Efficient Universal Cycle Constructions for Weak Orders

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A weak order is a way n competitors can rank in an event, where ties are allowed. A weak order can also be thought of as a relation on $\{1, 2, \ldots, n\}$ that is transitive and complete. We provide the first efficient algorithms to construct universal cycles for weak orders by considering both rank and height representations. Each algorithm constructs the universal cycles in $O(n)$ time per symbol using $O(n)$ space.

An ordering of how n competitors can rank in an event, where ties are allowed, is known as a *weak order*. As an example, the times for the 100m men's butterfly final in the 2016 Summer Olympics were:

1	Sadovnikov	RUS	51.84	8
2	Phelps	USA		2
3	Li	CHN	51.26	5
4	Schooling	SGP	50.39	1
5	Le Clos	RSA		2
6	Cseh	HUN		2
7	Shields	USA	51.73	7
8	Metella	FRA	51.58	6

The result was a three way tie for the silver medal. No bronze was awarded. This outcome corresponds to the weak ordering 82512276 that represents the *rank* of each competitor. Let $\mathsf{W}_{\mathbf{r}}(n)$ denote the set of weak orders in a competition with n competitors (teams) under this rank representation. For example, when $n = 3$, the 13 different weak orders are

$$\mathsf{W}_{\mathbf{r}}(3) = \{111, 113, 131, 311, 122, 212, 221, 123, 132, 213, 231, 312, 321\}.$$

The number of weak orders of order n are also known as the the ordered Bell numbers or Fubini numbers and their enumeration sequence is A000670 in the Online Encyclopedia of Integer Sequences [9]. The first six terms in this sequence starting at $n = 1$ are 1, 3, 13, 75, 541, and 4683 respectively.

Given a set of strings S of length n, a *universal cycle* for S is a sequence of length $|\mathsf{S}|$ that when considered cyclicly contains each string in S as a substring. Note this definition implies that each string in S will appear as a substring exactly once. As an example, a universal cycle for $\mathsf{W}_{\mathbf{r}}(3)$ is 1113212213123. The existence of universal cycles for $\mathsf{W}_{\mathbf{r}}(n)$ was proved by Leitner and Godbole [8] using the terminology *ranked permutations*. Using a height-based representation for weak orders defined later in this section, Diaconis and Graham [2] discuss the existence of universal cycles using the terminology *permutations with ties*; subsequently, Horan and Hurlbert [5] prove their existence using standard graph techniques, extending their results to related objects called s-overlap cycles. However, the more difficult problem of efficiently constructing universal cycles for weak orders, which was posed by Diaconis and Ruskey in Problem 477 of [6], remained open. In recent work, Jacques and Wong [7] proposed greedy constructions, but they require exponential space.

In this paper we present the first efficient universal cycle constructions for weak orders by considering both rank and height representations, thus answering the open problem described above. Our algorithms apply the k-ary universal cycle framework developed in [4], which generalizes a binary framework in [3], to construct the universal cycles using $O(n)$ time per symbol and $O(n)$ space. Implementations of our algorithms in C are available for download at `http://debruijnsequence.org`.

A *weak order* can be thought of as a binary relation on $= \{1, 2, \ldots, n\}$ that is transitive and complete (or connex). The latter property meaning that x y or y x (or both) for each $x, y \in$. We write x y if x y and y x, and we write x y if x y but $y \not\preceq x$. Using this notation, a weak order can be written as a permutation where each element is separated by either or . For example

$$4 \quad 2 \quad 5 \quad 6 \quad 3 \quad 8 \quad 7 \quad 1$$

corresponds to the weak ordering from our earlier Summer Olympics example. We will use this ordering to formally define our two representations for weak orders.

The *height* of element j is the number of symbols that precede j in the weak order. By replacing each element j by its height, the weak order 4 2 5 6 3 8 7 1 can be represented by 51201143. Let $\mathsf{W}_{\mathbf{h}}(n)$ denote the set of all weak orders of order n using this height representation. This is the representation used in [2, 5, 6]. As an example,

$$\mathsf{W}_{\mathbf{h}}(3) = \{000, 001, 010, 100, 011, 101, 110, 012, 120, 201, 021, 210, 102\}.$$

The *rank* of element j is one plus the number of elements that precede the rightmost symbol to the left of j in the weak order. By replacing each element j by its rank, as done in [8], the weak order 4 2 5 6 3 8 7 1 can be represented by 82512276. This rank-based representation is equivalent to the strings we described to define our set $\mathsf{W}_{\mathbf{r}}(n)$.

