuscript*. The statement of the formula for the sum begins with the word *rūpoṇā*, so that summa-

(9) In mathematics *dhana* means an affirmative quantity or *plus*. This probably explains the use of this term to denote the elements of a series which have to be added together.

(10) See chapter xiii of the *Gaṇita Kaumudī* (MS.) of NĀRĀYAṆA.

(11) For instance, see the *Triloka-sāra* of NEMICANDRA (*c.* 975).

tion is indicated by the terms *rūpoṇā karaṇena* (" by the operation *rūpoṇā* etc.") throughout the work. In the statement of the solution of problems, the first term, the common difference and the number of terms are written together and the resulting sum after these, as follows :

" *ā*	1	*u*	1	*pa*	19
	1		1		1

rūpoṇā karaṇena phalam $\frac{190}{1}$ "

In the above *ā* stands for *ādi* (" first term "), *u* for *uttara* (" common difference ") and *pa* for *pada* (" number of terms "). The above quotation may be translated thus (12) :

" The first term is 1/1, the common difference is 1/1 and the number of terms is 19/1, therefore, performing *rūpoṇā* etc., the sum is 190/1."

ĀRYABHAṬA I (499) states the formula (13) for finding the middle term, the rth term and the sum of any desired number of terms as follows :

" The desired number of terms minus one, halved and multiplied by the common difference when added to the initial term gives the middle term. The number of terms that precede any desired term (*i.e.*, $r - 1$, if the rth term is desired) multiplied by the common difference and being added to the first term gives (the desired term). The middle term multiplied by the number of terms, or the sum of the first and last terms multiplied by half the number of terms gives the sum of the desired number of terms."

Let

$$a_1 + a_2 + a_3 + \ldots + a_r + a_{r+1} + \ldots + a_{n+r-1} + a_{n+r} + \ldots$$

(12) Note how the denominator 1 is written in the case of all the integral quantities. This is to show that the quantities involved may have fractional values also.

(13) *A*, ii. 19. The translation given here differs from that given by CLARK, (*Āryabhaṭīya*, Chicago, 1930, p. 35). The manuscript of the *Āryabhaṭīya* with the commentary of BHĀSKARA I (522), in our possession gives the reading *samukham madhyam* in the place of *samukhamadhyam* in KERN's edition. BHĀSKARA I remarks, " *atra bahūni sūtrāṇi* " *i.e.*, " this contains several rules." The commentators SŪRYADEVA and PARAMEŚVARA also make the same remark. They all agree with the interpretation given here.

be a series in A. P., and let the common difference of the series be b. Then taking any n terms, say,

$$a_r + a_{r+1} + \ldots + a_{r+n-1},$$

the rule given above states :

(i) middle term $= \left\{ a_r + \frac{n-1}{2} b \right\} = \text{M (say)}$

(ii) nth term $= \{ a_r + (n-1) b \}$

(iii) Sum of the desired number of terms $= \text{M}.n$

$$= \frac{a_r + a_{r+n-1}}{2} . n$$

BRAHMAGUPTA (14) says :

" The last term is equal to the number of terms minus one multiplied by the common difference, (and then) added to the first term. The middle term is half the sum of the first and the last terms; this multiplied by the number of terms is the sum."

BRAHMAGUPTA's results are the same as those of ĀRYABHAṬA I. Similar statements occur in the works of ŚRĪDHARA (15), ĀRYABHAṬA II (16), BHĀSKARA II (17) and others. MAHĀVĪRA points out that the common difference may be a positive or negative quantity (18).

The particular case

$$\sum_1^n r = \frac{n(n+1)}{2}$$

is mentioned in all the Hindu works (19).

ORDINARY PROBLEMS IN A. P.—The problems of finding out (1) the common difference, (2) any desired term and (3) the number of terms are common to all Hindu works. They occur first in the *Bakhshālī Manuscript* (20). The problem of finding the number of terms requires the solution of a quadratic

(14) *BrSpSi*, p. 186.

(15) *Triś*, p. 28.

(16) *MSi*, p. 158.

(17) *L*, p. 27.

(18) *GSS*, p. 102.

(19) It is sometimes mentioned in connection with addition, as in ŚRĪDHARA's *Triśatikā* and the *Gaṇita-sāra-saṁgraha* of MAHĀVĪRA.

(20) See p. 25 & p. 35, problem 9; and p. 36 problem 10. The solution of this problem is incorrectly printed.

equation (21). Some indeterminate problems in which more than one of the above quantities are missing also occur in the *Bakhshālī Manuscript*, the *Gaṇita-sāra-saṁgraha* of MAHĀVĪRA and the *Gaṇita Kaumudī* of NĀRĀYAṆA. A typical example of such problems is the finding out of an arithmetic series that will have a given sum and a given number of terms (22).

