ON FINITE SUMS OF UNIT FRACTIONS

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1. Introduction

THE Rhind Papyrus of A'hmosè is one of the oldest known mathematical writings (*c.* 1650 B.C.) and has been the subject of much study (cf. (**3**)) during the past eighty years. From the evidence presented in this papyrus it is generally recognized that Egyptian mathematicians of that time were able to represent any 'reasonable-sized' positive rational number as a finite sum of distinct unit fractions, i.e. fractions with numerator 1, although no algorithm was given to accomplish this representation. The first proof that any positive rational could be so represented appears to be due to Leonardo Pisano (**4**) in 1202, whilst later proofs were given by J. J. Sylvester (**6**) in 1880, and others (cf. (**3**)). In 1954 it was shown by B. M. Stewart (**5**) and independently by R. Breusch (**1**) that if $p/q > 0$ and q is odd, then p/q is the sum of a finite number of reciprocals of distinct *odd* integers. In this paper we present a theorem which considerably generalizes these results. An exact statement of this theorem requires a certain amount of terminology and will be postponed until § 3. Roughly speaking, the theorem gives some very simple necessary and sufficient conditions for a rational number to be the finite sum of reciprocals of distinct positive integers taken from a fixed sequence M, where M is any sequence which belongs to a certain rather large class of sequences of positive integers. For example, it follows from this result that a rational number p/q can be expressed as a finite sum of reciprocals of distinct squares of integers if and only if

$$\frac{p}{q} \in \left[0, \frac{\pi^2}{6} - 1\right) \cup \left[1, \frac{\pi^2}{6}\right).$$

(For more examples, see § 4.)

2. Preliminaries

Let $S = (s_1, s_2, \ldots)$ be a sequence (possibly finite) of positive real numbers.

DEFINITION 1. $P(S)$ is defined to be the set of all sums of the form $\sum_{k=1}^{\infty} \epsilon_k s_k$, where $\epsilon_k = 0$ or 1 and all but a finite number of the ϵ_k are 0.

DEFINITION 2. S is said to be *complete* if all sufficiently large integers belong to $P(S)$.

DEFINITION 3. S is said to be *entirely complete* if all positive integers belong to $P(S)$.

We shall use the following theorem due to J. L. Brown (**2**).

THEOREM. *A non-decreasing sequence $S = (s_1, s_2, \ldots)$ of positive integers is entirely complete if and only if for all $n \geqslant 0$, $\sum_{k=1}^{n} s_k \geqslant s_{n+1} - 1$ (where a sum of the form $\sum_{k=a}^{b}$ is taken to be 0 for $b < a$).*

DEFINITION 4. S^{-1} is defined to be the sequence $(s_1^{-1}, s_2^{-1}, \ldots)$.

DEFINITION 5. Let $S = (s_1, s_2, \ldots)$ be complete. The *threshold of completeness of S* is defined to be the least non-negative integer θ such that any integer greater than or equal to θ belongs to $P(S)$.

Note that S is entirely complete if and only if the threshold of completeness of S is 0.

DEFINITION 6. Let $S = (s_1, s_2, \ldots)$ be a sequence of positive integers. $M(S)$ is defined to be the monotone increasing sequence formed from the set of all products $\prod_{i=1}^{m} s_{k_i}$, where $m = 1, 2, 3, \ldots$ and $k_1 < k_2 < \ldots < k_m$. (Thus, all the terms of $M(S)$ are distinct.)

We conclude this section with a final definition and a lemma.

DEFINITION 7. Let $S = (s_1, s_2, \ldots)$ be a sequence of real numbers. A real number α is said to be *S-accessible* if, for any $\epsilon > 0$, there exists p in $P(S)$ such that $0 \leqslant p - \alpha < \epsilon$.

LEMMA 1. *Let $S = (s_1, s_2, \ldots)$ be a sequence of real numbers such that*

$$s_1 > s_2 > \ldots \quad \textit{and} \quad s_n \downarrow 0,$$

and suppose that α is S-accessible. Then, for any $\epsilon > 0$, there exist integers $m \geqslant 0$ and $k_1, k_2, \ldots, k_m$ such that

$$s_{k_1} > s_{k_2} > \ldots > s_{k_m}$$

and
$$0 \leqslant \alpha - \sum_{i=1}^{m} s_{k_i} < \min(s_{k_m}, \epsilon).$$

Proof. Note that we must have $\alpha \geqslant 0$. Suppose that $A = (a_1, \ldots, a_n)$ is a finite subsequence of S such that

$$\alpha - \sum_{k=1}^{n} a_k < 0 \leqslant \alpha - \sum_{k=1}^{n-1} a_k.$$

Call such a subsequence 'special'.