The sets $\mathsf{W}_{\mathbf{r}}(n)$ and $\mathsf{W}_{\mathbf{h}}(n)$ are both closed under string rotation.

In this section we recall elements from the framework developed in [4] that are applied to develop efficient universal cycle constructions for weak orders. We begin by introducing the most general results that applied in our universal cycle construction for $\mathsf{W}_{\mathbf{h}}(n)$. Then we present a simplified special case that is applied in the construction of a universal cycle for $\mathsf{W}_{\mathbf{r}}(n)$.

Let denote a finite alphabet $\{1, 2, \ldots, k\}$ and assume that n, k 2. A function $f : {}^n \to$ is said to be a *feedback function*. A feedback function f is a *UC-successor* of S, a subset of ^{n}, if there exists a universal cycle U for S such that each string $\omega \in \mathsf{S}$ is followed by the symbol $f(\omega)$ in U. A partition of S into subsets $\mathsf{S}_1, \mathsf{S}_2, \ldots, \mathsf{S}_m$ is a *UC-partition with respect to f* if f is a UC-successor for each S_i where $i \in \{1, 2, \ldots, m\}$.

Consider the feedback function f defined by $f(w_1 w_2 \cdots w_n) = w_1$. Observe that f is a UC-successor for each equivalence class of strings under rotation. Thus,

$$\mathsf{S}_1 = \{111\}, \mathsf{S}_2 = \{113, 131, 311\}, \mathsf{S}_3 = \{122, 221, 212\}, \mathsf{S}_4 = \{123, 231, 312\}, \mathsf{S}_5 = \{132, 321, 213\}$$

is a UC-partition of $\mathbf{W}_{\mathbf{r}}(3)$ with respect to f.

Let $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_m$ be an ordered partition of $\mathbf{S}$. For $2 \quad i \quad m$, let $x_i, y_i, z_i \in$ and let $\beta_i \in {}^{n-1}$. A sequence of tuples $(\beta_2, x_2, y_2, z_2), (\beta_3, x_3, y_3, z_3), \ldots, (\beta_m, x_m, y_m, z_m)$ is a spanning sequence of the partition if for each (β_i, x_i, y_i, z_i):

(i) $y_i\beta_i \in \mathbf{S}_i$,

(ii) *if $i = \mathit{first}(\beta_i)$ then $x_i\beta_i \in \mathbf{S}_j$ for some $j < i$,*

(iii) *$x_iy_iz_i$ is a substring of the cyclic string created by starting with $x_{first(\beta_i)}$ then appending each y_j from tuples (β_j, x_j, y_j, z_j) where $\beta_j = \beta_i$ in increasing order of index j,*

where $\mathit{first}(\beta_i)$ is the smallest index of a tuple containing β_i.

The two main results from [4] are Theorem 2.8 and Theorem 2.9. They apply the above definitions to give a framework for constructing universal cycles. We combine them in the following theorem.

[4] *Let $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_m$ be a UC-partition of $\mathbf{S}$ with respect to f with spanning sequence*

$$(\beta_2, x_2, y_2, z_2), (\beta_3, x_3, y_3, z_3), \ldots, (\beta_m, x_m, y_m, z_m)$$

for some $m \quad 2$. Then the following feedback functions g and g' are UC-successors for $\mathbf{S}$:

$$g(\omega) = \begin{cases} f(y_i\beta_i) & \text{if } \omega = x_i\beta_i \text{ for some } i \in \{2,3,\ldots,m\} \text{ and } i = first(\beta_i); \\ f(z_i\beta_i) & \text{if } \omega = y_i\beta_i \text{ for some } i \in \{2,3,\ldots,m\}; \\ f(\omega) & \text{otherwise,} \end{cases}$$

$$g'(\omega) = \begin{cases} f(x_i\beta_i) & \text{if } \omega = y_i\beta_i \text{ for some } i \in \{2,3,\ldots,m\} \text{ and } i = first(\beta_i); \\ f(y_i\beta_i) & \text{if } \omega = z_i\beta_i \text{ for some } i \in \{2,3,\ldots,m\}; \\ f(\omega) & \text{otherwise.} \end{cases}$$

Next we present a special case for the definition of a spanning sequence and the above theorem when each β_i is unique. This simplified result will be used to develop a universal cycle for $\mathbf{W}_{\mathbf{r}}(n)$ in the next section. The more general result will be used for $\mathbf{W}_{\mathbf{h}}(n)$.