As illustrations of some other types of Hindu problems on arithmetical progression may be mentioned the following :

(1) There were number a of *utpala* flowers representable as the sum of a series in arithmetical progression, whereof 2 is the first term and 3 the common difference. A number of women divided these flowers equally among themselves. Each woman had 8 for her share. How many were the women and how many the flowers (23)?

(2) A person travels with velocities beginning with 4 and increasing successively by the common difference 8. Again, a second person travels with velocities beginning with 10 and increasing successively by the common difference 2. When do they meet (24)?

(3) The continued product of the first term, the number of terms and the common difference is 12. If the sum of the series is 10, find it (25).

(4) A man starts with a certain velocity and a certain acceleration per day. After 8 days another man follows him with a different velocity and an acceleration of 2 per day. They meet twice on the way. After how many days do these meetings occur (26)?

GEOMETRIC SERIES.—MAHĀVĪRA gives the formula

$$S = \frac{a(r^n - 1)}{r - 1}$$

for the sum of a geometric series whose first term is a and common ratio r. He says :

" The first term when multiplied by the continued product

(21) Such problems occur in the *Bakhshālī Manuscript* also.
(22) For some indeterminate problems see COLEBROOKE, p. 293.
(23) *GSS*, vi. 295.
(24) *GSS*, vi. 323½.
(25) *GK*, *śreḍhī vyavahāra*, ex. under rule 6.
(26) *Ibid.*, ex. under rule 9.

of the common ratio, taken as many times as the number of terms, gives rise to the *guṇadhana*. And it has to be understood that this *guṇadhana*, when diminished by the first term and then divided by the common ratio lessened by one, becomes the sum of the series in geometric progression " (27).

The same result is stated by him in the following alternative form :

" In the process of successive halving of the number of terms, put zero or one according as the result is even or odd. (Whenever the result is odd subtract one). Multiply by the common ratio when unity is subtracted and multiply so as to obtain square (when otherwise, *i.e.*, when the half is even). When the result of this operation is diminished by one and is then multiplied by the first term and (is then) divided by the common ratio lessened by one, it becomes the sum of the series " (28).

If n be the number of terms and r the common ratio, the first half of the above rule gives r^n. This process of finding the nth power of a number was used by PIṄGALA (29) (*c.* 200 B.C.) to find 2^n. The second half of the rule then gives

$$S = \frac{a(r^n - 1)}{r - 1}.$$

The above formula for the sum is stated by PṚTHUDAKASVĀMĪ (30), ĀRYABHAṬA II (31) and BHĀSKARA II (32) in the second form which appears to be the traditional method of stating the result.

MAHĀVĪRA has given rules for finding the first term, the common ratio or the number of terms, one of these being unknown and the others as well as the sum being given (33).

As illustrations of problems on geometric series may be mentioned the following :

(1) Having first obtained 2 golden coins in a certain city, a man goes on from city to city, earning everywhere three times

(27) *GSS*, ii. 93.
(28) *GSS*, ii. 94, also vi. 311½, where the rule is applied to the case in which the common ratio is fractional.
(29) *Chandaḥ sūtra*, Calcutta, 1892, viii. 20 ff.
(30) *BrSpSi*, p. 186 Com.
(31) *MSi*, p. 159.
(32) *L*, p. 31.
(33) *GSS*, ii. 97-103.

of what he earned immediately before. Say, how much he will make on the eighth day (34)?

(2) When the first term is 3, the number of terms 6 and the sum is 4095, what is the value of the common ratio (35)?

(3) The common ratio is 6, the number of terms is 5, and the sum is 3110. What is the first term here (36)?

(4) How many terms are there in a geometric series whose first term is 3, the common ratio is 5 and the sum is 22888183593 (37)?

SERIES OF SQUARES.—The series whose terms are the squares of the natural numbers, seems to have attracted attention at a fairly early date in India. The formula

$$\sum_{1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

occurs in the *Āryabhaṭīya* (38) (499), where it is stated in the following form :

" The sixth part of the product of the three quantities consisting of the number of terms, the number of terms plus one and twice the number of terms plus one, is the sum of the squares." This formula occurs in all the known Hindu works (39).

MAHĀVĪRA gives the sum of a series whose terms are the squares of the terms of a given arithmetic series.

Let

$$a + (a+b) + \dots + (a + \overline{r-1}b) + \dots + (a + \overline{n-1}b)$$

be an arithmetic series. Then according to him

$$a^2 + (a+b)^2 + \dots + (a+\overline{r-1}b)^2 + \dots + (a + \overline{n-1}b)^2$$
$$= n\left[\left\{\frac{2n-1}{6}b^2 + ab\right\}(n-1) + a^2\right].$$

NĀRĀYAṆA (40) gives the above result in the following form :

(34) *GSS*, ii. 96.
(35) *GSS*, ii. 102 first half.
(36) *GSS*, ii. 102 second half.
(37) *GSS*, ii. 105 last part.
(38) *Ā*, p. 39.
(39) Except the *Triśatikā*. The rule along with many others must have been given in SRĪDHARA's bigger work of which the *Triśatikā* is an abridgment.
(40) *GK*, *śreḍhī vyavahāra*, $17\frac{1}{2}$ and the first half of 18.

$$\sum_{1}^{n}(a + \overline{r-1}b)^2 = a\sum_{1}^{n}\{a + 2(r-1)b\} + b^2\sum_{1}^{n-1} r^2.$$

SERIES OF CUBES.—ĀRYABHAṬA I states the formula giving the sum of the series formed by the cubes of the natural numbers as follows :

" The square of the sum of the original series is the sum of the cubes " (41).