Let us first assume that there exists only a finite number of 'special' finite subsequences of S (where n is allowed to range over all positive integers), say A_j for $j = 1, 2, \ldots, t$. (Since α is S-accessible, it is clear that there exists at least one 'special' subsequence of S.) Consider the set of all quantities

$$\sum_{a \in A_j} a - \alpha \quad \text{for} \quad j = 1, 2, \ldots, t.$$

Since this is a non-empty finite set of positive numbers, it has some least element, say $\delta > 0$. Since α is S-accessible, there exist integers $m \geqslant 0$ and $k_1, k_2, \ldots, k_m$ such that

$$k_1 < k_2 < \ldots < k_m$$

and

$$0 \leqslant \sum_{i=1}^{m} s_{k_1} - \alpha < \frac{\delta}{2}.$$

Consider the quantities

$$\gamma_j = \sum_{i=1}^{m-j+1} s_{k_i} - \alpha \quad \text{for} \quad j = 1, 2, 3, \ldots.$$

From the above we have $\gamma_1 \geqslant 0$. If $\gamma_1 = 0$ then the lemma is proved. Thus, assume that $\gamma_1 > 0$. Now, if $\gamma_2 < 0$ then we reach a contradiction. For in this case we have

$$\gamma_1 = \sum_{i=1}^{m} s_{k_i} - \alpha > 0 > \gamma_2 = \sum_{i=1}^{m-1} s_{k_i} - \alpha$$

and consequently the subsequence $(s_{k_1}, s_{k_2}, \ldots, s_{k_m})$ is 'special', which is impossible since

$$\sum_{i=1}^{m} s_{k_i} - \alpha < \frac{\delta}{2}.$$

Thus we must have $\gamma_2 \geqslant 0$. If $\gamma_2 = 0$ then the lemma is proved. Therefore assume that $\gamma_2 > 0$. Now, if $\gamma_3 < 0$ then, as above, we see that $(s_{k_1}, \ldots, s_{k_{m-1}})$ is a 'special' subsequence, which is impossible since

$$\sum_{i=0}^{m-1} s_{k_i} - \alpha \leqslant \sum_{i=1}^{m} s_{k_i} - \alpha < \frac{\delta}{2}.$$

If $\gamma_3 = 0$ then the lemma is proved. We may therefore assume that $\gamma_3 > 0, \ldots,$ &c. By repeating this argument we can conclude that either $\gamma_r = 0$ for some r (in which case the lemma is proved) or $\gamma_r > 0$ for $r = 1, 2, 3, \ldots$. However, this latter case is impossible since $\gamma_{m+1} = -\alpha \leqslant 0$.

To complete the proof, assume now that there are infinitely many 'special' subsequences, say A_j for $j = 1, 2, \ldots$. Let $\epsilon > 0$. Then some A_n must contain a least term $s_u < \epsilon$, since there are only a finite number of the s_k which are greater than or equal to ϵ and hence only a finite number of the

A_j can consist entirely of these terms. Therefore

$$0 \leqslant \alpha - \sum_{\substack{a \in A_n \\ a \neq s_u}} a < s_u < \epsilon$$

since A_n is 'special'. This proves the lemma.

3. The main theorems

In this section we proceed to the main results of the paper. *All italic symbols will denote positive integers* unless otherwise specified.

THEOREM 1. *Let* $S = (s_1, s_2, \ldots)$ *be a sequence of positive integers such that*

(1) $M(S)$ *is complete,*

(2) s_n *is unbounded,*

(3) s_{n+1}/s_n *is bounded.*

Suppose that p/q *is a rational number, with* $(p, q) = 1$, *such that*

(4) p/q *is* $(M(S))^{-1}$*-accessible,*

(5) q *divides some term of* $M(S)$.

Then $p/q \in P((M(S))^{-1})$.