Let $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_m$ be an ordered partition of $\mathbf{S}$. For $2 \quad i \quad m$, let $x_i, y_i \in$ and let $\beta_i \in {}^{n-1}$. A sequence of tuples $(\beta_2, x_2, y_2), (\beta_3, x_3, y_3), \ldots, (\beta_m, x_m, y_m)$ is a simplified spanning sequence of the partition if each β_i is unique and for each i the string $y_i\beta_i \in \mathbf{S}_i$ implies the string $x_i\beta_i \in \mathbf{S}_j$ for some $j < i$.

[4] *Let $\mathbf{S}_1, \mathbf{S}_2, \ldots, \mathbf{S}_m$ be a UC-partition of $\mathbf{S}$ with respect to f with simplified spanning sequence $(\beta_2, x_2, y_2), (\beta_3, x_3, y_3), \ldots, (\beta_m, x_m, y_m)$ for some $m \quad 2$. Then the following feedback function $g(\omega)$ is a UC-successor for $\mathbf{S}$:*

$$g(\omega) = \begin{cases} f(x_i\beta_i) & \text{if } \omega = y_i\beta_i \text{ for some } i \in \{2,3,\ldots,m\}; \\ f(y_i\beta_i) & \text{if } \omega = x_i\beta_i \text{ for some } i \in \{2,3,\ldots,m\}; \\ f(\omega) & \text{otherwise.} \end{cases}$$

In the above theorem, the modifications of f to get g correspond to repeatedly applying a standard cycle joining technique. The way the cycles are joined are directed by the simplified spanning sequence. Such a cycle joining (gluing) approach has been exploited in many de Bruijn sequence constructions (see [3, 4]).

$\mathbf{W}_{\mathsf{r}}(n)$

In this section we apply Theorem 2.4 to develop a UC-successor for $\mathsf{W}_{\mathbf{r}}(n)$. Let $\mathsf{W}'_{\mathbf{r}}(n)$ be the subset of all weak orders in $\mathsf{W}_{\mathbf{r}}(n)$ that have no repeating symbol except for possibly the symbol 1. For example, $\mathsf{W}'_{\mathbf{r}}(3) = \mathsf{W}_{\mathbf{r}}(3) \setminus \{122, 212, 221\}$. Additionally, let $num_\omega(v)$ denote the number of occurrences of symbol v in the string ω.

Let $\mathsf{S}_1, \mathsf{S}_2, \ldots, \mathsf{S}_m$ be a UC-partition of $\mathsf{W}_{\mathbf{r}}(n)$ with respect to $f(w_1 w_2 \cdots w_n) = w_1$. Let the lexicographically smallest representatives for each part be given by $\alpha_1, \alpha_2, \ldots, \alpha_m$ respectively. Let $\mathsf{R}_{\mathbf{r}}(n) = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$. Consider the partition to be ordered first by the number of 1s in the representatives (smallest to largest), and then by reverse lexicographic order of the representatives. Note, that $\mathsf{S}_1 = \{1^n\}$ for all n. Define a sequence $S_r = (\beta_2, x_2, y_2), (\beta_3, x_3, y_3), \ldots, (\beta_m, x_m, y_m)$ for this ordered partition where each (β_i, x_i, y_i) is defined as follows assuming $\alpha_i = a_1 a_2 \cdots a_n$:

$$(\beta_i, x_i, y_i) = \begin{cases} (a_{j+1} \cdots a_n a_1 \cdots a_{j-1},\ 1,\ a_j) & \text{if } \alpha_i \in \mathsf{W}'_{\mathbf{r}}(n); \\ (a_{k+1} \cdots a_n a_1 \cdots a_{k-1},\ a_k + num_{\alpha_i}(a_k) - 1,\ a_k) & \text{otherwise,} \end{cases}$$

where j is the index of the unique symbol $(num_{\alpha_i}(1) + 1)$, and k is the largest index[1] such that $a_k \neq 1$ and $num_{\alpha_i}(a_k) > 1$.