Thus according to him

$$\sum_{1}^{n} r^3 = \left(\sum_{1}^{n} r\right)^2 = \left\{\frac{n(n+1)}{2}\right\}^2.$$

The above formula occurs in all the Hindu works. The general case in which the terms of the series are the cubes of the terms of a given arithmetic series, has been treated by MAHĀVĪRA (42).

Let

$$S = \sum_{1}^{n}(a + \overline{r-1}b)$$

be an arithmetic series whose first term is a and the common difference b. Then according to MAHĀVĪRA

$$\sum_{1}^{n}(a + \overline{r-1}b)^3 = b \cdot S^2 \pm Sa(a \sim b)$$

according as $a >$ or $< b$.

NĀRĀYAṆA (43) has also given the above result in the same form as MAHĀVĪRA.

SERIES OF SUMS.—Let

$$N_n = 1 + 2 + 3 + \dots + n,$$

then, the series

$$\sum_{1}^{n} N_r,$$

formed by taking successively the sum up to 1, 2, 3,... terms of the series of natural numbers, is given in all the Hindu works (44), beginning with that of ĀRYABHAṬA I, who says :

(41) *Ā*, p. 39.
(42) *GSS*, vi. 303.
(43) *GK*, *śreḍhī vyavahāra*, $18\frac{1}{2}$ f.
(44) With the exception of the *Triśatikā*.

"In the case of an *upaciti* which has one for the first term and one for the common difference between the terms, the product of three terms having the number of terms (n) for the first term and one for the common difference, divided by six is the *citighana*. Or the cube of the number of terms plus one, minus the cube root of the cube (45), divided by six" (46).

The above rule states that

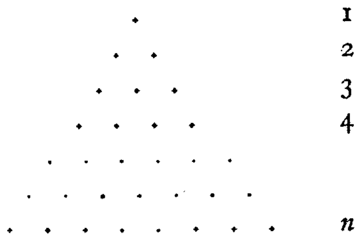
$$\sum_{1}^{n} N_r = \frac{n(n+1)(n+2)}{6}$$

or

$$= \frac{(n+1)^3 - (n+1)}{6}$$

The sum of the series $\sum_{1}^{n} N_r$ has been called by ĀRYABHAṬA *citighana* which means the "solid content of a pile in the shape of a pyramid on a triangular base." The pyramid is constructed as follows :

Form a triangle with $\sum_{1}^{n} r$ things arranged as below :



Total $= n(n+1)/2$.

Form a similar triangle with $\sum_{1}^{n-1} r$ things and place it on top of the first, then form another such trianlge with $\sum_{1}^{n-2} r$ things and place it on top of the first two. Proceed as above till there is one thing at the top. The figure obtained in this manner will be a pyramid of layers such that the base layer consists of $\sum_{1}^{n} r$

(45) This means $\{(n+1)^3\}^{\frac{1}{3}} = (n+1)$. Recourse is taken to this form of expression for the sake of metrical convenience.

(46) *Ā*, p. 39.

things, the next higher layer consists of $\sum_{1}^{n-1} r$ things, and so on. The number of things in the solid pyramid

$$citighana = \sum_{1}^{n} N_r ,$$

where

$$N_r = \sum_{m=1}^{m=r} m.$$

The base of the pyramid is called *upaciti*, so that

$$upaciti = \sum_{m=1}^{n} m.$$

The above *citighana* is the series of figurate numbers. The Hindus obtained the formula for the sum of the series of figurate numbers as early as the fifth century A.D. and they knew the sum of the series of natural numbers in the fifth century B.C. It can not be said with certainty whether they used the representation of the sum by triangles or not. The subject of piles of shots, however, has been given great prominence in the Hindu works, all of which contain a section dealing with *citi* ("piles"). It will not be a matter of surprise if the geometrical representation of figurate numbers is traced to Hindu sources.