Proof. (a) By hypothesis there exists an A such that

$$\frac{s_{n+1}}{s_n} < A \quad \text{for} \quad n = 1, 2, 3, \ldots,$$

and A can be taken greater than 1. Let d be an arbitrary positive integer. Denote the sequence $M(S)$ by $(m_1, m_2, \ldots)$ (recalling that $m_1 < m_2 < \ldots$). Choose x so that

$$\frac{m_x}{m_d} > A.$$

Let M denote the finite sequence $(m_1, m_2, \ldots, m_x)$. Since each m_i is the product of a finite number of the s_k, there exists a y such that s_j does not occur in any of the terms

$$m_1, m_2, \ldots, m_x$$

for $j \geqslant y$. Choose u so that

$$u > y \quad \text{and} \quad s_u > \theta,$$

where θ is the threshold of completeness of $M(S)$. By condition (5), there exist r and e such that

$$\frac{p}{q} = \frac{pe}{qe} = \frac{pe}{s_1 s_2 \ldots s_r}.$$

Hence, by condition (4) and Lemma 1, we can find integers w_1 and n_1, and terms t_j of $M((s_1, s_2, \ldots, s_{w_1}))$ for $j = 1, 2, \ldots, n_1$, so that

$$t_1 < t_2 < \ldots < t_{n_1}$$

and

$$0 \leqslant \frac{p}{q} - \sum_{j=1}^{n_1} \frac{1}{t_j} = \frac{R}{s_1 s_2 \ldots s_{w_1}} < \frac{1}{s_1 s_2 \ldots s_u},$$

and such that

$$\frac{R}{s_1 s_2 \ldots s_{w_1}} < \frac{1}{t_{n_1}}.$$

If $R = 0$ then the theorem is proved. Hence we may assume that $R > 0$. By condition (2), there exists $w_2 > w_1$ such that

$$s_{w_1+1} s_{w_1+2} \ldots s_{w_2} \geqslant \theta.$$

By condition (1), there exists w_3 such that if

$$\theta \leqslant k \leqslant m_d s_u + \theta$$

then

$$k \in P(M((s_1, s_2, \ldots, s_{w_3}))).$$

Choose w so that

$$w > w_2 + w_3 + u \quad \text{and} \quad s_u s_{u+1} < s_w.$$

(Again, condition (2) guarantees the existence of such a w.)

(b) Let

$$R^* = R \,.\, s_{w_1+1} s_{w_1+2} \ldots s_w.$$

Note that

$$R^* \geqslant 1 \,.\, s_{w_1+1} s_{w_1+2} \ldots s_{w_2} \geqslant \theta.$$

Also,

$$\frac{R^*}{s_1 \ldots s_w} = \frac{R}{s_1 \ldots s_{w_1}} < \frac{1}{s_1 \ldots s_u}$$

and hence $R^* < s_{u+1} \ldots s_w$. Therefore,

$$0 \leqslant R^* - \theta < s_{u+1} \ldots s_w.$$

(c) Form the finite sequence $L = (L_1, L_2, \ldots, L_b)$ (whose terms are considered to be just *formal products* of the s_j) in the following way:

(1) $L_1 = s_u$.

(2) Suppose that

$$L_k = s_v \circ s_{v+1} \circ \ldots \circ s_w$$

for some v satisfying $u + 2 \leqslant v \leqslant w$. Then L_{k+1} is defined to be the expression

$$s_u \circ s_{u+1} \circ s_v \circ s_{v+1} \circ s_{v+2} \circ \ldots \circ s_{w-1}.$$

(3) Suppose that

$$L_k = s_{i_1} \circ s_{i_2} \circ \ldots \circ s_{i_h},$$

where

$$u \leqslant i_1 < i_2 < \ldots < i_h \leqslant w,$$

and where there is a largest m such that

$$1 \leqslant m \leqslant h \quad \text{and} \quad w \geqslant i_m + 1 \neq i_j \quad \text{for} \quad j = 1, 2, \ldots, h.$$

Then L_{k+1} is defined to be the expression

$$s_{i_1} \circ \ldots \circ s_{i_{m-1}} \circ s_{i_m+1} \circ s_{i_{m+1}} \circ \ldots \circ s_{i_h}.$$

The formation of L_{k+1} from L_k is continued until the term

$$L_b = s_{u+1} \circ s_{u+2} \circ \ldots \circ s_w$$

is reached.