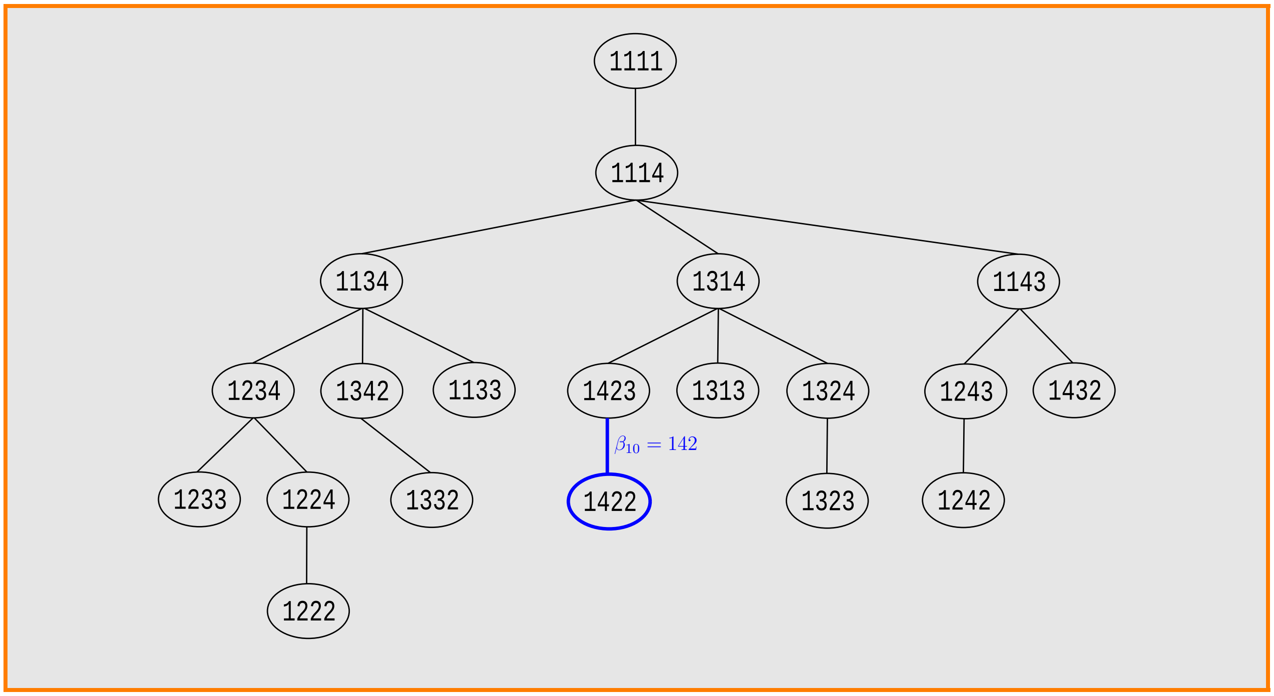
Consider the UC-partition $\mathsf{S}_1, \mathsf{S}_2, \ldots, \mathsf{S}_{20}$ of $\mathsf{W}_{\mathbf{r}}(4)$ with respect to $f(w_1 w_2 \cdots w_n) = w_1$ ordered first by the number of 1s in the representatives, then in reverse lexicographic order. The following illustrates this ordered partition by its representatives $\alpha_1, \alpha_2, \ldots, \alpha_{20}$ along with the sequence S_r defined above.

i	α_i	(β_i, x_i, y_i)
1	1111	-
2	1114	(111,1,4)
3	1314	(141,1,3)
4	1313	(131,4,3)
5	1143	(114,1,3)
6	1134	(411,1,3)
7	1133	(113,4,3)
8	1432	(143,1,2)
9	1423	(314,1,2)
10	1422	(142,3,2)

i	α_i	(β_i, x_i, y_i)
11	1342	(134,1,2)
12	1332	(213,4,3)
13	1324	(413,1,2)
14	1323	(132,4,3)
15	1243	(431,1,2)
16	1242	(124,3,2)
17	1234	(341,1,2)
18	1233	(123,4,3)
19	1224	(412,3,2)
20	1222	(122,4,2)

The sequence S_r induces the following tree where the nodes are the representatives α_i of each S_i. Observe that the parent of the representative of each $y_i\beta_i$ is the representative for $x_i\beta_i$. Also if α_j is the parent of α_i, then $j < i$. For instance for $\alpha_{10} = 1422$ we have $\beta_{10} = 142$ and its parent is $\alpha_9 = 1423$ which is a rotation of $x_{10}\beta_{10} = 3142$.

