Determinantal formulas for sum of generalized arithmetic-geometric series

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Abstract. The main purpose of this paper is to give some closed form expressions by determinants for the sum of generalized arithmetic-geometric series. This will be done by solving the recurrence relation with combinatorial auto-convolution, satisfied by these sums. In a particular case, such a recurrence relation was obtained by L. Boulton and M. H. Rosas in this Boletin, [1]. Other recurrence relations of this type was solved by the author in [2] and [4], and was applied to study a new kind of equations - the differential recurrence equations with auto-convolution, both linear and combinatorial. The Catalan numbers also verify a recurrence relation with linear auto-convolution. See [5] or [7]. Applications to usual arithmetic-geometric series, sequences of considered sums that are natural numbers, Fubini's numbers, Eulerian numbers and polynomials and some examples of Z-transforms are given. Another closed form expressions for the sum of generalized arithmetic- geometric series was given by R. Stalley, [10], using the properties of Stirling's numbers of second kind. A direct and elementary proof for this result was given by the author in [3]

Resumen. El objectivo principal de este trabajo es dar a algunas expresiones de forma cerrada por los factores determinantes para la suma de la generalizada aritmetica-geometrica serie. Esto se hara mediante la resolucion de la relacion de recurrencia con la combinatorial auto-convolution, satisfecho por estas cantidades. En un caso particular, como una esto relacion de recurrencia fue obtenido por L. Boulton y M. H. Rosas en este Boletin, [1]. Otras relaciones de recurrencia de este tipo fue resuelto por el autor in [2] y [4], y fue utilizado para estudiar un nuevo tipo de ecuaciones - las ecuaciones diferenciales de recurrencia con auto-convolution, lineales y combinatorica. Los numeros de Catalan tambien verificar una relacion de recurrencia con lineal auto-convolution. Ver [5] o [7]. Aplicaciones de la usual aritmetica-geometrica serie, las secuencias de las sumas considera que son numeros naturales, numeros de Fubini, numeros y polinomios de Euler y para algunos ejemplos de la Transformada Z se dan. Otra expresion de forma cerrada para la suma de la generalizada aritmtica-geomtrica serie fue dada por R. Stalley [10], utilizando las propiedades de los numeros de Stirling de segunda clase. Una prueba de directa y elementar de este resultado fue dada por el autor en [3].

1 Introduction

We consider the sum of the generalized arithmetic-geometric series

$$S_n(z) = \sum_{j=1}^{n} j^n z^j,\ z \in \mathbb{C},\ |z| < 1,\ n = 0, 1, 2, ... \tag{1}$$

It is also the generating function of the numerical sequence $(j^n,\ j = 1, 2, ...\ )$. For $n = 0$, it reduces to the sum of geometric progression

$$S_0(z) = \sum_{j=1}^{\infty} z^j = \frac{z}{1-z}\ . \tag{2}$$

which first appeared, for $z = \frac{1}{2}$, around 1650 BC in the Rhind papyrus, as a problem on areas of lots obtained by dividing a rectangular field in two equal parts, then one of these again in two equal parts, and so on. See [1]. Usual, the sum (1) can be calculated by successive derivation of the uniform convergent serie (2), for $|z| \le r < 1$, and by multiplication with z. A closed form expression for these sums using this procedure and the properties of the Stirling numbers of second kind was given by R. Stalley, [10]. Another proof, based on Cauchy-Mertens theorem was given by author in [3]. In this paper will be presented another two closed form expressions for the sum of generalized arithmetic-geometric series, based on determinants.

2 Discrete linear and combinatorial convolutions

Being given the sequences $a = (a_n)$, $b = (b_n)$, and $c = (c_n)$, where $n = 0, 1, 2, \ldots$, we say that c is *linear*, respectively *combinatorial convolution* of a and b, and we denote $c = a \star b$, respectively $c = a \star_C b$, if $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, respectively $c_n = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}$, for $n = 0, 1, 2, \ldots$. If $\widetilde{a} = \left(\frac{a_n}{n!}\right)$, $\widetilde{b} = \left(\frac{b_n}{n!}\right)$ and $\widetilde{c} = \left(\frac{c_n}{n!}\right)$, we have $c = a \star_C b$ if and only if $\widetilde{c} = \widetilde{a} \star \widetilde{b}$. If $c = a \star b$, then $\sum_{n=0}^{\infty} a_n \sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} c_n$ and $\sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} c_n z^n$. If the series factors are absolutely convergent, then their product is also absolutely convergent (Cauchy). If one series factor is absolutely convergent and the other convergent, then their product is convergent (Mertens, [6]). See also [8].