As an example of L, suppose that $u = 5$ and $w = 9$. Then

$$L = (s_5, s_6, s_7, s_8, s_9, s_5 \circ s_6, s_5 \circ s_7, s_5 \circ s_8, s_5 \circ s_9, s_6 \circ s_9,$$
$$s_7 \circ s_9, s_8 \circ s_9, s_5 \circ s_6 \circ s_8, s_5 \circ s_6 \circ s_9, s_5 \circ s_7 \circ s_9, s_5 \circ s_8 \circ s_9,$$
$$s_6 \circ s_8 \circ s_9, s_7 \circ s_8 \circ s_9, s_5 \circ s_6 \circ s_7 \circ s_8, s_5 \circ s_6 \circ s_7 \circ s_9, s_5 \circ s_6 \circ s_8 \circ s_9,$$
$$s_5 \circ s_7 \circ s_8 \circ s_9, s_6 \circ s_7 \circ s_8 \circ s_9).$$

(d) For

$$L_j = s_{i_1} \circ s_{i_2} \circ \ldots \circ s_{i_n}$$

let $|L_j|$ denote the numerical value of $s_{i_1} s_{i_2} \ldots s_{i_n}$, i.e. $|L_j|$ is just the ordinary product of the integers $s_{i_1}, s_{i_2}, \ldots, s_{i_n}$. Form the finite sequence $F = (f_1, f_2, \ldots, f_z)$ as follows:

(1) $f_1 = |L_1|$.

(2) f_{k+1} is defined to be the smallest integer g such that

$$g > f_k \quad \text{and} \quad g = |L_j|$$

for some $j \leqslant b$.

Thus,

$$f_1 < f_2 < \ldots < f_z$$

and f_k is in $M((s_u, s_{u+1}, \ldots, s_w))$ for $k = 1, 2, \ldots, z$. Now, for L_k in (c)2 we have

$$\frac{|L_{k+1}|}{|L_k|} = \frac{s_u s_{u+1}}{s_w} < 1 < A.$$

Also, for L_k in (c)3 we have

$$\frac{|L_{k+1}|}{|L_k|} = \frac{s_{i_m+1}}{s_{i_m}} < A.$$

Thus,

$$\frac{|L_{k+1}|}{|L_k|} < A \quad \text{for} \quad k = 1, 2, \ldots, b-1.$$

Therefore, for f_n in F, where $n = 1, 2, \ldots, z-1$, it follows from the definition of f_{n+1} that

$$\frac{f_{n+1}}{f_n} < A.$$

Note that

$$f_1 = s_u \quad \text{and} \quad f_z \geqslant s_{u+1} s_{u+2} \ldots s_w.$$

(e) Suppose that

$$R^* \in P(M((s_1, s_2, \ldots, s_w))).$$

Then there exist n_2 and $u_1, u_2, \ldots, u_{n_2}$ in $M((s_1, s_2, \ldots, s_w))$ such that

$$u_1 < u_2 < \ldots < u_{n_2}$$

and

$$\sum_{k=1}^{n_2} u_k = R^*.$$

Thus,

$$\frac{R^*}{s_1 s_2 \ldots s_w} = \sum_{k=1}^{n_2} \frac{u_k}{s_1 s_2 \ldots s_w} = \sum_{k=1}^{n_2} \frac{1}{t'_k},$$

where t'_k belongs to $M((s_1, s_2, \ldots, s_w))$ and

$$t'_1 > t'_2 > \ldots > t'_{n_2}.$$

Since, by (a),

$$\frac{R^*}{s_1 s_2 \ldots s_w} = \frac{R}{s_1 s_2 \ldots s_{w_1}} < \frac{1}{t_{n_1}},$$

we have

$$\frac{1}{t'_{n_2}} < \frac{1}{t_{n_1}}.$$

Therefore,

$$\frac{p}{q} = \sum_{k=1}^{n_1} \frac{1}{t_k} + \frac{R^*}{s_1 s_2 \ldots s_w} = \sum_{k=1}^{n_1} \frac{1}{t_k} + \sum_{k=1}^{n_2} \frac{1}{t'_k},$$

where all the denominators are distinct elements of $M((s_1, s_2, \ldots, s_w))$. Therefore,

$$\frac{p}{q} \in P((M(S))^{-1}).$$

Thus, to prove the theorem it suffices to show that

$$R^* \in P(M((s_1, s_2, \ldots, s_w))).$$

(f) By (b), we have

$$0 \leqslant R^* - \theta < s_{u+1} s_{u+2} \ldots s_w \leqslant f_z \leqslant m_x f_z.$$

