

A Loopless Gray Code for Minimal Signed-Binary Representations

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Abstract. A string $\dots a_2a_1a_0$ over the alphabet $\{-1, 0, 1\}$ is said to be a minimal signed-binary representation of an integer n if $n = \sum_{k \geq 0} a_k 2^k$ and the number of non-zero digits is minimal. We present a loopless (and hence a Gray code) algorithm for generating all minimal signed binary representations of a given integer n .

1 Introduction

A string $\dots a_2a_1a_0$ is said to be a signed-binary representation (SBR) of an integer n if $n = \sum_{k \geq 0} a_k 2^k$ and $a_k \in \{-1, 0, 1\}$ for all k . A *minimal* SBR has the least number of non-zero digits. For example, 45 has five minimal SBRs: 101101, 110 $\bar{1}$ 01, 10 $\bar{1}$ 0 $\bar{1}$ 01, 10 $\bar{1}$ 00 $\bar{1}$ $\bar{1}$ and 1100 $\bar{1}$ $\bar{1}$, where $\bar{1}$ denotes -1 . Our main result is a *loopless* algorithm that generates *all* minimal SBRs for an integer n in Gray code order. See Fig. 1 for an example. Our algorithm requires linear time for generating the first string. Thereafter, only $O(1)$ time is required in the worst-case for identifying the portion of the current string to be modified for generating the next string¹.

Volumes 3 and 4 of Knuth's *The Art of Computer Programming* are devoted entirely to algorithms for generation of combinatorial objects. For the output of such an algorithm to be considered a Gray code, successive objects must differ by a constant amount. However, the time required to obtain each new object may be $\omega(1)$. A generation algorithm is said to be loopless if after the initial object is generated, successive objects may be obtained in $O(1)$ time in the worst-case. For a survey of Gray code generation algorithms, see Savage [20].

The earliest algorithm for listing all minimal SBRs is due to Ganesan and Manku [8]; however they did not consider the efficiency of implementing their algorithm. By modifying their technique, Sawada [21] was able

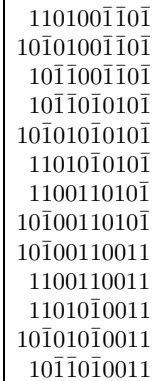


Fig. 1. A Gray code listing of minimal SBRs for 819. Successive strings differ in three adjacent positions.

¹ See <http://www.cs.stanford.edu/~manku/projects/graycode/index.html> for source code in C.

generate all minimal SBRs in constant amortized time. Additionally, the output constitutes a Gray code. However, the algorithm is not loopless, since successive strings require linear time in the worst case.

Our approach is novel — we first identify the *canonical* minimal SBR (see §2 for its definition). The canonical SBR is split into disjoint “chains”. Individual chains are handled by a Gray code algorithm which never outputs certain forbidden strings (see §4). The cross-product of all the chains is handled by a generalization of the Binary Reflected Gray Code (BRGC) [10, 3] (see §3 and §5). A detailed history of SBRs is presented in §6.

2 A Loopless Gray Code for Minimal SBRs

From earlier work by Sawada [21], we know that any minimal signed binary representation (SBR) for an integer n can be transformed into another minimal SBR for the same integer by repeated application of the following re-write rules: $10\bar{1} \rightarrow 011$, $011 \rightarrow 10\bar{1}$, $\bar{1}01 \rightarrow 0\bar{1}\bar{1}$, and $0\bar{1}\bar{1} \rightarrow \bar{1}01$. Our strategy for listing minimal SBRs in Gray code order is the following. We study the structural properties of a specific minimal SBR, popularly known as the *canonical* SBR. We then develop a procedure for listing all strings that result from repeated application of the four re-write rules to the canonical SBR.