3 A recurrence relation with combinatorial auto-convolution

Theorem 1. *For $z \in \mathbb{C}$, $|z| < 1$, the sums $S_n(z)$ satisfy the recurrence relation with combinatorial auto-convolution*

$$S_{n+1}(z) = S_n(z) + \sum_{k=0}^{n} \binom{n}{k} S_k(z) S_{n-k}(z),\ n = 0, 1, 2, ... \tag{3}$$

Proof . Performing the change of index $q = j - i$ and using Newton's binomial theorem, we have

$$S_{n+1}(z) = \sum_{j=1}^{\infty} j^{n+1} z^j = \sum_{j=1}^{\infty} j^n z^j + \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} j^n z^j = S_n(z) + \sum_{i=1}^{\infty} \sum_{q=1}^{\infty} (i+q)^n z^{i+q} =$$

$$= S_n(z) + \sum_{i=1}^{\infty} \sum_{q=1}^{\infty} \sum_{k=0}^{n} \binom{n}{k} i^k q^{n-k} z^i z^q = S_n(z) + \sum_{k=0}^{n} \binom{n}{k} \sum_{i=1}^{\infty} i^k z^i \sum_{q=1}^{\infty} q^{n-k} z^q =$$

$$= S_n(z) + \sum_{k=0}^{n} \binom{n}{k} S_k(z) S_{n-k}(z).$$

■

Remark. Theorem 1 was given for $z = \frac{1}{2}$ in [1].

Examples. Using formulas (2) and (3), for $z \in \mathbb{C},\ |z| < 1$, we obtain

$$S_1(z) = S_0(z) + S_0^2(z) = \frac{z}{(1-z)^2}\ , \tag{4}$$

$$S_2(z) = S_1(z) + 2S_0(z)S_1(z) = \frac{z^2 + z}{(1-z)^3}\ ,$$

$$S_3(z) = S_2(z) + 2S_0(z)S_2(z) + S_1^2(z) = \frac{z^3 + 4z^2 + z}{(1-z)^4}\ .$$

Corollary 1. *For $z \in \mathbb{C},\ |z| < 1$, the numbers*

$$x_n(z) = \frac{1}{n!} S_n(z),\ n = 0, 1, 2, \ldots, \tag{5}$$

satisfy the recurrence relation with linear auto-convolution

$$(n+1)x_{n+1}(z) = x_n(z) + \sum_{k=0}^{n} x_k(z) x_{n-k}(z),\ n = 0, 1, 2, \ldots, \tag{6}$$

and the initial condition

$$x_0(z) = \frac{z}{1-z}\ . \tag{7}$$

Proof . Substituting (5) in (3) and multiplying with $\frac{1}{n!}$ it results (6). From(5) and (2) results (7). ■

4 The exponential generating function

For $z \in \mathbb{C},\ |z| < 1$, we denote

$$G(z,p) = \sum_{n=0}^{\infty} x_n(z)p^n = \sum_{n=0}^{\infty} \frac{1}{n!} S_n(z)p^n,\ \forall p \in \mathbb{C},\ |p| < 1, \tag{8}$$

the *generating function* of the sequence $x_n(z)$, that is also the *exponential generating function* of the sequence $S_n(z)$.