If

$$m_1 f_1 > R^* - \theta$$

then

$$\theta \leqslant R^* < m_1 f_1 + \theta \leqslant m_d s_u + \theta.$$

By the definition of w, we have

$$R^* \in P(M((s_1, s_2, \ldots, s_w)))$$

and hence, by (e), the theorem is proved in this case. Therefore, suppose that

$$m_1 f_1 \leqslant R^* - \theta.$$

Since $m_x f_j > m_1 f_{j+1}$ for $j = 1, 2, \ldots, z-1$, and

$$m_1 f_1 \leqslant R^* - \theta < m_x f_z,$$

there exists at least one integer $p' \leqslant x-1$ such that there is an $f_{a'}$ in F such that

$$m_{p'} f_{a'} \leqslant R^* - \theta < m_{p'+1} f_{a'}.$$

Let p^* be the largest integer such that there exists an f_a in F such that

$$m_{p^*}f_a \leqslant R^* - \theta < m_{p^*+1}f_a.$$

There are two cases:

(i) Suppose that $p^* < d$. Then we must have

$$m_d f_1 > R^* - \theta.$$

For suppose not, i.e. suppose that

$$m_d f_1 \leqslant R^* - \theta.$$

Then

$$m_{d+1} f_1 \leqslant R^* - \theta$$

since if

$$m_{d+1} f_1 > R^* - \theta$$

then

$$m_d f_1 \leqslant R^* - \theta < m_{d+1} f_1,$$

which is in contradiction to the definition of p^*. Similarly,

$$m_{d+2} f_1 \leqslant R^* - \theta,$$
$$m_{d+3} f_1 \leqslant R^* - \theta,$$
$$\cdot \quad \cdot \quad \cdot \quad ,$$
$$m_x f_1 \leqslant R^* - \theta.$$

But

$$m_x f_1 > m_d f_2$$

since

$$\frac{m_x}{m_d} > A > \frac{f_2}{f_1}.$$

Therefore

$$m_d f_2 \leqslant R^* - \theta.$$

Thus, as before, we must have

$$m_{d+1} f_2 \leqslant R^* - \theta,$$
$$m_{d+2} f_2 \leqslant R^* - \theta$$
$$\cdot \quad \cdot \quad \cdot \quad ,$$
$$m_x f_2 \leqslant R^* - \theta.$$

But

$$\frac{m_x}{m_d} > A > \frac{f_3}{f_2}.$$

Therefore

$$m_d f_3 \leqslant R^* - \theta$$

and

$$m_{d+1} f_3 \leqslant R^* - \theta,$$
$$m_{d+2} f_3 \leqslant R^* - \theta,$$
$$\cdot \quad \cdot \quad \cdot \quad ,$$

and finally,

$$m_x f_z \leqslant R^* - \theta,$$

which is a contradiction to our hypothesis on the size of $R^*-\theta$. Consequently, we must have

$$m_d f_1 > R^* - \theta.$$

Therefore,

$$\theta \leqslant R^* < m_d f_1 + \theta = m_d s_u + \theta.$$

Hence, by hypothesis on w, we know that

$$R^* \in P(M((s_1, s_2, \ldots, s_w))),$$

and, by (e), the theorem is proved in this case.

(ii) Suppose that $p^* \geqslant d$. Again we have two cases:

(1) Suppose that there exists c such that

$$k \geqslant c \Rightarrow m_1 + m_2 + \ldots + m_k \geqslant m_{k+1}.$$

Now d was chosen to be an arbitrary positive integer. Hence we can assume that d was chosen so that $d \geqslant c$. Consider the quantities

$$R^* - m_{p^*} f_a,$$
$$R^* - (m_{p^*} + m_{p^*-1}) f_a,$$
$$\cdot \quad \cdot \quad \cdot \quad ,$$
$$R^* - (m_{p^*} + m_{p^*-1} + \ldots + m_1) f_a.$$

Since $p^* \geqslant d$,

$$m_1 + m_2 + \ldots + m_{p^*} \geqslant m_{p^*+1}.$$

Therefore,

$$R^* - (m_{p^*} + \ldots + m_1) f_a \leqslant R^* - m_{p^*+1} f_a < \theta$$

by the definition of p^*.

Let k be the smallest integer such that

$$R^* - (m_{p^*} + m_{p^*-1} + \ldots + m_k) f_a \geqslant \theta.$$

Thus

$$2 \leqslant k \leqslant p^* \quad \text{and} \quad R^* - (m_{p^*} + m_{p^*-1} + \ldots + m_{k-1}) f_a < \theta.$$