DEFINITION (Canonical SBR). *Let S denote the binary representation of a given integer, padded with two leading zeros. For instance, integer 45 would correspond to the string $S = 00101101$. $S_{\text{canonical}}$ is the unique minimal SBR for S such that the product of any two adjacent digits is 0. Thus we never have 11, $1\bar{1}$, $\bar{1}1$ or $\bar{1}\bar{1}$ as a substring. For example,*

$$S = 00101011111010000001010110101000010100$$

$$S_{\text{canonical}} = 010\bar{1}0\bar{1}0000\bar{1}010000010\bar{1}0\bar{1}0\bar{1}0101000010100$$

$S_{\text{canonical}}$ has been used by previous authors (Reitwiesner [19], Chang and Tsao-Wu [6], Jedwab and Mitchell [12] and Prodinger [18]). In fact, $S_{\text{canonical}}$ for integer n can be obtained by “bit-wise subtracting $n/2$ from $3n/2$ ” (Prodinger [18]). Starting with $S_{\text{canonical}}$ is critical to the simplicity of our approach.

DEFINITION (Blocks). *A maximally long bit-sequence of $(01)^+$ and $(0\bar{1})^+$ in $S_{\text{canonical}}$ is called a block. The following string has eight blocks (each block has been underlined):*

$$\underline{01} \underline{0\bar{1}0\bar{1}} 000 \underline{0\bar{1}} \underline{01} 0000 \underline{01} \underline{0\bar{1}0\bar{1}0\bar{1}} \underline{0101} 000 \underline{0101} 00$$

DEFINITION (Chains). *A chain is a maximally long sequence of two or more adjacent blocks. The following string has three chains (each chain has been underlined):*

$$\underline{01 \ 0\bar{1}0\bar{1}} 000 \underline{0\bar{1} \ 01} 0000 \underline{01 \ 0\bar{1}0\bar{1}0\bar{1} \ 0101} 000 \ 0101 \ 00$$

Two chains are separated by one or more 0s. Therefore, none of the four rewrite rules, when applied to one chain, affects another chain. This proves the following:

Theorem 1. *The set of minimal SBRs of S corresponds to the cross product of the sets of minimal SBRs for individual chains of $S_{\text{canonical}}$.*

We now develop two loopless algorithms: one for generating the minimal SBRs of a chain in Gray code order (see §5), and another for generating the cross-product of Gray codes (see §3).

3 Gray Codes for Cross-Products

Consider the cross product of m combinatorial objects: $X_m \times X_{m-1} \times \cdots \times X_1$, where object X_i has $t_i \geq 2$ members which can be listed in Gray code order. Clearly, there is a 1-1 correspondence between members of the cross product and tuples of the form $(a_m, a_{m-1}, \dots, a_1)$, where $a_i \in [1, t_i]$ represents the a_i -th object in the Gray code of X_i . When each $t_i = 2$, one possible Gray code for the set of tuples is the Binary Reflected Gray Code (BRGC) [10]. A generalization of the BRGC, developed by Bitner, Ehrlich, and Reingold [3], handles arbitrary values of $t_i \geq 2$. Procedure BRGC (displayed in Fig. 2) is such an algorithm.

Procedure BRGC maintains three tuples: $(a_m, a_{m-1}, \dots, a_1)$ is the *current-tuple*, $(d_m, d_{m-1}, \dots, d_1)$ is the *direction-tuple*, and $(p_{m+1}, p_m, \dots, p_1)$ is the *pointer-tuple*.

INITIALIZE initializes the three tuples. The current-tuple has $a_i = 1$ or $a_i = t_i$, chosen arbitrarily. The direction-tuple has initial value $d_i = 1$ if $a_i = 1$; otherwise $d_i = -1$. The pointer-tuple has initial value $(m+1, m, m-1, \dots, 1)$.

NEXT(i) updates $a_i \leftarrow a_i + d_i$.

IS_TERMINAL(i) returns TRUE iff $(a_i = t_i \text{ and } d_i = 1)$ or $(a_i = 1 \text{ and } d_i = -1)$.

The pointer-tuple lies at the heart of procedure BRGC. If $p_1 = m+1$, procedure BRGC terminates. Otherwise, let $i = p_1$. Then a_i , the i -th member of the current-tuple, is modified. The direction-tuple indicates whether to increment ($d_i = 1$) or decrement ($d_i = -1$) the value of a_i .

Sample output produced by the algorithm is shown in Table 1(A).

Procedure BRGC can easily be adapted to generate members of $X_m \times X_{m-1} \times \cdots \times X_1$ in Gray code order. Clearly, such an algorithm is loopless if the algorithm that generates members of each X_i in Gray code order is loopless.