Theorem 2. *The exponential generating function $G(z,p)$ of the sequence $S_n(z),\ n = 0,1,2,\ldots,$ is given by formula*

$$G(z,p) = \frac{ze^p}{1-ze^p},\ \forall z,p \in \mathbb{C},\ |z| < 1,\ |p| < 1. \tag{9}$$

Proof . Multiplying relation (6) with p^n, and summing for $n = 0,1,2,\ldots$, we obtain

$$\sum_{n=0}^{\infty}(n+1)x_{n+1}(z)p^n = \sum_{n=0}^{\infty} x_n(z)p^n + \sum_{n=0}^{\infty}\sum_{k=0}^{n} x_k(z)x_{n-k}(z)p^n\ . \tag{10}$$

In conformity with Cauchy-Mertens theorem about the multiplication of power series, the relation (10) reduces to differential equation

$$\frac{d}{dp}G(z,p) = G(z,p) + G^2(z,p)\ . \tag{11}$$

Putting the equation (11) in the form

$$\frac{1}{G(z,p)}\frac{dG(z,p)}{dp} - \frac{1}{G(z,p)+1}\frac{dG(z,p)}{dp} = 1$$

and integrating, it results $\dfrac{G(z,p)}{G(z,p)+1} = e^{p+C}$, so

$$G(z,p) = \frac{e^{p+C}}{1-e^{p+C}}\ , \tag{12}$$

were $C \in \mathbb{C}$ is an arbitrary constant. From (7), (8) and (12), the last two relations for $p = 0$, it results

$$G(z,0) = x_0(z) = \frac{z}{1-z} = \frac{e^C}{1-e^C}\ ,$$

hence

$$e^C = z. \tag{13}$$

From (12) and (13) we obtain (9). ∎

5 Linear recurrence relation

Theorem 3. *For $z \in \mathbb{C}$, $|z| < 1$, the sequence $S_n(z)$ satisfies the linear recurrence relation with variable coefficients*

$$S_n(z) = \frac{z}{1-z}\left[1 + \sum_{k=0}^{n-1}\binom{n}{k} S_k(z)\right],\ n = 1, 2, \dots \,. \tag{14}$$

Proof . From (9) results $G(z,p)\left(\frac{1}{z} - e^p\right) = e^p$. According with (8), this relation becomes

$$\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{n!}S_n(z)p^n - \sum_{n=0}^{\infty}\frac{1}{n!}S_n(z)p^n\sum_{n=0}^{\infty}\frac{1}{n!}p^n = \sum_{n=0}^{\infty}\frac{1}{n!}p^n,\ \forall p \in \mathbb{C},\ |p| < 1.$$

Using again Cauchy-Mertens theorem about the multiplication of power series, the last relation takes the form

$$\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{n!}S_n(z)p^n - \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{k!(n-k)!}S_k(z)p^n = \sum_{n=0}^{\infty}\frac{1}{n!}p^n,\ \forall p \in \mathbb{C},\ |p| < 1. \tag{15}$$

Identifying in (15) the coefficients of p^n, it results

$$\frac{1}{z}\frac{1}{n!}S_n(z) - \sum_{k=0}^{n}\frac{1}{k!(n-k)!}S_k(z) = \frac{1}{n!},\ n = 0, 1, 2, \dots \,. \tag{16}$$

Multiplying (16) with $n!$, we obtain the relation

$$\frac{1-z}{z}S_n(z) - \sum_{k=0}^{n-1}\binom{n}{k} S_k(z) = 1,\ n = 1, 2, \dots, \tag{17}$$

from which it results (14). ■

Examples. Using formulas (2) and (14), for $z \in \mathbb{C}$, $|z| < 1$, we obtain

$$S_1(z) = \frac{z}{1-z}\big[1 + S_0(z)\big] = \frac{z}{(1-z)^2},\ \ S_2(z) = \frac{z}{1-z}\big[1 + S_0(z) + 2S_1(z)\big] = \frac{z(z+1)}{(1-z)^3},$$

$$S_3(z) = \frac{z}{1-z}\big[1 + S_0(z) + 3S_1(z) + 3S_2(z)\big] = \frac{z(z^2+4z+1)}{(1-z)^4}.$$

6 First determinantal formula for the sum of generalized arithmetic-geometric series

Theorem 4. *For $z \in \mathbb{C},\ |z| < 1$, the sums $S_n(z)$, $n = 2, 3, ...$, are given by formula*

$$S_n(z) = \frac{n!z^n}{(1-z)^{n+1}} \begin{vmatrix} \frac{1-z}{z} & 0 & \dots & 0 & 0 & 1 \\ -1 & \frac{1-z}{z} & \dots & 0 & 0 & \frac{1}{2!} \\ . & . & \dots & . & . & . \\ -\frac{1}{(n-2)!} & -\frac{1}{(n-3)!} & \dots & -1 & \frac{1-z}{z} & \frac{1}{(n-1)!} \\ -\frac{1}{(n-1)!} & -\frac{1}{(n-2)!} & \dots & -\frac{1}{2!} & -1 & \frac{1}{n!} \end{vmatrix}. \quad (18)$$