[1] The smallest index (among other possible choices for k) also works, producing an alternate simplified spanning sequence



S_r is a simplified spanning sequence of the ordered partition $\mathsf{S}_1, \mathsf{S}_2, \ldots, \mathsf{S}_m$.

Proof. Based on the definition of a simplified spanning sequence we must show three things about S_r: (1) each β_i is unique, (2) each $y_i\beta_i \in \mathsf{S}_i$, and (3) each $x_i\beta_i \in \mathsf{S}_j$ for some $j < i$. Consider (β_i, x_i, y_i) for some $2 \leq i \leq m$. Since $y_i\beta_i$ is a rotation of α_i it is in S_i, thus satisfying (2). Observe that the definitions of the indices j and k imply that $y_i > 1$. Furthermore, by the definition of these indices, observe that $x_i\beta_i$ is in $\mathsf{W}_{\mathbf{r}}(n)$, and its corresponding representative α_j either has more 1s or is lexicographically larger than α_i. Thus $x_i\beta_i$ is in some S_j where $j < i$, thus satisfying (3). Finally, to demonstrate (1), suppose there exists $j \neq i$ such that $\beta_j = \beta_i$. Since $\alpha_i \neq \alpha_j$, this means $y_j \neq y_i$. If y_i is not found in β_i then $num_{\alpha_i}(y_i) = 1$, α_i must be in $\mathsf{W}'_{\mathbf{r}}(n)$, and $y_i = num_{\alpha_i}(1) + 1$. By the definition of a weak order, the only other possible value for y_j such that $y_j\beta_j$ is in $\mathsf{W}_{\mathbf{r}}(n)$ is $y_j = 1$. However this contradicts our earlier claim that all $y_j > 1$. Otherwise, assume y_i appears in β_i which means $num_{\alpha_i}(y_i) > 1$. Because $\alpha_i \in \mathsf{W}_{\mathbf{r}}(n)$ there is no symbol $y_i + num_{\alpha_i}(y_i) - 1$ in α_i. Thus since $y_j\beta_j$ is in $\mathsf{W}_{\mathbf{r}}(n)$, either $y_j = y_i$ (a contradiction) or $y_j = y_i + num_{\alpha_i}(y_i) - 1$. But from the previous argument since $y_i + num_{\alpha_i}(y_i) - 1$ would have to be unique in α_j, which was just ruled out in the first case. Thus each β_i is unique, satisfying (1). □

Using the simplified spanning sequence S_r, illustrated in Example 2, we can immediately apply Theorem 2.4 to define a UC-successor g_r for $\mathsf{W}_{\mathbf{r}}(n)$. However, storing the simplified spanning sequence will require an exponential amount of memory, and hence the UC-successor will not be efficient. Thus we need to determine when a weak order $\omega = w_1w_2\cdots w_n \in \mathsf{W}_{\mathbf{r}}(n)$ belongs to the set $\mathsf{U} = \{x_i\beta_i \mid i \in \{2,3,\ldots,m\}\} \cup \{y_i\beta_i \mid i \in \{2,3,\ldots,m\}\}$. We consider the following four cases noting that $\mathsf{X}_1 \cup \mathsf{Y}_1 \cup \mathsf{X}_2 \cup \mathsf{Y}_2 = \mathsf{U}$:

$$\mathsf{X}_1 = \{x_i\beta_i \mid \alpha_i \in \mathsf{W}'_{\mathbf{r}}(n)\},$$

$$\mathsf{Y}_1 = \{y_i\beta_i \mid \alpha_i \in \mathsf{W}'_{\mathbf{r}}(n)\},$$

$$\mathsf{X}_2 = \{x_i\beta_i \mid \alpha_i \notin \mathsf{W}'_{\mathbf{r}}(n)\},$$

$$\mathsf{Y}_2 = \{y_i\beta_i \mid \alpha_i \notin \mathsf{W}'_{\mathbf{r}}(n)\}.$$