4 Gray Codes for Cross-Products with Forbidden Tuples

Let R_m denote the set of m -tuples $(a_m, a_{m-1}, \dots, a_1)$ satisfying

- 1) $\forall m \geq i \geq 1: a_i \in [1, t_i]$, with $t_i \geq 2$
- 2) $\forall m \geq i > 1: (a_i = t_i) \Rightarrow (a_{i-1} = 1)$

BRGC

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INITIALIZE

WHILE TRUE DO
  last  $\leftarrow 1$ 
   $i \leftarrow p_{\text{last}}$ 
  IF ( $i = m + 1$ ) THEN exit

  NEXT( $i$ )

  IF (IS_TERMINAL( $i$ )) THEN
     $d_i \leftarrow -d_i$ 
     $j \leftarrow i + 1$ 
     $p_i \leftarrow p_j$ 
     $p_j \leftarrow j$ 

  IF ( $i \neq \text{last}$ ) THEN  $p_{\text{last}} \leftarrow \text{last}$ 

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Fig. 2. A generalization of the Binary Reflected Gray Code [10,3]. See Table 1(A) for sample output.

Table 1. Output of BRGC (Fig 2) and BRGC-RESTRICT (Fig 3) for $t_3 = 2$, $t_2 = 3$, $t_1 = 3$. The initial tuple is $(a_3, a_2, a_1) = (1, 3, 1)$. The output is generated after each iteration of the WHILE loop. For simplicity we use ‘-’ to represent -1.

(A) With BRGC										(B) With BRGC-RESTRICT									
a_3	a_2	a_1	p_4	p_3	p_2	p_1	d_3	d_2	d_1	a_3	a_2	a_1	p_4	p_3	p_2	p_1	d_3	d_2	d_1
1	3	1	4	3	2	1	1	-	1	1	3	1	4	3	2	1	1	-	1
1	3	2	4	3	2	1	1	-	1	1	2	1	4	3	2	1	1	-	1
1	3	3	4	3	2	2	1	-	-	1	2	2	4	3	2	1	1	-	1
1	2	3	4	3	2	1	1	-	-	1	2	3	4	3	2	2	1	-	-
1	2	2	4	3	2	1	1	-	-	1	1	3	4	3	3	1	1	1	-
1	2	1	4	3	2	2	1	-	1	1	1	2	4	3	3	1	1	1	-
1	1	1	4	3	3	1	1	1	1	1	1	1	4	3	2	3	1	1	1
1	1	2	4	3	3	1	1	1	1	2	1	1	4	4	2	1	-	1	1
1	1	3	4	3	2	3	1	1	-	2	1	2	4	4	2	1	-	1	1
2	1	3	4	4	2	1	-	1	-	2	1	3	4	3	2	4	-	1	-
2	1	2	4	4	2	1	-	1	-										
2	1	1	4	4	2	2	-	1	1										
2	2	1	4	4	2	1	-	1	1										
2	2	2	4	4	2	1	-	1	1										
2	2	3	4	4	2	2	-	1	-										
2	3	3	4	3	4	1	-	-	-										
2	3	2	4	3	4	1	-	-	-										
2	3	1	4	3	2	4	-	-	1										

For example, with $t_3 = 2$, $t_2 = 3$, and $t_1 = 3$, R_m consists of 3-tuples listed in Table 1(B). We now develop a loop-free algorithm for listing R_m in Gray code order. This algorithm will be used in §5 for listing minimal SBRs of chains.

Let G_m denote a Gray code for R_m . Then the reversal of G_m , denoted $\overline{G}_m$, is also a Gray code. We define G_m recursively as follows. The base cases are $G_0 = ()$, the empty tuple, and $G_1 = (1), (2), \dots, (t_1)$. For $m \geq 1$, G_{m+1} depends upon the parity (odd/even) of both t_{m+1} and t_m . Four cases arise; the sequence of $(m+1)$ -tuples for G_{m+1} for the four cases is defined below.