Hence,

$$R^* - (m_{p^*} + m_{p^*-1} + \ldots + m_k) f_a < \theta + m_{k-1} f_a.$$

But

$$m_{k-1} f_a + \theta < m_k f_a$$

since

$$f_a \geqslant s_u > \theta \quad \text{and} \quad m_{k-1} + 1 \leqslant m_k,$$

and consequently

$$(m_k - m_{k-1}) f_a \geqslant 1 . s_u > \theta.$$

Therefore,

$$R^{**} = R^* - (m_{p^*} + m_{p^*-1} + \ldots + m_k) f_a < m_k f_a$$

and consequently

$$R^* = m_{p^*} f_a + m_{p^*-1} f_a + \ldots + m_k f_a + R^{**},$$

where

$$\theta \leqslant R^{**} < m_k f_a \quad \text{and} \quad R^{**} < R^*.$$

Thus, to show that

$$R^* \in P(M((s_1, s_2, \ldots, s_w)))$$

it suffices to show that

$$R^{**} \in P(M((s_1, s_2, \ldots, s_w))).$$

For, any term used in representing R^{**} cannot exceed R^{**} and hence must be less than $m_k f_a$, and from the definition of u it follows immediately that if m is in M and f is in F then

$$mf \in M((s_1, s_2, \ldots, s_w)).$$

Now we return to the beginning of (f), replace R^* by R^{**}, and repeat the preceding argument (which is possible since $\theta \leqslant R^{**} < R^* < m_x f_z$).

(2) Suppose that for any c there exists k_c such that

$$k_c \geqslant c \quad \text{and} \quad m_1 + m_2 + \ldots + m_{k_c} < m_{k_c+1}.$$

Now, by hypothesis, $M(S)$ is complete. Thus, if there exists an n (where, of course, n denotes a positive integer) such that

$$n \notin P(M(S))$$

then we have

$$\sum_{j=1}^{k_c} m_j - n \notin P((m_1, \ldots, m_{k_c})) \quad \text{for} \quad c = 1, 2, 3, \ldots.$$

(For it is clear that if $A = (a_1, \ldots, a_i)$ is any finite sequence then $n \notin P(A)$ if and only if $\sum_{k=1}^{i} a_k - n \notin P(A)$.) But

$$\sum_{j=1}^{k_c} m_j < m_{k_c+1} < m_{k_c+2} < \ldots$$

so that

$$\sum_{j=1}^{k_c} m_j - n \notin P(M(S)) \quad \text{for} \quad c = 1, 2, 3, \ldots.$$

Therefore, there are infinitely many positive integers which do not belong to $P(M(S))$. This is a contradiction to the assumption that $M(S)$ is complete. Hence there cannot exist an n such that

$$n \notin P(M(S)).$$

In other words, $M(S)$ is entirely complete and thus $\theta = 0$. Consider the quantities

$$R^* - m_{p^*} f_a,$$

$$R^* - (m_{p^*} + m_{p^*-1}) f_a,$$

$$\cdot \quad \cdot \quad \cdot \quad ,$$

$$R^* - (m_{p^*} + m_{p^*-1} + \ldots + m_1) f_a.$$

Since $M(S)$ is entirely complete, by the theorem of Brown mentioned in § 2 we have

$$\sum_{j=1}^{k} m_j \geqslant m_{k+1} - 1 \quad \text{for all } k.$$

Therefore,

$$R^* - (m_{p^*} + \ldots + m_1) f_a \leqslant R^* - (m_{p^*+1} - 1) f_a = R^* - m_{p^*+1} f_a + f_a < 0 + f_a$$
$$= f_a \leqslant m_1 f_a.$$

Let k be the smallest integer such that

$$R^* - (m_{p^*} + m_{p^*-1} + \ldots + m_k) f_a \geqslant 0.$$

Then $1 \leqslant k \leqslant p^*$. If $k = 1$, then

$$R^{**} = R^* - (m_{p^*} + \ldots + m_1) f_a < m_1 f_a$$

and

$$R^{**} < R^*.$$

Thus,

$$R^* = m_{p^*} f_a + \ldots + m_1 f_a + R^{**}.$$

Hence, it suffices to show that

$$R^{**} \in P(M((s_1, s_2, \ldots, s_w))).$$

We return to the beginning of (f), replace R^* by R^{**}, and repeat the preceding argument (which is possible since $0 = \theta \leqslant R^{**} < R^* < m_x f_z$). If $k > 1$, then

$$R^* - (m_{p^*} + m_{p^*-1} + \ldots + m_{k-1}) f_a < 0.$$

Thus,

$$R^{**} = R^* - (m_{p^*} + m_{p^*-1} + \ldots + m_k) f_a < m_{k-1} f_a < m_k f_a$$