Proof . From (5) and (16) results relation

$$\frac{1-z}{z}x_n(z) - \sum_{k=0}^{n-1} \frac{1}{(n-k)!}x_k(z) = \frac{1}{n!},\ \ n = 1, 2, ... \quad (19)$$

For n given, the relation (7) and first n relations (19) form the linear algebraic system

$\frac{1-z}{z}x_0(z) = 1$,

$-x_0(z) + \frac{1-z}{z}x_1(z) = 1$,

$-\frac{1}{2!}x_0(z) - x_1(z) + \frac{1-z}{z}x_2(z) = \frac{1}{2!}$,

. . . .

$-\frac{1}{n-1!}x_0(z) - \frac{1}{(n-2)!}x_1(z) - \ldots - x_{n-2}(z) + \frac{1-z}{z}x_{n-1}(z) = \frac{1}{(n-1)!}$,

$-\frac{1}{n!}x_0(z) - \frac{1}{(n-1)!}x_1(z) - \ldots - \frac{1}{2!}x_{n-2}(z) - x_{n-1}(z) + \frac{1-z}{z}x_n(z) = \frac{1}{n!}$.

Using Cramer's rule, we obtain following expression for the last unknown of the above system

$$x_n(z) = \frac{z^{n+1}}{(1-z)^{n+1}} \begin{vmatrix} \frac{1-z}{z} & 0 & 0 & \cdots & 0 & 0 & 1 \\ -1 & \frac{1-z}{z} & 0 & \cdots & 0 & 0 & 1 \\ -\frac{1}{2!} & -1 & \frac{1-z}{z} & \cdots & 0 & 0 & \frac{1}{2!} \\ . & . & . & \cdots & . & . & . \\ \frac{-1}{(n-1)!} & \frac{-1}{(n-2)!} & \frac{-1}{(n-3)!} & \cdots & -1 & \frac{1-z}{z} & \frac{1}{(n-1)!} \\ \frac{-1}{n!} & \frac{-1}{(n-1)!} & \frac{-1}{(n-2)!} & \cdots & -\frac{1}{2!} & -1 & \frac{1}{n!} \end{vmatrix}.$$

Adding the last column to first, it results

$$x_n(z) = \frac{z^{n+1}}{(1-z)^{n+1}} \begin{vmatrix} \frac{1}{z} & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & \frac{1-z}{z} & 0 & \cdots & 0 & 0 & 1 \\ 0 & -1 & \frac{1-z}{z} & \cdots & 0 & 0 & \frac{1}{2!} \\ . & . & . & \cdots & . & . & . \\ 0 & \frac{-1}{(n-2)!} & \frac{-1}{(n-3)!} & \cdots & -1 & \frac{1-z}{z} & \frac{1}{(n-1)!} \\ 0 & \frac{-1}{(n-1)!} & \frac{-1}{(n-2)!} & \cdots & -\frac{1}{2!} & -1 & \frac{1}{n!} \end{vmatrix} .$$

Developing the determinant after its first column, and taking into account the relation (5), we obtain the formula (18).

■

Examples. Applying (18), we have

$$S_2(z) = \frac{2z^2}{(1-z)^3} \begin{vmatrix} \frac{1-z}{z} & 1 \\ -1 & \frac{1}{2} \end{vmatrix} = \frac{z(1+z)}{(1-z)^3} ,$$

$$S_3(z) = \frac{6z^3}{(1-z)^4} \begin{vmatrix} \frac{1-z}{z} & 0 & 1 \\ -1 & \frac{1-z}{z} & \frac{1}{2} \\ -\frac{1}{2} & -1 & \frac{1}{6} \end{vmatrix} = \frac{z(1+4z+z^2)}{(1-z)^4} ,$$

$$S_4(z) = \frac{24z^4}{(1-z)^5} \begin{vmatrix} \frac{1-z}{z} & 0 & 0 & 1 \\ -1 & \frac{1-z}{z} & 0 & \frac{1}{2} \\ -\frac{1}{2} & -1 & \frac{1-z}{z} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{2} & -1 & \frac{1}{24} \end{vmatrix} = \frac{z(1+11z+11z^2+z^3)}{(1-z)^5} .$$