(even, even)	(even, odd)	(odd, even)	(odd, odd)
$1\overline{G}_m,$	$1\overline{G}_m,$	$1G_m,$	$1G_m,$
$2\overline{G}_m,$	$2\overline{G}_m,$	$2\overline{G}_m,$	$2\overline{G}_m,$
$3\overline{G}_m,$	$3\overline{G}_m,$	$3G_m,$	$3G_m,$
$4\overline{G}_m,$	$4\overline{G}_m,$	$4\overline{G}_m,$	$4\overline{G}_m,$
$\dots$	$\dots$	$\dots$	$\dots$
$t_m\overline{G}_m,$	$t_m\overline{G}_m,$	$t_m\overline{G}_m,$	$t_m\overline{G}_m,$
$t_{m+1}1\overline{G}_{m-1}$	$t_{m+1}1G_{m-1}$	$t_{m+1}1\overline{G}_{m-1}$	$t_{m+1}1G_{m-1}$

The notation xG_i denotes a sequence of tuples with $i+1$ members: the first member of each tuple is x ; the remaining members of the tuple constitute G_i . The last tuple in G_m is the same as the first tuple in $\overline{G}_m$ and *vice versa*. Thus,

<u>BRGC-RESTRICT</u>	Procedure INITIALIZE:
INITIALIZE	FOR $i \leftarrow m + 1$ DOWNT0 1 DO $p_i \leftarrow i$
WHILE TRUE DO	$a_m \leftarrow d_m \leftarrow 1$
$\text{last} \leftarrow \text{MAP}(1)$	IF (EVEN(t_m)) THEN $\text{rev} \leftarrow \text{true}$
$i \leftarrow \text{MAP}(p_{\text{last}})$	ELSE $\text{rev} \leftarrow \text{false}$
IF ($i = m + 1$) THEN exit	FOR $i \leftarrow m - 1$ DOWNT0 1 DO
NEXT(i)	IF $\text{rev} = \text{false}$ THEN
IF (IS_TERMINAL(i)) THEN	$a_i \leftarrow d_i \leftarrow 1$
$d_i \leftarrow -d_i$	IF (EVEN(t_i)) THEN $\text{rev} \leftarrow \text{true}$
$j \leftarrow \text{MAP}(i + 1)$	ELSE
$p_i \leftarrow p_j$	$a_i \leftarrow t_i$
$p_j \leftarrow j$	$d_i \leftarrow -1$
	$i \leftarrow i - 1$
	$a_i \leftarrow d_i \leftarrow 1$
IF ($i \neq \text{last}$) THEN $p_{\text{last}} \leftarrow \text{last}$	IF (EVEN(t_i)) THEN $\text{rev} \leftarrow \text{false}$

Fig. 3. A loopless algorithm for listing restricted cross products. See Table 1(B) for sample output.

since the first tuple in each listing begins with a one, G_{m+1} for $m \geq 1$ is indeed a Gray code for R_{m+1} .

Theorem 2. *Procedure BRGC-RESTRICT in Fig. 3 is a loopless algorithm for producing the Gray code G_m .*

BRGC-RESTRICT (Fig. 3) differs from BRGC (Fig. 2) in two important aspects:

1. The initial string $(a_m, a_{m-1}, \dots, a_1)$ has to be initialized appropriately (see procedure INITIALIZE). We begin by assigning $a_m \leftarrow 1$. The recursive definition of G_m then helps us determine the initial values for each a_i , where $m - 1 \geq i \geq 1$. To do this we need only keep track of whether or not a_i is the first member in the first i -tuple of G_i or $\overline{G}_i$. The variable rev is used to determine the list. Recall that the direction d_i is initialized to 1 if $a_i = 1$. If $a_i = t_i$, then d_i is initialized to -1 . The initialization for the “pointer-tuple” p is the same as before: $(m + 1, m, m - 1, \dots, 1)$.
2. We employ a function MAP which is defined as follows:

$$\text{MAP}(i) = \begin{cases} i + 1 & \text{if } (m > i \geq 1) \text{ and } (a_i = 1) \text{ and } (a_{i+1} = t_{i+1}) \\ i & \text{otherwise} \end{cases}$$

If $\text{MAP}(i)$ always returns i , then BRGC-RESTRICT would be identical to BRGC.

An interesting special case corresponds to $t_i = 2$ for all i . Then G_m consists of m -digit strings using the digits $\{1, 2\}$ in which 22 is a forbidden substring. The total number of such strings equals the $(m + 1)^{\text{st}}$ Fibonacci number.