and

$$R^{**} < R^*.$$

Hence, as before, it suffices to show that

$$R^{**} \in P(M((s_1, s_2, \ldots, s_w))).$$

Now we return to the beginning of (f), replace R^* by R^{**}, and repeat the preceding argument (which is possible since $0 = \theta \leqslant R^{**} < R^* < m_x f_z$). Now, each time the argument of (f) is applied the 'new' remainder $R^{*(n+1)}$ is strictly less than the 'starting' remainder $R^{*(n)}$. But all the remainders $R^{*(j)}$ are always non-negative integers. Hence the argument must terminate in a finite number of steps, resulting in the conclusion that

$$R^* \in P(M((s_1, s_2, \ldots, s_w))).$$

Therefore, by (e), we have

$$\frac{p}{q} \in P((M(S))^{-1}).$$

This proves the theorem.

THEOREM 2. *Condition* (2) *in Theorem* 1 *can be replaced by*

(2′) s_n *is bounded and there exist infinitely many* s_k *which are not equal to* 1.

Proof. By hypothesis it follows that there exist

$$k, n_1, n_2, n_3, \ldots$$

such that

$$n_1 < n_2 < n_3 < \ldots \quad \text{and} \quad 1 < k = s_{n_j} \quad \text{for } j = 1, 2, 3, \ldots.$$

Let

$$S^* = (s_1, k, s_2, k, k^2, s_3, k, k^2, k^3, s_4, \ldots).$$

Then it follows at once that

$$M(S) = M(S^*).$$

Thus

$$(M(S^*))^{-1} = (M(S))^{-1} \quad \text{and} \quad P((M(S^*))^{-1}) = P((M(S))^{-1}).$$

But in S^* we have $s_n{}^*$ is unbounded and

$$\frac{s_{n+1}{}^*}{s_n{}^*} \leqslant \max(k, s_1, s_2, \ldots) < \infty.$$

Thus, S^* satisfies conditions (1), (2), and (3) of Theorem 1. Therefore, since

$$M(S^*) = M(S),$$

the theorem is proved.

Remark. Note that if there exist only finitely many n such that $s_n \neq 1$, then $M(S)$ must contain just a finite number of terms and therefore $M(S)$ is not complete. Combining this fact with Theorems 1 and 2 we have

THEOREM 3. *Condition* (2) *in Theorem* 1 *can be omitted.*

THEOREM 4. *Let* $S = (s_1, s_2, \ldots)$ *and suppose that* $p/q \in P((M(S))^{-1})$, *where* $(p, q) = 1$. *Then*

(1) p/q *is* $(M(S))^{-1}$*-accessible,*

(2) q *divides some term of* $M(S)$.

Proof. Condition (1) is immediate from the definition of accessibility. To get condition (2), we have by hypothesis

$$\frac{p}{q} = \frac{1}{s_{i_1} \ldots s_{i_k}} + \ldots + \frac{1}{s_{j_1} \ldots s_{j_m}} = \frac{r}{s_1 s_2 \ldots s_n}$$

for some r and n. Therefore,

$$p s_1 s_2 \ldots s_n = qr.$$

But we also have

$$(p, q) = 1.$$

Thus q divides $s_1 s_2 \ldots s_n$, where $s_1 s_2 \ldots s_n$ is a term of $M(S)$. This proves the theorem.

We now combine the preceding theorems to obtain the main result of this paper.

THEOREM 5. *Let* $S = (s_1, s_2, \ldots)$ *be a sequence of positive integers such that*

(1) $M(S)$ *is complete,*

(2) s_{n+1}/s_n *is bounded.*

Then

$$\frac{p}{q} \in P((M(S))^{-1})$$

(*where* $(p,q) = 1$) *if and only if*

(3) p/q *is* $(M(S))^{-1}$-*accessible,*

(4) q *divides some term of* $M(S)$.

Proof. The proof follows immediately from Theorems 3 and 4.

Remark. It might be surmised that in Theorem 5 either condition (1) or condition (2) could be weakened or even removed. While no examples are known which show that condition (2) cannot be omitted, the following example shows that condition (1) cannot be omitted.