MAHĀVĪRA'S SERIES.—MAHĀVĪRA (850) has generalized the series of sums in the following manner :

Let $\alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n$,
be a series in arithmetical progression, the first term being α_1 and the common difference β, so that $\alpha_r = (\alpha_1 + \overline{r-1}\,\beta)$. MAHĀVĪRA considers the following series

$$\sum_{r=1}^{r=n}\left(\sum_{m=1}^{m=\alpha_r} m\right),$$

and gives its sum as

$$\frac{n}{2}\left[\left\{\frac{(2n-2)\beta^2}{6} + \frac{\beta}{2} + \alpha_1\beta\right\}(n-1) + \alpha_1(\alpha_1+1)\right]. \quad (47)$$

NĀRĀYAṆA gives the above result in another form. According to him

$$\sum_{r=1}^{n}\left(\sum_{m=1}^{\alpha_r} m\right) = \left(\sum_{1}^{\alpha_1+\beta} m - \sum_{1}^{\alpha_1} m\right)\sum_{1}^{n-1} m + n\sum_{1}^{\alpha_1} m + \beta^2\sum_{1}^{n-2}\left(\sum_{1}^{r} m\right).$$

(47) *GSS*, vi. 305-305½.

Denoting by N_r the sum of r terms of the series of natural numbers, NĀRĀYAṆA's result may be written in the form

$$\sum_{r=1}^{n} N_{\alpha_r} = (N_{\alpha_1+\beta} - N_{\alpha_1})\, N_{n-1} + n\, N_{\alpha_1} + \beta^2 \sum_{1}^{n-2} N_r$$

$$= \left\{\frac{(\alpha_1+\beta)(\alpha_1+\beta+1)}{2} - \frac{\alpha_1(\alpha_1+1)}{2}\right\} \frac{n(n-1)}{2} + \frac{n\cdot\alpha_1(\alpha_1+1)}{2} + \beta^2\,\frac{(n-2)(n-1)\cdot n}{6}$$

which can be easily reduced to MAHĀVĪRA's form.

NĀRĀYAṆA'S SERIES.—NĀRĀYAṆA has given formulae for the sum of a series whose terms are formed successively by taking the partial sums of other series in the following manner :

Let the symbol ${}^nV_1$ denote the arithmetic series of natural numbers up to n terms; *i.e.*, let

$${}^nV_1 = 1 + 2 + 3 + \ldots + n.$$

Let ${}^nV_2$ denote the series formed by taking the partial sums of the series ${}^nV_1$. Then

$${}^nV_2 = \sum_{r=1}^{n} {}^rV_1.$$

Similarly, let

$${}^nV_3 = \sum_{r=1}^{n} {}^rV_2,$$

$${}^nV_m = \sum_{r=1}^{n} {}^rV_{m-1}.$$

The series ${}^nV_m$ has been called by NĀRĀYAṆA m-*vāra-saṁkalita* ("*m*-order-series") meaning therby that the operation of forming a new series by taking the partial sums of a previous series has been repeated m times. The number m may be called the order *(vāra)* of the series.

NĀRĀYAṆA states the sum ${}^nV_m$ as follows :

"The terms of the sequence beginning with the *pada* (number of terms, *i.e.* n) and increasing by one taken up to the order *(vāra)* plus one times are successively the numerators, and the terms of the sequence beginning with unity and increasing by one are respectively the denominators. The continued product of these (fractions) gives the *vāra-saṁkalita* ("sum of the iterated series of a given order").

Thus according to the above, n being the number of terms of the iterated series, and m the order, we get the following sequence of numbers

$$\frac{n}{1},\ \frac{n+1}{2},\ \frac{n+2}{3},\ \ldots\ldots,\ \frac{n+m}{m+1}.$$

The sum of the series is the continued product of the above sequence, *i.e.*,

$${}^nV_m = \frac{n(n+1)(n+2)\ldots,(n+m)}{1\ .\ 2\ .\ 3\ .\ \ldots(m+n)}.$$

Putting $m = 1, 2, 3, \ldots$ we get

$${}^nV_1 = \sum_{1}^{n} r = \frac{n(n+1)}{1.\ 2},$$

$${}^nV_2 = \sum_{1}^{n} {}^rV_1 = \frac{n(n+1)(n+2)}{1.\ 2.\ 3.},$$

$${}^nV_3 = \sum_{1}^{n} {}^rV_2 = \frac{n(n+1)(n+2)(n+3)}{1.\ 2.\ 3.\ 4.},$$

and so on.

NĀRĀYAṆA (1350) has made use of the numbers of the *vāra-saṁkalita* in the theory of combinations, in chapter XIII of his work, the *Gaṇita Kaumudī*. The series discussed above are the series of figurate numbers. They seem to have been first studied in the West by PASCAL (1665).

GENERALIZATION.—NĀRĀYAṆA has considered the more general series obtained in the same way as above from a given arithmetical progression.

Let

$${}^nS_1 = \sum_{1}^{n} \alpha_r = \alpha_1 + \alpha_2 + \alpha_3 + \ldots + \alpha_n,$$

where $\sum_{1}^{n} \alpha_r$ is an arithmetic series whose first term is α_1 and common difference β. As above, let us define the iterated series ${}^nS_2$, ${}^nS_3, \ldots$ ${}^nS_k$ as follows :

$${}^nS_2 = \sum_{r=1}^{n} {}^rS_1,$$

$${}^nS_3 = \sum_{r=1}^{n} {}^rS_2,$$

NĀRĀYAṆA states the formula for the sum of the series thus:

"The sum of the iterated series of the given order derived from the natural numbers equal to the given number minus one, is put down at two places. These become the multipliers. The order as increased by unity being divided by the given number of terms as diminished by unity is a multiplier of the first (of these multipliers). The first term and the common difference multiplied respectively by the two quantities and (the results) added together give the required sum of the iterated series."