5 A Loopless Gray Code for Chains

We begin with two examples for illustration of our approach.

EXAMPLE (Chain with 2 Blocks). Let $B_2B_1 = (0\bar{1})^s(01)^t$. A rewrite rule is applicable only where the two blocks join: $\bar{1}01 \rightarrow 0\bar{1}\bar{1}$, to obtain $(0\bar{1})^{s-1}00\bar{1}\bar{1}(01)^{t-1}$. Now, we could apply the inverse rule ($0\bar{1}\bar{1} \rightarrow \bar{1}01$) to obtain the previous string, or we can apply the same rule again to the unique substring $\bar{1}01$ in the new representation. This pattern will repeat until we reach the end of the chain. The number of minimal SBRs for this chain is $t + 1$ and is independent of s . As an example, if $s = 2$ and $t = 3$, then the 4 minimal SBRs of $0\bar{1}0\bar{1}010101$ will be: $0\bar{1}0\bar{1}010101, 0\bar{1}00\bar{1}\bar{1}0101, 0\bar{1}00\bar{1}0\bar{1}\bar{1}01$ and $0\bar{1}00\bar{1}0\bar{1}0\bar{1}\bar{1}$. Only B_1 is changing, except for the rightmost digit of B_2 that changes after the first rewrite. $\square$

EXAMPLE (Chain with 3 Blocks). Without loss of generality, let $B_3B_2B_1 = (0\bar{1})^s(01)^t(0\bar{1})^u$. In this case, we can again apply the rewrite rules between B_3 and B_2 as with the two block case, but now we can also apply similar rewrite rules between B_2 and B_1 . The only difference is that the rewrite rules between B_2 and B_1 can only be applied if the state of B_2 has not been altered to its final state where it ends with $\bar{1}\bar{1}$. In that case, no rewrite rules are possible between the two blocks (block B_1 must remain in its initial form: $(0\bar{1})^u$). If we ignore the leftmost block, observe that this problem is an instance of the restricted cross products (where $m = 2$) described in §4. $\square$

To generalize the above observations, we define

$$s(k, \ell) = \begin{cases} (01)^k & \text{if } \ell = 1 \\ (\bar{1}0)^{\ell-2}\bar{1}\bar{1}(01)^{k-\ell+1} & \text{if } 1 < \ell \leq k+1 \end{cases}$$

For block $B_i = (01)^k$ (that is not the leftmost block of a chain), the sequence $s(k, 1), s(k, 2), \dots, s(k, k+1)$ corresponds to the $k+1$ different strings that the block B_i may cycle through. The string $\bar{s}(k, \ell)$ is defined similarly, with 1 and $\bar{1}$ interchanged throughout the string. Examples:

$$\begin{array}{lll} s(4, 1) = 01 & \bar{s}(4, 1) = 0\bar{1}0\bar{1}0\bar{1}0\bar{1} \\ s(4, 2) = \bar{1}\bar{1} & \bar{s}(4, 2) = 110\bar{1}0\bar{1}0\bar{1} \\ s(4, 3) = \bar{1}0\bar{1}\bar{1} & \bar{s}(4, 3) = 10110\bar{1}0\bar{1} \\ s(4, 4) = \bar{1}0\bar{1}0\bar{1}\bar{1} & \bar{s}(4, 4) = 1010110\bar{1} \\ s(4, 5) = \bar{1}0\bar{1}0\bar{1}0\bar{1}\bar{1} & \bar{s}(4, 5) = 10101011 \end{array}$$

Using these strings we can now formally map the problem of cycling through all minimal SBRs of a chain $B_{m+1}B_m \cdots B_1$ to the problem of generating restricted m -tuples. Without loss of generality assume that m is odd and that each B_i is initially defined as follows:

$$\begin{array}{lll} B_{m+1} = \bar{s}(k_{m+1}, 1) = (0\bar{1})^{k_{m+1}}, \\ B_m = s(k_m, 1) = (01)^{k_m}, \\ B_{m-1} = \bar{s}(k_{m-1}, 1) = (0\bar{1})^{k_{m-1}}, \\ \dots & \dots & \dots \\ B_2 = s(k_2, 1) = (01)^{k_2}, \\ B_1 = \bar{s}(k_1, 1) = (0\bar{1})^{k_1}. \end{array}$$

Then a listing of all minimal SBRs for the chain is a subset of the cross-product of strings in blocks $B_m, B_{m-1}, \dots, B_1$, satisfying two constraints for $m \geq i > 1$:

- (1) If the string in block B_i equals $s(k_i, k_i + 1)$, then the string in block B_{i-1} must equal $\bar{s}(k_{i-1}, 1)$.
- (2) If the string in block B_i equals $\bar{s}(k_i, k_i + 1)$, then the string in block B_{i-1} must equal $s(k_{i-1}, 1)$.