7 Second determinantal formula for the sum of generalized arithmetic-geometric series

Theorem 5. *For $Z \in \mathbb{C}$, $|z| < 1$, the sums $S_n(z)$, $n = 2, 3, \ldots$, are given by formula*

$$S_n(z) = \frac{z^{n-1}}{(1-z)^{n+1}} \begin{vmatrix} z+1 & \frac{z-1}{z} & 0 & \cdots & 0 & 0 \\ 2z+1 & \binom{3}{2} & \frac{z-1}{z} & \cdots & 0 & 0 \\ . & . & . & \cdots & . & . \\ (n-3)z+1 & \binom{n-2}{2} & \binom{n-2}{3} & \cdots & \frac{z-1}{z} & 0 \\ (n-2)z+1 & \binom{n-1}{2} & \binom{n-1}{3} & \cdots & \binom{n-1}{n-2} & \frac{z-1}{z} \\ (n-1)z+1 & \binom{n}{2} & \binom{n}{3} & \cdots & \binom{n}{n-2} & \binom{n}{n-1} \end{vmatrix} \tag{20}$$

Proof . The relation (2) and first n relations (17) form the linear algebraic system

$S_0(z) = \dfrac{z}{1-z}$,

$S_0(z) - \dfrac{1-z}{z} S_1(z) = -1$,

$S_0(z) + \binom{2}{1} S_1(z) - \dfrac{1-z}{z} S_2(z) = -1$,

. . . .

$S_0(z) + \binom{n-1}{1} S_1(z) + \cdots + \binom{n-1}{n-2} S_{n-2}(z) - \dfrac{1-z}{z} S_{n-1}(z) = -1$.

$S_0(z) + \binom{n}{1} S_1(z) + \binom{n}{2} S_2(z) + \cdots + \binom{n}{n-1} S_{n-1}(z) - \dfrac{1-z}{z} S_n(z) = -1$.

Applying Cramer's rule, we obtain the following expression for the last unknown of the above system

$$S_n(z) = \frac{(-1)^n z^n}{(1-z)^n} \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \frac{z}{1-z} \\ 1 & \frac{z-1}{z} & 0 & \cdots & 0 & 0 & -1 \\ 1 & \binom{2}{1} & \frac{z-1}{z} & \cdots & 0 & 0 & -1 \\ . & . & . & \cdots & . & . & . \\ 1 & \binom{n-2}{1} & \binom{n-2}{2} & \cdots & \frac{z-1}{z} & 0 & -1 \\ 1 & \binom{n-1}{1} & \binom{n-1}{2} & \cdots & \binom{n-1}{n-2} & \frac{z-1}{z} & -1 \\ 1 & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n-2} & \binom{n}{n-1} & -1 \end{vmatrix}$$

Adding the first column to the last, and developing the determinant obtained about the last column, it results

$$S_n(z) = \frac{z^n}{(1-z)^{n+1}} \begin{vmatrix} 1 & \frac{z-1}{z} & 0 & \cdots & 0 & 0 \\ 1 & \binom{2}{1} & \frac{z-1}{z} & \cdots & 0 & 0 \\ . & . & . & \cdots & . & . \\ 1 & \binom{n-2}{1} & \binom{n-2}{2} & \cdots & \frac{z-1}{z} & 0 \\ 1 & \binom{n-1}{1} & \binom{n-1}{2} & \cdots & \binom{n-1}{n-2} & \frac{z-1}{z} \\ 1 & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{n-2} & \binom{n}{n-1} \end{vmatrix}$$

Subtracting the first row of the other, we obtain .