Suppose it to be required to find $^{n}S_{m}$, where n is the *pada* (number of terms) and m is the *vāra* (order) of the iterated series. Let, as before, $^{n}V_{r}$ denote the iterated series of the rth order derived from the series of n natural numbers. Then, taking $^{(n-1)}V_{m}$ at two places, and multiplying the first of these by $\frac{m+1}{n-1}$ as directed, we get

$$\frac{m+1}{n-1}\, {}^{(n-1)}V_{m} \quad \text{and} \quad {}^{(n-1)}V_{m}.$$

Multiplying the first term, α_1, and the common difference, β, by these two respectively and adding we get the required sum

$$^{n}S_{m} = \alpha_1 \frac{m+1}{n-1}\, {}^{(n-1)}V + \beta\, {}^{(n-1)}V_{m}.$$

Rationale. The above formula has been evidently obtained by NĀRĀYAṆA as follows:

$$^{n}S_{1} = \sum_{1}^{n} \alpha_r = \alpha_1 + (\alpha_1 + \beta) + \ldots\ldots + (\alpha_1 + \overline{n-1}\,\beta).$$

$$= n\left\{\alpha_1 + \frac{n-1}{2}\beta\right\},$$

$$^{n}S_{2} = \sum_{1}^{n} {}^{r}S_{1} = \alpha_1 \sum_{1}^{n} r + \beta \sum_{1}^{n} \frac{r(r-1)}{2}$$

$$= \alpha_1\, {}^{n}V_{1} + \beta\, {}^{(n-1)}V_{2},$$

$$^{n}S_{3} = \sum_{1}^{n} {}^{r}S_{2} = \alpha_1 \sum_{1}^{n} {}^{r}V_{1} + \beta \sum_{1}^{n} {}^{(r-1)}V_{2}$$

$$= \alpha_1\, {}^{n}V_{2} + \beta\, {}^{(n-1)}V_{3}$$

. . . .

$$^{n}S_{k} = \sum_{r=1}^{n} {}^{r}S_{k-1}.$$

$$\therefore\quad {}^{n}S_{m} = \alpha_1\, {}^{n}V_{m-1} + \beta\, {}^{(n-1)}V_{m}$$

$$\text{But } {}^{n}V_{m-1} = \frac{m+1}{n-1}\, {}^{(n-1)}V_{m}$$

$$\therefore\quad {}^{n}S_{m} = \alpha_1 \frac{m+1}{n-1}\, {}^{(n-1)}V_{m} + \beta\, {}^{(n-1)}V_{m}.$$

NĀRĀYAṆA'S PROBLEM.—The above series has been investigated by NĀRĀYAṆA in order to solve the following type of problems:

"A cow gives birth to one calf every year. The calves become young and themselves begin giving birth to calves when they are three years old. O learned man, tell me the number of progeny produced during twenty years by one cow."

Solution:

(i) The number of calves produced during twenty years by the cow is 20.

(ii) The first calf becomes a cow in 3 years and begins giving birth to calves every year, so that the number of its progeny during the period under consideration is (20 — 3) = 17. Similarly the second calf becoming a cow produces during the period under consideration (19 — 3) = 16 calves, and so on. Thus the total number of calves of the second generation is

$$\sum_{1}^{17} r = {}^{17}V_{1}.$$

(iii) The first calf of the eldest cow (of the group of 17) in (ii) produces during the period under consideration (17 — 3) = 14 calves; the second calf of the same group produces 13 calves, and so on. The total progeny (of the second generation) of the group of 17 in (ii) is $(14 + 13 + 12 + \ldots + 1) = {}^{14}V_{1}$. Similarly the total progeny of the group of 16 in (ii) is $^{13}V_{1}$, of the group of 15 in (ii) is $^{12}V_{1}$, and so on. Thus the total progeny of the third generation is

$$\sum_{1}^{14} {}^{r}V_{1} = {}^{14}V_{2}.$$

Similarly the total progeny of the fourth generation is

$$\sum_{1}^{(14-3)} {}^{r}V_{2} = {}_{3}{}^{11}V\,;$$

and so on.