EXAMPLE. Let S be the sequence

$$(4, 1, 3, 3^2, 3^3, \ldots, 3^n, \ldots).$$

Then

$$M(S) = (1, 3, 4, 3^2, 4 \cdot 3, 3^3, 4 \cdot 3^2, \ldots)$$

and S satisfies condition (2) of Theorem 5 but not condition (1) (since, e.g., $2 \cdot 3^n \notin P(M(S))$ for $n = 0, 1, 2, \ldots$). Consider the rational number $\frac{1}{2}$. Since

$$\tfrac{1}{2} = \sum_{k=1}^{\infty} 3^{-k}$$

and

$$\sum_{k=m+2}^{\infty} 3^{-k} = 3^{-m-1} . 2^{-1} = (6 \cdot 3^m)^{-1} \quad \text{for any } m,$$

$$\sum_{k=1}^{m+1} 3^{-k} + (4 \cdot 3^m)^{-1} = \tfrac{1}{2} - (6 \cdot 3^m)^{-1} + (4 \cdot 3^m)^{-1} = \tfrac{1}{2} + (12 \cdot 3^m)^{-1}.$$

Since m is arbitrary, $\frac{1}{2}$ is $(M(S))^{-1}$-accessible. We also know that 2 divides 4, and since 4 is a term of $M(S)$ conditions (3) and (4) of Theorem 5 are satisfied. It is now asserted that

$$\tfrac{1}{2} \notin P((M(S))^{-1}).$$

For suppose that

$$\tfrac{1}{2} \in P((M(S))^{-1}).$$

Then

$$\tfrac{1}{2} = 3^{-a_1} + \ldots + 3^{-a_m} + 4^{-1}(3^{-b_1} + \ldots + 3^{-b_n}),$$

where $0 \leqslant a_1 < a_2 < \ldots < a_m$ and $0 \leqslant b_1 < b_2 < \ldots < b_n$. Now, $a_1 \geqslant 1$ since $a_1 = 0$ implies that

$$\tfrac{1}{2} = 1 + \ldots,$$

which is impossible. If $a_m < b_n$ then

$$3^{b_n - a_m} \equiv 0 \pmod 3.$$

Thus

$$\frac{3^{b_n}}{2} = 3^{b_n - a_1} + \ldots + 3^{b_n - a_m} + 4^{-1}(3^{b_n - b_1} + \ldots + 1),$$

$$2 \cdot 3^{b_n} = 4 \cdot 3^{b_n - a_1} + \ldots + 4 \cdot 3^{b_n - a_m} + (3^{b_n - b_1} + \ldots + 1),$$

which is a contradiction since the left-hand side is divisible by 3 while the right-hand side is not. A similar contradiction is reached if we assume that $a_m > b_n$. Hence we must have $a_m = b_n$ (note that $a_m \geqslant a_1 \geqslant 1$). This implies that

$$\frac{3^{b_n}}{2} = 3^{b_n - a_1} + \ldots + 3^{b_n - a_m} + 4^{-1}(3^{b_n - b_1} + \ldots + 1) = 3k + 1 + 4^{-1}(3d + 1)$$

for some k and d. Therefore

$$2 \cdot 3^{b_n} = 12k + 4 + 3d + 1,$$

which again is a contradiction for the same reason as above. Thus,

$$\tfrac{1}{2} \notin P((M(S))^{-1}),$$

which proves the assertion.

4. Concluding remarks

Theorem 5 can now be applied to a variety of sequences S which satisfy conditions (1) and (2). While the proofs involved in these applications will be left to a later paper, several results will be stated here.

(1) Let a and b be arbitrary positive integers and let p/q denote some positive rational with $(p, q) = 1$.

Then

$$\frac{p}{q} = \sum_{i=1}^{n} \frac{1}{ak_i + b}$$

for some positive integers n and $k_1 < k_2 < \ldots < k_n$ if and only if

$$\left(\frac{q}{(q,(a,b))}, \frac{a}{(a,b)}\right) = 1$$

(where (x, y) denotes the g.c.d. of x and y). (This result is obtained by considering the sequence $(a+1, 2a+1, 3a+1, \ldots)$.)

(2) A rational number p/q can be expressed as a finite sum of reciprocals of distinct squares of integers if and only if

$$\frac{p}{q} \in \left[0, \frac{\pi^2}{6} - 1\right) \cup \left[1, \frac{\pi^2}{6}\right).$$

(3) For any positive integer n, every sufficiently small positive rational can be expressed as the finite sum of reciprocals of distinct nth powers of integers.

(4) A positive rational p/q with $(p, q) = 1$ can be expressed as the finite sum of reciprocals of distinct square-free integers if and only if q is square-free.

(5) Let T be a set of integers which contains all sufficiently large prime numbers and all sufficiently large squares. Then *every* positive rational number can be expressed as a finite sum of reciprocals of distinct integers taken from T.

It may be remarked that (1) and (5) settle two questions raised by H. S. Wilf in (**7**).

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