$$S_n(z) = \frac{z^n}{(1-z)^{n+1}} \begin{vmatrix} 1 & \frac{z-1}{z} & 0 & \cdots & 0 & 0 \\ 0 & \frac{z+1}{z} & \frac{z-1}{z} & \cdots & 0 & 0 \\ . & . & . & \cdots & . & . \\ 0 & \frac{(n-3)z+1}{z} & \binom{n-2}{2} & \cdots & \frac{z-1}{z} & 0 \\ 0 & \frac{(n-2)z+1}{z} & \binom{n-1}{2} & \cdots & \binom{n-1}{n-2} & \frac{z-1}{z} \\ 0 & \frac{(n-1)z+1}{z} & \binom{n}{2} & \cdots & \binom{n}{n-2} & \binom{n}{n-1} \end{vmatrix}$$

Developing the determinant after its first column, it results the formula (20). ∎

Examples. Applying (20), we have

$$S_2(z) = \frac{z(z+1)}{(1-z)^3},\ S_3(z) = \frac{z^2}{(1-z)^4} \begin{vmatrix} z+1 & \frac{z-1}{z} \\ 2z+1 & 3 \end{vmatrix} = \frac{z(z^2+4z+1)}{(1-z)^4}\ ,$$

$$S_4(z) = \frac{z^3}{(1-z)^5} \begin{vmatrix} z+1 & \frac{z-1}{z} & 0 \\ 2z+1 & 6 & \frac{z-1}{z} \\ 3z+1 & 6 & 4 \end{vmatrix} = \frac{z(z^3+11z^2+11z+1)}{(1-z)^5}\ .$$

8 Applications

8.1 Arithmetic-geometric series

Using the formulas (2) and (4), for $a, q, z \in \mathbb{C},\ z \neq 0,\ |z| < 1$, the sum of usual arithmetic-geometric series is

$$\sum_{j=0}^{\infty} (a + jq)\, z^j = a(1 + S_0(z)) + qS_1(z) = a\left(1 + \frac{z}{1-z}\right) + q\frac{z}{(1-z)^2} = \frac{a + (q-a)z}{(1-z)^2}\ .$$

8.2 The sequences $S_n\left(\frac{m}{m+1}\right)$ of natural numbers

For $m = 1, 2, \ldots$, we take $z = \frac{m}{m+1}$. Obviously, in this case we have $|z| < 1$. From (2) it results

$$S_0\left(\frac{m}{m+1}\right) = m\ . \tag{21}$$

The linear recurrence relation (14) takes the form

$$S_n\left(\frac{m}{m+1}\right) = m\left[1 + \sum_{k=0}^{n-1}\binom{n}{k} S_k\left(\frac{m}{m+1}\right)\right],\ n = 1, 2, \ldots\ . \tag{22}$$

From (21) and (22) results by mathematical induction that $S_n\left(\frac{m}{m+1}\right),\ n = 0, 1, 2, \ldots$, is a sequence of natural numbers. In conformity with (1), (18) and (20), we have

$$S_n\left(\frac{m}{m+1}\right) = \sum_{j=1}^{\infty} \frac{j^n m^j}{(m+1)^j} =$$

$$= n!m^n(m+1)\begin{vmatrix} \frac{1}{m} & 0 & \cdots & 0 & 0 & 1 \\ -1 & \frac{1}{m} & \cdots & 0 & 0 & \frac{1}{2!} \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdot \\ \frac{-1}{(n-2)!} & \frac{-1}{(n-3)!} & \cdots & -1 & \frac{1}{m} & \frac{1}{(n-1)!} \\ \frac{-1}{(n-1)!} & \frac{-1}{(n-2)!} & \cdots & -\frac{1}{2!} & -1 & \frac{1}{n!} \end{vmatrix} =$$

$$
= m^{n-1}(m+1)\begin{vmatrix}
2m+1 & -\frac{1}{m} & 0 & \cdots & 0 & 0 \\
3m+1 & \binom{3}{2} & -\frac{1}{m} & \cdots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
(n-2)m+1 & \binom{n-2}{2} & \binom{n-2}{3} & \cdots & -\frac{1}{m} & 0 \\
(n-1)m+1 & \binom{n-1}{2} & \binom{n-1}{3} & \cdots & \binom{n-1}{n-2} & -\frac{1}{m} \\
nm+1 & \binom{n}{2} & \binom{n}{3} & \cdots & \binom{n}{n-2} & \binom{n}{n-1}
\end{vmatrix} \tag{23}
$$

Particular case. For $m = 1$, the sequence of integer numbers $S_n\left(\frac{1}{2}\right)$, with its first terms $1, 2, 6, 26, \ldots$, is Sloane's [9] sequence A 000629. In loc. cit. are given the generating function formula (8) and recurrence relation (22), for the considered particular case.

