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Fibonacci Identities as Binomial Sums II

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Abstract

As in [2], our goal in this article is to write some more prominent and fundamental identities regarding Fibonacci numbers as binomial sums.

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1. Introduction

The most well-known linear homogeneous recurrence relation of order two with constant coefficients is

$$F_{n+2} = F_{n+1} + F_n, \quad \text{where} \quad F_0 = 0, \ F_1 = 1, \ \text{and } n \geq 0.$$

This recurrence relation produces the most popular and widely-used integer sequence $0, 1, 1, 2, 3, 5, 8, 13, ...,$ namely, the famous Fibonacci sequence. As in [2], to facilitate rapid numerical calculations of identities pertaining to Fibonacci numbers we write some of these fundamental identities as binomial sums.

Hundreds of Fibonacci identities have been developed over the centuries by numerous mathematicians and number enthusiasts. They have been published in various journals and books for at least the past two centuries. The *Fibonacci Quarterly* is a good source for those Fibonacci identities that have been published since 1962. An impressive collection of over 200 known Fibonacci identities, and in most cases along with the name of the original author, can be found in [15], by Thomas Koshy. Another source for some well-known Fibonacci identities is [4], by Marjorie Bicknel and Verner E. Hoggatt. Like many ideas in mathematics it may not be possible to find the true and genuine author of some of the Fibonacci identities. However, the following individuals authored at least one of the identities that we have presented in this paper: R. H. Anglin [1], G. Candido [5], L. Carlitz [6], J. Ginsburg [7], H. W. Gould [8], R. L. Graham [9], J. H. Halton [10], V. E. Hoggatt Jr. [11, 12], V. E. Hoggatt Jr. and G. E. Bergum [13], J. A. H. Hunter [14], T. Koshy [15-17], D. Lind [18], P. Mana [19], G. C. Padilla [20, 21], C. B. A. Peck [22], C. W. Raine [23], K. S. Rao [24], R. S. Seamons [25], M. N. S. Swamy [27, 28], G. Wulczyn [29], C. C. Yalavigi [30], D. Zeitlin and F. D. Parker [31].

2. Identities

It is known that the left-hand side of Fibonacci identities in Theorems 2.2-2.7 can be written as a (power of a) single Fibonacci number. We acknowledge that we have not independently verified the validity of some of these identities. To proceed, first we recall the following theorem from [1].

Theorem 2.1 [1]. If F_n is any Fibonacci number, then

$$F_{n+1} = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + ... + \binom{n - \lfloor \frac{n}{2} \rfloor + 1}{\lfloor \frac{n}{2} \rfloor - 1} + \binom{n - \lfloor \frac{n}{2} \rfloor}{\lfloor \frac{n}{2} \rfloor}$$
$$= \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}, \quad n \geq 0.$$

To prove Theorems 2.2-2.7 we can simply use Theorem 2.1, and the fact that each Fibonacci identity on the left-hand side can be written as a (power of a) single Fibonacci number. Or, we could use the principle of mathematical induction, combinatorial arguments, or just simple algebra to prove these theorems. However, we caution the reader that some of these identities have been somewhat modified to fit a desired format and they may not look exactly as they appear in the literature.

Theorem 2.2.

(i) $F_{n+1}(F_n F_{n+2} - F_{n+1}^2) = (-1)^{n+1} \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}$

(ii) $F_n^2 + F_{n+4}^2 - \left(4F_{n+2}^2 + F_{n+3}^2\right) = \frac{1}{2}\left[1 + (-1)^n\right] + \sum_{i=0}^{n} F_i F_{i+1}$

$= F_{n+1}F_{n+2} - \sum_{i=0}^{n} F_i^2 = \frac{1}{2}\left[1 + (-1)^n\right] + (n+1)F_{n+1}F_{n+2} - \sum_{i=0}^{n+1} i F_i^2$

$$= \left[\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}\right]^2$$

$$(iii) \quad F_{n+3}^3 + 2F_{n+2}^3 - \frac{1}{3}(F_n^3 + F_{n+4}^3) = \left[\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}\right]^3$$

$$(iv) \quad F_{n+2}^2(F_{n+2}^2 - 4F_nF_{n+1}) + F_n^2(2F_{n+1}^2 - F_n^2)$$
$$= \frac{1}{2}(F_n^2 + F_{n+1}^2 + F_{n+2}^2)^2 - (F_n^4 + F_{n+2}^4) = 2\left[2F_{n+1}^2 - (-1)^n\right]^2 - (F_n^4 + F_{n+2}^4)$$
$$= \left[\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}\right]^4$$

$$(v) \quad F_{n+2}^5 + 5F_nF_{n+1}F_{n+2}(F_nF_{n+1} - F_{n+2}^2) - F_n^5 = \left[\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}\right]^5$$

$$(vi) \quad 2[2F_{n+1}^2 - (-1)^n]^3 + 3F_n^2F_{n+1}^2F_{n+2}^2 - (F_n^6 + F_{n+2}^6) = \left[\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}\right]^6$$

$$(vii) \quad 8F_n^2F_{n+1}^2(F_n^4 + F_{n+1}^4 + 4F_n^2F_{n+1}^2 + 3F_nF_{n+1}F_{2n+1}) - (F_n^8 + F_{n+2}^8)$$
$$+2[2F_{n+1}^2 - (-1)^n]^4 = \left[\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i}\right]^8$$