The total number of cows and calves at the end of 20 years is

$$1 + 20 + {}^{17}V_1 + {}^{14}V_2 + {}^{11}V_3 + {}^{8}V_4 + {}^{5}V_5 + {}^{2}V_6.$$

$$= 1 + 20 + \frac{17.18}{1.2} + \frac{14.15.16}{1.2.3} + \frac{11.12.13.14}{1.2.3.4} + \frac{8.9.10.11.12}{1.2.3.4.5} + \frac{5.6.7.8.9.10}{1.2.3.4.5.6} + \frac{2.3.4.5.6.7.8}{1.2.3.4.5.6.7}$$

$$= 1 + 20 + 153 + 560 + 1001 + 792 + 210 + 8$$

$$= 2745.$$

After giving the solution of the problem NĀRĀYAṆA remarks: "An alternative method of solution is by means of the *Meru* used in the theory of combinations in connection with (the calculations regarding) metre. This I have given later on."

MISCELLANEOUS RESULTS.—The following results have been given by ŚRĪDHARA, MAHĀVĪRA and NĀRĀYAṆA:

(1) (48) $n^2 = 1 + 3 + 5 + \ldots$ to n terms,

(2) (49) $n^3 = \sum_1^n \left\{3r(r-1)+1\right\}$

(3) $\quad = 3\sum_1^n r(r-1) + n,$

(3) (50) $n^3 = n + 3n + 5n + \ldots$ to n terms,

(4) (51) $n^3 = n^2(n-1) + \sum_1^n (2r-1),$

(5) (52) $\left\{(n+3)\frac{n}{4}+1\right\}(n^2+n) = \sum_1^n r + \sum_1^n r^2 + \sum_1^n r^3 + \sum_1^n\left(\sum_1^r m\right) = \sum_1^n r\left(1+r+r^2+\frac{r+1}{2}\right)$

(6) (53) $\sum_1^n r + n^2 = \sum_1^n \left\{2+3(r-1)\right\}$

(7) $\quad \sum_1^n r + n^3 = \sum_1^n \left\{(n+1)+(2n+1)(r-1)\right\}$

(48) *Triś*, p. 5; *GSS*, ii. 29; *GK*, i. 18.
(49) *Triś*, p. 6; *GSS*, ii. 45; *GK*, i. 22.
(50) *GSS*, ii. 44; *GK*, *śreḍhī-vyavahāra*, 10-11.
(51) *Ibid.*
(52) *GSS*, vii. 309½.
(53) (6), (7) and (8) follow from a rule given by NĀRĀYAṆA, *GK*, *l.c.* 11.

(8) $\quad n^2 + n^3 = \sum_1^n \left\{(n+1) + 2(n+1)(r-1)\right\}$

(9) (54) $S \pm \left(\frac{S}{a} - n\right) \cdot \frac{m}{r-1} = a + (ar \pm m) + \left\{(ar \pm m) r \pm m\right\} + \left[\left\{(ar \pm m) r \pm m\right\} r \pm m\right] + \ldots$ to n terms

where $S = a + ar + ar^2 + \ldots$ to n terms.

BINOMIAL SERIES.—The development of $(a + b)^n$ for integral values of n has been known in India from very early times. The case $n = 2$ was known to the authors of the *Śulba Sūtras* (1500 to 1000 B.C.). The series formed by the binomial coefficients

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n$$

seems to have been studied at a very early date. PIṄGALA (*c.* 200 B.C.) a writer on metrics knew the sum of the above series to be 2^n (55). This result is found also in the works of MAHĀVĪRA (56) (850), PṚTHUDAKASVĀMĪ (57) (860) and all later writers.

PASCAL TRIANGLE.—The so-called PASCAL triangle was known to PIṄGALA who explained the method of formation of the triangle in short aphorisms (*sūtra*). These aphorisms have been explained by the commentator HALĀYUDHA thus:

"Draw one square at the top; below it draw two squares so that half of each of them lies beyond the former on either side of it. Below them, in the same way, draw three squares; then below them four; and so on up to as many rows as are desired: this is the preliminary representation of the *Meru*. Then putting down 1 in the first square, the figuring should be started. In the next two squares put 1 in each. In the third row put 1 in each of the extreme squares, and in the middle square put the sum of the two numbers in the two squares of the second row. In the fourth row put 1 in each of the two extreme squares: in an intermediate square put the sum of the numbers in the two squares of the previous row which lie just above it. Putting

(54) *GSS*, vi. 314.
(55) PIṄGALA, *Chandah Sūtra*, viii. 23-27.
(56) *GSS*, ii. 94.
(57) *BrSpSi*, xii. 17 commentary.

down of numbers in the other rows should be carried on in the same way. Now the numbers in the second row of squares show the monosyllabic forms : there are two forms each consisting of one long and one short syllable. The numbers in the third row give the disyllabic forms : in one form all syllables are long, in two forms one syllable is short (and the other long), and in one all syllables are short. In this row of the squares we get the number of variations of the even verse. The numbers in the fourth row of squares represent trisyllabic forms. There one form has all syllables long, three have one syllable short, three have two short syllables, and one has all syllables short. And so on in the fifth and succeeding rows; the figure in the first square gives the number of forms with all syllables long, that in the last all syllables short, and the figures in the successive intermediate squares represent the number of forms with one, two, etc. short syllables."