Pseudocode for the UC-successor $g_r(\omega)$ where $\omega = w_1 w_2 \quad w_n$ for the set $\mathsf{W}_{\mathbf{r}}(n)$.

```
g_r(ω)

ω 2 W^∅_r(n)    w_1 = num_ω(1) + 1         1     ▷ ω 2 Y_1
ω 2 W^∅_r(n)    w_1 = 1           num_ω(1)       ▷ ω 2 X_1

▷ ω 2 Y_2
ω 2̸ W^∅_r(n)    num_ω(w_1) > 1    w_1 > 1
    num_ω(w_i) = 1    w_i = 1
              2    i < RepStart(ω)        w_1 + num_ω(w_1)    1

▷ ω 2 X_2
num_ω(w_1) = 1     w_1 > 1
  p     the largest symbol in ω less than w_1
    p 6= 1
      w_i 6= p     (num_ω(w_i) = 1    w_i = 1)
                 2    i < RepStart(p w_2 w_3    w_n)        p

     w_1
```

We start by examining X_1 and Y_1. From our definition of S_r, a string ω will belong to Y_1 if it in $\mathsf{W}'_{\mathbf{r}}(n)$ (since ω is a rotation of α_i) and $w_1 = num_\omega(1) + 1$. For each such ω, by replaci yields a string ω' in X_1. Since ω' will also belong to $\mathsf{W}'_{\mathbf{r}}(n)$ it will belong to X_1 if and only i and $w_1 = 1$. Testing for membership in X_2 and Y_2 is a bit more complicated. Let t deno of ω such that $w_t \quad w_n w_1 \quad w_{t-1}$ is the representative of ω. Let the function RepStart(ω) ex t such that $w_t \quad w_n w_1 \quad w_{t-1}$ is the representative of ω. For example RepStart(825122 ow from our definition of S_r, ω will belong to Y_2 if it is not in $\mathsf{W}'_{\mathbf{r}}(n)$, w_1 6= 1, and num_ω and every symbol in $w_2 w_3 \quad w_{t-1}$ is equal to 1 or appears exactly once in ω. For each such ω, rep with $w'_1 = w_1 + num_\omega(w_1) \quad 1$ will yield a string ω' in X_2. Note that since ω is a weak order, n = 1 and $w'_1 > 1$. Furthermore, it is possible that ω' is in $\mathsf{W}'_{\mathbf{r}}(n)$ even though ω (which is a rotati α_i) is not. Given the former two constraints ω' will be in X_2 if and only if every symbol in $w_2 w_3$ not equal to w_1 and is either equal to 1 or appears exactly once in ω.

Using the membership conditions for the sets X_1, Y_1, X_2 and Y_2 outlined above, Algori pseudocode for a UC-successor $g_r(\omega)$ for $\mathsf{W}_{\mathbf{r}}(n)$ by applying Theorem 2.4.

The function $g_r : \quad ^n \; ! \quad$ *presented in Algorithm 1 is a UC-successor for* $\mathsf{W}_{\mathbf{r}}$

Pseudocode in Algorithm 2 applies the UC-successor $g_r(\omega)$ given in Algorithm 1 to co rsal cycle for $\mathsf{W}_{\mathbf{r}}(n)$ starting with the weak order 1^n. The algorithm initializes ω to 1

Applying the UC-successor $g_r(\omega)$ to construct a universal cycle for $\mathsf{W}_{\mathbf{r}}(n)$.

```
    UC( n)
ω = w1w2    wn    1^n
    i      2    n      num_ω(i)    0
num_ω(1)    n

    PRINT(w1)
    v    g(ω)
    num_ω(w1)    num_ω(w1)    1
    num_ω(v)    num_ω(v) + 1
    ω    w2w3    wn v
      num_ω(1) = n
```

A universal cycle for $\mathsf{W}_{\mathbf{r}}(n)$ can be constructed using the successor $g_r(\omega)$ presented in Algorithm 1 starting from any initial weak order ω in $O(n)$ time per symbol using $O(n)$ space.

The following are the universal cycles for $\mathsf{W}_{\mathbf{r}}(n)$ generated by Algorithm 2 for $n = 3$ and $n = 4$:

- ▷ $n = 3$: 1113213122123;
- ▷ $n = 4$: 111143214312421243114132313241313142214231411331134213321341222122412331234.