A Gray code for the chain can be obtained by setting $t_i = k_i + 1$ for $m \geq i \geq 1$ and using BRGC-RESTRICT outlined in §4. There is a 1-1 correspondence between tuples generated by BRGC-RESTRICT and strings assigned to blocks. A tuple $(a_m, a_{m-1}, a_{m-2}, \dots, a_1)$ generated by BRGC-RESTRICT corresponds to the following configuration: string $s(k_m, a_m)$ in block B_m , string $\bar{s}(k_{m-1}, a_{m-1})$ in block B_{m-1} , string $s(k_{m-2}, a_{m-2})$ in block B_{m-2} , and so on. The only special consideration is that rightmost bit in the leftmost block B_{m+1} must be changed to 0 iff B_m is not in its original state. This is a trivial constant time operation.

Since BRGC-RESTRICT (Fig. 3) is loopless, we have a loopless algorithm to list all minimal SBRs for a given chain. For cross-product of chains (see Theorem 1) we apply procedure BRGC (Fig. 2).

Theorem 3. *A listing of all minimal SBRs for a given integer n can be generated by a loopless algorithm.*

6 A Brief History of Signed Binary Representations

Signed-digit representations have been investigated by both mathematicians and computer scientists (see Hwang [11], Parhami [16] and Knuth [13]). Signed-binary representations using the digits $\{-1, 0, 1\}$ were first investigated by Reitwiesner [19] and Avizienis [2] in the context of digital hardware. Reitwiesner presented an algorithm for identifying the *canonical* signed-binary representation, which is that representation in which no two adjacent digits are non-zero. Over the years, similar algorithms have been re-discovered by several authors (Chang and Tsao-Wu [6], Jedwab and Mitchell [12] and Prodinger [18]). A technique for identifying *all* minimal signed-binary representations, not just the canonical, was discovered by Ganesan and Manku [8]. Sawada [21] adapted this technique to list all minimal SBRs in Gray code order in constant amortized time.

The average weight of minimal signed-binary representations of b -bit numbers approaches $b/3$ for large b . This result has been re-discovered several times, using different proof techniques (Reitwiesner [19], Arno and Wheeler [1], Prodinger [18] and Ganesan and Manku [8]).

Sloane and Plouffe's sequence M0103 and Sloane's sequence A007302 correspond to the weights of minimal signed-binary representations of natural numbers. Sloane's Sequence A005578 are numbers n at which the weight of minimal signed-binary representations of n increases. Sloane's sequence A057526 is the number of zeros in minimal signed-binary representations of natural numbers.

For $m \geq 2$, $(\dots a_2 a_1 a_0)_m$ is said to be a "signed-digit representation" of n if $n = \sum_{k \geq 0} a_k m^k$ and $m_k \in \{0, \pm 1, \pm 2, \dots, \pm (m-1)\}$. A minimal representation

has the least number of non-zero digits. The general case $m \geq 2$ has appeared in early work by Avizienis [2]. Clark and Liang [7] defined a *canonical* representation as one satisfying two additional constraints: (a) $|a_{i+1} + a_i| < m$ for all i , and (b) $|a_i| < |a_{i+1}|$, if $a_{i+1}a_i < 0$, where $|a_i|$ denotes the absolute value of a_i . Such a representation is also known as a *generalized non-adjacent form* (GNAF) since it possesses the property that no two consecutive digits are non-zero for $m = 2$. The GNAF for any integer is minimal and unique. An algorithm for identifying the GNAF was presented in [7]. The average weight for b -digit numbers was shown to be asymptotically $\frac{m-1}{m+1}b$ by Arno and Wheeler [1]. Wu and Hasan [26] derive closed-form formulae for the same. These results were re-discovered by Ganesan and Manku [8].