Thus according to the above, the number of variations of a metre containing n syllables will be obtained from the representation of the *Meru* as follows :

Number of syllables		Total number of variations
	1	
1 . . .	1 1	. . $2 = 2^1$
2 . . .	1 2 1	. . $4 = 2^2$
3 . . .	1 3 3 1	. . $8 = 2^3$
4 . . .	1 4 6 4 1	. . $16 = 2^4$
5 . . .	1 5 10 10 5 1	. . $32 = 2^5$
6 . . .	1 6 15 20 15 6 1	. . $64 = 2^6$

(Meru prastāra)

From the above it is clear that PIṄGALA knew the result

$$ {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_{n-1} + {}^nC_n = 2^n. $$

INFINITE SERIES.—The results given above regarding finite series indicate the extent to which the theory of series had advanced in Northern India by the middle of the fourteenth century. The mathematicians of South India, especially those of Kerala seem to have made great progress and appear to have been in possession of the infinite geometric series and other simple infinite series by the end of the fourteenth century. Sometime during the first half of the fifteenth century they discovered what is now known as GREGORY's series. Use of this series seems to have been made for the calculation of π, and in astronomy. As the works of this period are not available, it is not possible to trace the gradual evolution of the infinite series in India.

TALAKULATTURĀ'S SERIES.—TALAKULATTURĀ NAMBUTIRĪ (1432) discovered an infinite series for the arc of a circle in terms of its sine and cosine and the radius of the circle. He says :

" By the method stated before for the calculation of the circle, the arc corresponding to a given value of the sine can be found. Multiply the given value *(iṣṭa)* of the sine *(jyā)* by the radius and divide by the cosine *(koṭijyā)*. The result thus obtained is the first quotient. Then operating again and again with the square of the (given) sine as the multiplier and the square of the cosine as the divisor, obtain from the first quotient other quotients. Divide the successive quotients by the odd numbers 1, 3, 5, etc., respectively. Now subtract the even order of quotients from the odd ones. The remainder is the arc required " (58).

That is to say, if r denote the radius of a circle, α an arc of it and θ the angle subtended at the centre by that arc, then

$$ r\theta = \alpha = \frac{r \operatorname{Sin}\theta}{1 . \operatorname{Cos}\theta} - \frac{r . \operatorname{Sin}^3\theta}{3 . \operatorname{Cos}^3\theta} + \frac{r . \operatorname{Sin}^5\theta}{5 . \operatorname{Cos}^5\theta} - \frac{r \operatorname{Sin}^7\theta}{7 . \operatorname{Cos}^7\theta} + \dots $$

The series is convergent if $\sin\theta < \cos\theta$, i.e., $\theta < \frac{\pi}{4}$. But if $\theta > \frac{\pi}{4}$, the series is divergent and so the rule appears to fail. If in that case, however, we take $\sin\left(\frac{\pi}{2} - \theta\right)$ as given instead of $\sin\theta$, then in accordance with the rule, we get the series

(58) The occurrence of this series in the *Tantra saṁgraha* by NAMBUTIRI was first pointed out by WHISH, *Trans. Royal Asiatic Soc.*, III (1835).

$$\frac{r\pi}{2}-\alpha=\frac{r\,\mathrm{Sin}\left(\frac{\pi}{2}-\theta\right)}{1.\,\mathrm{Cos}\left(\frac{\pi}{2}-\theta\right)}-\frac{r\,\mathrm{Sin}^3\left(\frac{\pi}{2}-\theta\right)}{3.\,\mathrm{Cos}^3\left(\frac{\pi}{2}-\theta\right)}-\frac{r\,\mathrm{Sin}^5\left(\frac{\pi}{2}-\theta\right)}{5.\,\mathrm{Cos}^5\left(\frac{\pi}{2}-\theta\right)}-\ldots$$

or $$\frac{r\pi}{2}-\alpha=\frac{r\,\mathrm{Cos}\,\theta}{1.\,\mathrm{Sin}\,\theta}-\frac{r\,\mathrm{Cos}^3\theta}{3.\,\mathrm{Sin}^3\theta}+\frac{r\,\mathrm{Cos}^5\theta}{5.\,\mathrm{Sin}^5\theta}-\ldots$$

which is convergent. Knowing the value of $\frac{r\pi}{2}-\alpha$, we can easily calculate the value of α. Thus the rule gives the desired result even in case $\theta>\frac{\pi}{4}$. Hence the author remarks:

"Of the arc and its complement, take the sine of the smaller as given *(iṣṭa)*; then performing repeatedly the operations (described), one obtains a similar result."

The above series is stated also by PATHUMANA SOMAYĀJĪ (1733) and ŚAṄKARAVARMAṆA (*c.* 1700?). The former writes:

"Find the first quotient by dividing by the cosine the given sine as multiplied by the radius. Then get the other quotients by multiplying the first and those successively resulting by the square of the sine and dividing them in the same way by the square of the cosine. Now dividing these quotients respectively by 1, 3, 5, etc., subtract the sum of even ones (in the series) from the sum of the odd ones. Thus the sine will become the arc" (59).