$\mathbf{W}_{\mathsf{h}}(n)$

In this section we consider the height representation for weak orders and $\mathsf{W}_{\mathbf{h}}(n)$. For $n = 3$, the feedback function $f(w_1w_2 \quad w_n) = w_1$ partitions $\mathsf{W}_{\mathbf{h}}(n)$ into 5 sets with the lexicographically largest element from each set being 210, 201, 110, 100, and 000 respectively. In order to define a simplified spanning sequence, we must define four unique β_i. However there are only a total of three possible values for β_i, namely $f00, 01, 10g$. Thus in order to develop UC-successors for $\mathsf{W}_{\mathbf{h}}(n)$ using this feedback function, we define an appropriate spanning sequence and then apply Theorem 2.2.

Let $\mathsf{T}_1, \mathsf{T}_2, \ldots, \mathsf{T}_m$ be a UC-partition of $\mathsf{W}_{\mathbf{h}}(n)$ with respect to $f(w_1w_2 \quad w_n) = w_1$. Let the lexicographically largest representatives for each part be given by $\alpha_1, \alpha_2, \ldots, \alpha_m$ respectively. Let $\mathsf{R}_{\mathbf{h}}(n) = f\alpha_1, \alpha_2, \ldots, \alpha_m g$. Consider the partition to be ordered in lexicographic order with respect to their representatives α_i. Thus, $\mathsf{T}_1 = f0^n g$ for all n. Assuming $i > i$, let $\alpha_i = a_1a_2 \quad a_n$, $\alpha_i^- = (a_1 \quad 1)a_2a_3 \quad a_n$, and $\alpha_i^+ = (a_1{+}1)a_2a_3 \quad a_n$. We define a sequence $S_h = (\beta_2, x_2, y_2), (\beta_3, x_3, y_3), \ldots, (\beta_m, x_m, y_m)$ for this ordered partition where each (β_i, x_i, y_i, z_i) is defined as follows:

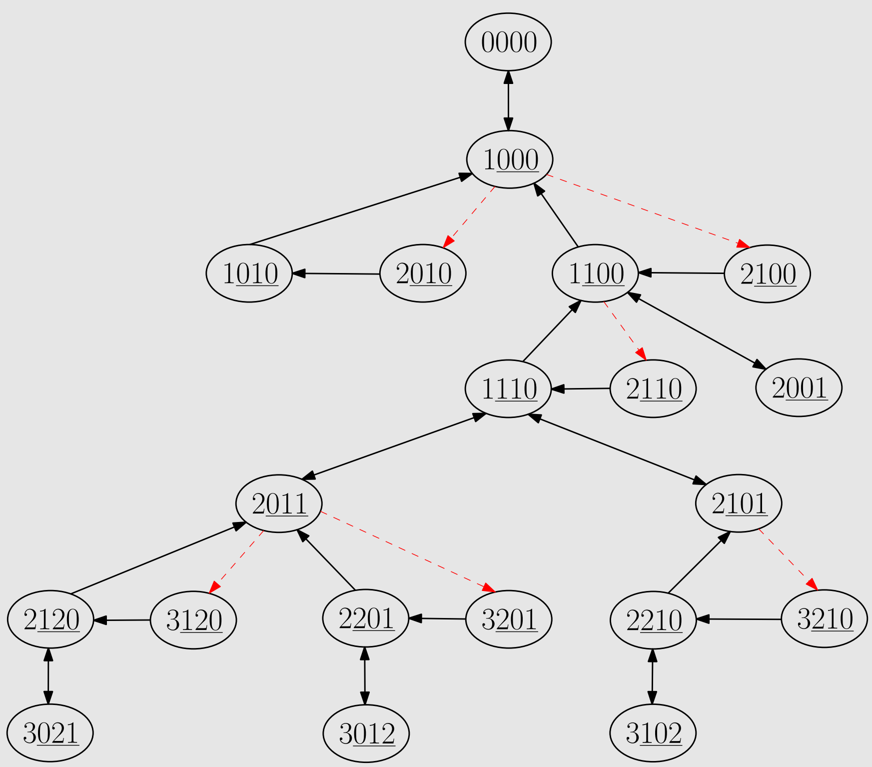
- ▷ $\beta_i = a_2a_3 \quad a_n$,
- ▷ $x_i = a_i \quad 1$,
- ▷ $y_i = a_1$,
- ▷ $z_i = \begin{cases} a_1 + 1 & \alpha_i^+ \ 2\ \mathsf{R}_{\mathbf{h}}(n) \\ a_1 \quad 2 & \alpha_i^- \ 2\ \mathsf{R}_{\mathbf{h}}(n) \\ a_1 \quad 1 & \text{otherwise.} \end{cases}$

When proving that S_h is a spanning sequence in the next lemma, we show that it is not possible for both α_i^+ and α_i^- to be in $\mathsf{R}_{\mathbf{h}}(n)$. Thus, the definition of z_i is well-defined.