6.1 Fast Exponentiation

Fast computation of $x^n \bmod r$ is very valuable in cryptography (see surveys by Koç [14] and Gordon [9]). Exponentiation can be studied in terms of addition chains and addition-subtraction chains.

An addition chain for integer n is a sequence of values $a_0 = 1, a_1, a_2, \dots, a_r = n$ with the property that for each $i > 0$, there exist j and k such that $a_i = a_j + a_k$. Then x^n can be computed with r multiplications. See Knuth [13] for a survey of addition chains. The best known lower-bound is $\log_2 n + \log_2 H(n) - 2.13$ by Schönhage [22]. An upper bound for the length of addition chains is $\lfloor \log_2 n \rfloor + H(n)$, where $H(n)$ denotes the Hamming weight of n (the number of 1-bits in binary representation of n). The upper bound is realized by the folklore “fast-multiplication algorithm”. For a randomly chosen b -bit exponent, $b/2$ bits are 1 on average; so the expected number of multiplications is $3b/2$. Several papers propose heuristics for reducing the average by discovering short addition chains (see Bos and Coster [4] and Yacobi [27], for example).

For evaluating $x^n \bmod r$ when x and r are fixed *a priori*, we can pre-compute $x^{-1} \bmod r$, enabling efficient “division” as well. Further, in elliptic curve cryptography, computing $x^{-1} \bmod r$ is as costly as computing $x \bmod r$. This leads us to the idea of addition-subtraction chains (described below), which reduces the average number of multiplications far below $3b/2$.

An addition-subtraction chain for integer n is a sequence of values $a_0 = 1, a_1, a_2, \dots, a_r = n$ with the property that for each $i > 0$, there exist j and k such that $a_i = \pm a_j \pm a_k$. Then x^n can be computed with r multiplications/divisions. Signed-binary representations correspond to addition-subtraction chains. For b -bit exponents, approximately $b/3$ bits are ± 1 ; so the average number of multiplications/divisions is roughly $4b/3$. Higher bases lead to further savings.

Addition-subtraction chains are useful for fast exponentiation in groups (Wu and Hasan [25], Brickell *et al* [5]). Their usefulness in elliptic curve cryptography was first pointed out by Morain and Olivos [15]. Conversion of an integer in binary to its minimal signed-digit representation is popularly known as *recoding*. Efficient software/hardware implementation of recoding presents its own unique challenges. This has led to a variety of recoding algorithms and generalizations of signed-digit representations by the cryptography community. For a good overview of recoding literature, see Phillips and Burgess [17].

6.2 Routing in Chord and CM-2

Weitzman [24] studied routing in the Connection Machine CM-2, developed by Thinking Machines in 1980s. CM-2 was a massively parallel computer using a hypercube-based inter-connection network for routing. Every processor could send a message to another processor a fixed distance $\pm 2^i$ away for any $i \geq 0$. Weitzman discovered that $F(n)$, the optimal cost of communication between two processors distance n away, was given by $F(0) = 0$, $F(2^k) = 1$ and $F(n) = 1 + \min(F(n - 2^k), F(2^{k+1} - n))$, for $2^k < n < 2^{k+1}$. The relationship between $F(n)$ and signed-binary representations was exposed by Ganesan and Manku [8]. They studied a peer-to-peer routing network called Chord [23]. In its simplest form, Chord is an undirected graph on 2^b nodes arranged in a circle, with edges connecting pairs of nodes that are 2^k positions apart for any $k \geq 0$. The shortest path for clockwise distance d can be identified by computing a minimal signed-binary representation of d' defined as follows [8]:

$$d' = \begin{cases} d & \text{if } d \leq \lfloor 2^b/3 \rfloor \\ 2^b - d & \text{if } d > \lfloor 2^b/3 \rfloor \\ d \text{ or } 2^b - d & \text{otherwise} \end{cases}$$

1 and $\bar{1}$ in the signed-binary representation correspond to clockwise and anti-clockwise traversals of Chord edges respectively. A variety of algorithms for solving the problem are presented in [8]. One of them is “LEFT-TO-RIGHT BIDIRECTIONAL GREEDY”, which is identical to Weitzman’s algorithm.

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