Theorem 2.3.

$$(i) \quad F_{n+1}^2 + F_nF_{n+3} = \left[\sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-i}{i}\right]^2$$

$$(ii) \quad F_n^3 + F_{n+1}^3 + 3F_nF_{n+1}F_{n+2} = \left[\sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-i}{i}\right]^3$$

$$(iii) \quad F_n^5 + F_{n+1}^5 + 5F_nF_{n+1}F_{n+2}[2F_{n+1}^2 - (-1)^n] = \left[\sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-i}{i}\right]^5$$

$$(iv) \quad F_n^7 + F_{n+1}^7 + 7F_nF_{n+1}F_{n+2}[2F_{n+1}^2 - (-1)^n]^2 = \left[\sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-i}{i}\right]^7$$

Theorem 2.4.

$$(i)\quad 2(F_{n+1}^2+F_{n+2}^2)-F_n^2=\left[\sum_{i=0}^{\lfloor\frac{n+2}{2}\rfloor}\binom{n+2-i}{i}\right]^2$$

$$(ii)\quad F_n^3+F_{3n+3}+3F_{n+1}F_{n+2}F_{n+3}=\left[\sum_{i=0}^{\lfloor\frac{n+2}{2}\rfloor}\binom{n+2-i}{i}\right]^3$$

$$(iii)\quad 55(F_{n+8}^4-F_{n+2}^4)-385(F_{n+6}^4-F_{n+4}^4)+F_n^4=\left[\sum_{i=0}^{\lfloor\frac{n+9}{2}\rfloor}\binom{n+9-i}{i}\right]^4$$

Theorem 2.5.

$$(i)\quad F_n^2+F_{n+1}^2=\sum_{i=0}^{n}\binom{2n-i}{i}$$

$$(ii)\quad F_{n+2}^2-F_n^2=F_{n+1}(F_n+F_{n+2})=\sum_{i=0}^{\lfloor\frac{2n+1}{2}\rfloor}\binom{2n+1-i}{i}$$

$$(iii)\quad F_{2n+2}^2(F_{2n+1}F_{2n+3}-F_{2n+2}^2)=\sum_{i=0}^{n}F_{4i+2}=\left[\sum_{i=0}^{\lfloor\frac{2n+1}{2}\rfloor}\binom{2n+1-i}{i}\right]$$

$$(iv)\quad F_{n+2}F_{n+3}-F_nF_{n+1}=\sum_{i=0}^{n+1}\binom{2n+2-i}{i}$$

$$(v)\quad (F_nF_{n+3})^2+(2F_{n+1}F_{n+2})^2=\left[\sum_{i=0}^{n+1}\binom{2n+2-i}{i}\right]^2$$

Theorem 2.6.

$$(i)\quad F_{n+1}\left[5F_{n+1}^2-3(-1)^n\right]=F_{n+3}^3-F_n^3-3F_{n+1}F_{n+2}F_{n+3}$$

$$=F_{n+1}^3+F_{n+2}^3-F_n^3=\sum_{i=0}^{\lfloor\frac{3n+2}{2}\rfloor}\binom{3n+2-i}{i}$$

$$(ii)\quad 10\sum_{i=0}^{n+1}F_i^3-[5+6(-1)^nF_n]=\sum_{i=0}^{\lfloor\frac{3n+4}{2}\rfloor}\binom{3n+4-i}{i}$$

$$(iii)\quad F_{3n}+4F_{3n+3}=\frac{1}{3}\left(F_{n+4}^3+F_n^3-3F_{n+2}^3\right)=\sum_{i=0}^{\lfloor\frac{3n+5}{2}\rfloor}\binom{3n+5-i}{i}$$

Theorem 2.7.

$$(i)\quad 3+25\sum_{i=1}^{n}\sum_{j=1}^{i}F_{2j-1}^2-5n(n+1)-F_{4n+3}=\sum_{i=0}^{2n}\binom{4n-i}{i}$$

$$(ii)\quad 1+2n+5\sum_{i=0}^{n}F_{2i}^2=\sum_{i=0}^{\lfloor\frac{4n+1}{2}\rfloor}\binom{4n+1-i}{i}$$

$$(iii)\quad 5\sum_{i=0}^{n}F_{2i+1}^2-2(n+1)=\sum_{i=0}^{\lfloor\frac{4n+3}{2}\rfloor}\binom{4n+3-i}{i}$$

3. Conclusion

Out of hundreds of Fibonacci identities that have been developed over the centuries, we have presented just a sample of known Fibonacci identities as binomial sums here and in [2]. Also, we are hoping that these two articles may serve as a catalyst for the reader to write her/his favorite Fibonacci identities as binomial sums. Additionally, in a forthcoming article we will present some other prominent identities involving Lucas or Lucas and Fibonacci numbers as binomial sums.

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