Consider the UC-partition $\mathsf{T}_1, \mathsf{T}_2, \ldots, \mathsf{T}_{20}$ of $\mathsf{W}_{\mathbf{h}}(4)$ with respect to $f(w_1 w_2 \quad w_n) = w_1$ where the sets are listed in lexicographic order with respect to their unique representatives in $\mathsf{R}_{\mathbf{h}}(n)$. The following table illustrates this ordered partition by its representatives $\alpha_1, \alpha_2, \ldots, \alpha_{20}$ along with its corresponding sequence S_h.

i	α_i	(β_i, x_i, y_i, z_i)	i	α_i	(β_i, x_i, y_i, z_i)
1	0000	-	11	2110	(110, 1, 2, 0)
2	1000	(000, 0, 1, 0)	12	2120	(120, 1, 2, 3)
3	1010	(010, 0, 1, 2)	13	2201	(201, 1, 2, 3)
4	1100	(100, 0, 1, 2)	14	2210	(210, 1, 2, 3)
5	1110	(110, 0, 1, 2)	15	3012	(012, 2, 3, 2)
6	2001	(001, 1, 2, 1)	16	3021	(021, 2, 3, 2)
7	2011	(011, 1, 2, 1)	17	3102	(102, 2, 3, 2)
8	2010	(010, 1, 2, 0)	18	3120	(120, 2, 3, 1)
9	2100	(100, 1, 2, 0)	19	3201	(201, 2, 3, 1)
10	2101	(101, 1, 2, 1)	20	3210	(210, 2, 3, 1)

The partition and sequence S_h is illustrated by the following graph. The nodes are the representatives α_i of each T_i where each β_i is underlined. Consider two nodes α_i and α_j where $j < i$. If β_i is unique, then there is a bi-directional edge (α_i, α_j) if $x_i\beta_i$ 2 T_j. If β_i is not unique, there is a uni-directional edge (α_i, α_j) if $x_i\beta_i$ 2 T_j and a dashed (red) uni-directional edge (α_j, α_i) if $z_i\beta_i$ 2 T_j.



S_h is a spanning sequence of the ordered partition $\mathsf{T}_1, \mathsf{T}_2, \ldots, \mathsf{T}_m$.

Proof. We demonstrate that S_h satisfies the three conditions of its definition. Consider $1 < i \quad m$. By definition $y_i\beta_i = \alpha_i$ and thus $y_i\beta_i$ 2 T_i, satisfying condition (i). Since α_i 2 $\mathsf{W}_{\mathbf{h}}(n)$, every symbol from 0 to a_1 appears in α_i. Thus, $x_i\beta_i = (a_1 \quad 1)\beta_i$ is a valid weak order and moreover, it must belong to some set T_j where $j < i$ by the ordering of the sets. Thus condition (ii) is satisfied. For condition (iii) we consider the following two cases:

A universal cycle for $\mathsf{W}_{\mathbf{h}}(n)$ can be constructed using either the successor $g_h(\omega)$ or $g'_h(\omega)$ starting from any initial weak order ω 2 $\mathsf{W}_{\mathbf{h}}(n)$. Each universal cycle can be constructed in $O(n)$ time per symbol using $O(n)$ space.

By applying the UC-successor g_h starting from 0^n, we obtain the following two universal cycles for $\mathsf{W}_{\mathbf{h}}(n)$ for $n = 3$ and $n = 4$:

- ▷ $n = 3$: 0001012011021;
- ▷ $n = 4$: 00001010201001200110121023102210321011202130212031201230122013201110211 0021.

By applying the UC-successor g'_h starting from 0^n, we obtain the following two universal cycles for $\mathsf{W}_{\mathbf{h}}(n)$ for $n = 3$ and $n = 4$:

- ▷ $n = 3$: 0001021012011;
- ▷ $n = 4$: 0000102010100210012001102110121032102310221011203120213021201320123012 20111.

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