8.3 Fubini numbers.

Sequence $F_0\left(\frac{1}{2}\right) = 1$, $F_n\left(\frac{1}{2}\right) = \frac{1}{2}S_n\left(\frac{1}{2}\right)$, $n = 1, 2, \ldots$, with first terms $1, 1, 3, 13, \ldots$, is *the Fubini sequence of numbers.* See Sloane [9], A 000670.

8.4 Eulerian numbers and polynomials.

Eulerian polynomials are defined by relation

$$
E_{n-1}(z) = \sum_{k=0}^{n-1} E(n,k)z^k = \frac{(1-z)^{n+1}}{z} S_n(z),\ \forall z \in \mathbb{C},\ |z| < 1,\ n = 1, 2, \ldots\ , \tag{24}
$$

the coefficients $E(0,0) = E(n,0) = E(n,n-1) = 1$, and

$$
E(n,k) = E(n, n-1-k),\ k = 0, 1, \ldots, n-1,\ n = 1, 2, \ldots\ , \tag{25}
$$

being by definition *Eulerian numbers.* See Sloane, [9], A 008292, [5] and [3]. The Euler polynomials and numbers can be calculated using formulas (3), (13), (17) or (20) from this paper.

Examples. Using above examples for $S_n(z)$, we obtain $E_0(z) = \frac{(1-z)^2}{z}S_1(z) = 1$, $E_1(z) = \frac{(1-z)^3}{z}S_2(z) = z + 1$, $E_2(z) = \frac{(1-z)^4}{z}S_3(z) = z^2 + 4z + 1$, $E_3(z) = \frac{(1-z)^5}{z}S_4(z) = z^3 + 11z^2 + 11z + 1$.

8.5 Z-transforms.

For $z \in \mathbb{C},\ |z| < 1$ and $n = 1, 2, \ldots$, from (18) and (20) results that Z-transform of the sequence of powers of natural numbers $(j^n : \ j = 1, 2, \ldots)$, is given by the formulas

$$Z\left((j^n)\right) = \sum_{j=1}^{\infty} \frac{j^n}{z^j} =$$

$$= \frac{n!z}{(z-1)^{n+1}} \begin{vmatrix} z-1 & 0 & \cdots & 0 & 0 & 1 \\ -1 & z-1 & \cdots & 0 & 0 & \frac{1}{2!} \\ . & . & \cdots & . & . & . \\ \frac{-1}{(n-2)!} & \frac{-1}{(n-3)!} & \cdots & -1 & z-1 & \frac{1}{(n-1)!} \\ \frac{-1}{(n-1)!} & \frac{-1}{(n-2)!} & \cdots & -\frac{1}{2!} & -1 & \frac{1}{n!} \end{vmatrix} =$$

$$= \frac{z}{(z-1)^{n+1}} \begin{vmatrix} z+1 & 1-z & 0 & \cdots & 0 & 0 \\ z+2 & \binom{3}{2} & 1-z & \cdots & 0 & 0 \\ . & . & . & \cdots & . & . \\ z+n-3 & \binom{n-2}{2} & \binom{n-2}{3} & \cdots & 1-z & 0 \\ z+n-2 & \binom{n-1}{2} & \binom{n-1}{3} & \cdots & \binom{n-1}{n-2} & 1-z \\ z+n-1 & \binom{n}{2} & \binom{n}{3} & \cdots & \binom{n}{n-2} & \binom{n}{n-1} \end{vmatrix}. \quad (26)$$

Examples. $Z((j)) = \frac{z}{(z-1)^2}$, $Z((j^2)) = \frac{z(1+z)}{(z-1)^3}$, $Z((j^3)) = \frac{z(1+4z+z^2)}{(z-1)^4}$.

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