ŚAṄKARAVARMAṆA says:

"Divide the product of the radius and the sine by the cosine. Divide this quotient and others resulting successively from it on repeated multiplication by the square of the sine and division by the square of the cosine, by 1, 3, 5, etc., respectively. Then subtract the sum of the even quotients (in the series) from the sum of the odd ones. The remainder is the arc (required)" (60).

Introducing the modern tangent function, the above series can be written as

$$\theta=\tan\theta-\frac{1}{3}\tan^3\theta+\frac{1}{5}\tan^5\theta-\ldots$$

The series was rediscovered by JAMES GREGORY in 1671 and

(59) *Karaṇapaddhati* (MS.), vi. 18.
(60) *Sadratnamālā* (MS.), iv. 11.

then by LEIBNITZ in 1673. It is now generally attributed to the former. But rightly speaking the series should be credited to TALAKULATTURĀ NAMBUTIRĪ.

SERIES FOR THE SINE AND COSINE OF AN ARC.—The Hindus discovered series for the sine and cosine of an angle in powers of its circular measure. For example, PATHUMANA writes:

"In the series of quotients obtained by dividing an arc of a circle severally by 2, 3, etc., times the radius, multiply the arc by the first (term); the resulting product by the second (term); this product again by the third (term); and so on. Put down the even terms of the sequence so obtained after the arc and the odd ones after the radius, and subtract the alternate ones. The remainders will respectively be the sine and cosine of that arc" (61).

Let α be an arc of a circle of radius r, and let it subtend an angle θ at the centre. Proceeding according to the above rule we get the series of quotients

$$\frac{\alpha}{2.r},\ \frac{\alpha}{3.r},\ \frac{\alpha}{4.r},\ \frac{\alpha}{5.r}\ \ldots\ldots \quad \text{(A)}$$

Then, multiplying as directed we get the following sequence:

$$\frac{\alpha^2}{2.r},\ \frac{\alpha^3}{2.3.\,r^2},\ \frac{\alpha^4}{2.3.4.\,r^3},\ \ldots\ldots \quad \text{(B)}$$

Putting down even terms of (B) after the arc (α), we have

$$\alpha,\ \frac{\alpha^3}{3!\,r^2},\ \frac{\alpha^5}{5!\,r^4},\ \frac{\alpha^7}{7!\,r^6},\ \ldots\ldots \quad \text{(C)}$$

and putting down odd terms after the radius (r), we get

$$r,\ \frac{\alpha^2}{2!\,r},\ \frac{\alpha^4}{4!\,r^3},\ \frac{\alpha^6}{6!\,r^5},\ \ldots\ldots \quad \text{(D)}$$

Then in (C) subtracting alternate terms we get

$$\textit{Jyā}\ \alpha=r\,\mathrm{Sin}\,\theta=\alpha-\frac{\alpha^3}{3!\,r^2}+\frac{\alpha^5}{5!\,r^4}-\ldots$$

or $$\mathrm{Sin}\,\theta=\theta-\frac{\theta^3}{3!}+\frac{\theta^5}{5!}-\ldots$$

(61) *Karanapaddhati*, vi. 12 f.

Similarly from (D) we have

$$Kotijy\hat{a}\ \alpha = r \operatorname{Cos} \theta = r - \frac{\alpha^2}{2!\, r} + \frac{\alpha^4}{4!\, r^3} - \frac{\alpha^6}{6!\, r^5} + \ldots$$

or

$$\operatorname{Cos} \theta = 1 - \frac{\theta^2}{2!} \quad \frac{\theta^4}{4!} \quad \frac{\theta^6}{6!} + \ldots$$

These series appear also in the works of ŚAṄKARAVARMAṆA (62).

LIST OF ABBREVIATIONS

1. *Ā* — *Āryabhaṭīya*, edited by H. KERN, Leiden, 1874.
2. *BrSpSi* — *Brâhma-sphuṭa Siddhānta*, edited by S. DVIVEDI, Benares, 1902.
3. COLEBROOKE — *Algebra with arithmetic and mensuration from the Sanscrit of Brahmegupta and Bhascara*, London, 1817.
4. *GK* — *Gaṇita Kaumudī* (unpublished) of NĀRĀYAṆA.
5. *GSS* — *Gaṇita-sāra-saṁgraha*, edited by M. RAṄGĀCĀRYA, Madras, 1912.
6. *L* — *Līlāvatī*, edited by S. DVIVEDI, Benares (reprint), 1912.
7. *MSi* — *Mahāsiddhānta*, edited by S. DVIVEDI, Benares, 1910.
8. *Triś* — *Triśatikā*, edited by S. DVIVEDI, Benares, 1899.

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(62) *Sadratnamālā*, iv. 5

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