

Generalized Euler–Seidel method for second order recurrence relations

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Abstract: We obtain identities for the generalized second order recurrence relation by using the generalized Euler–Seidel matrix with parameters x, y . As a consequence, we give some properties and generating functions of well-known special integer sequences.

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1 Introduction

Let (a_n) be a sequence. In [2], the Euler–Seidel matrix associated with this sequence is determined recursively by the formula

$$\begin{aligned} a_n^0 &= a_n \quad (n \geq 0) \\ a_n^k &= a_n^{k-1} + a_{n+1}^{k-1} \quad (n \geq 0, k \geq 1). \end{aligned} \quad (1)$$

From relation (1), it can be seen that the first row and the first column can be transformed into each other via the well known binomial inverse pair as,

$$a_0^n = \sum_{k=0}^n \binom{n}{k} a_k^0, \quad (2)$$

$$a_n^0 = \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} a_0^k. \quad (3)$$

Also any entry a_n^k can be written in terms of the initial sequence as:

$$a_n^k = \sum_{i=0}^k \binom{k}{i} a_{n+i}^0. \quad (4)$$

Proposition 1. (Euler) [4] Let

$$a(t) = \sum_{n=0}^{\infty} a_n^0 t^n$$

be the generating function of the initial sequence (a_n^0) . Then the generating function of the sequence (a_n^n) is

$$\bar{a}(t) = \sum_{n=0}^{\infty} a_n^n t^n = \frac{1}{1-t} a\left(\frac{t}{1-t}\right). \quad (5)$$

Proposition 2. (Seidel) [9] Let

$$A(t) = \sum_{n=0}^{\infty} a_n^0 \frac{t^n}{n!}$$

be the exponential generating function of the initial sequence (a_n^0) . Then the exponential generating function of the sequence (a_n^n) is

$$\bar{A}(t) = \sum_{n=0}^{\infty} a_n^n \frac{t^n}{n!} = e^t A(t). \quad (6)$$

In fact, it is possible to state a more general result than (6). The following equation gives relation between exponential generating function of columns (or rows) with the exponential generating function of the initial sequence (see [2]).

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n^k \frac{u^k}{k!} \frac{t^n}{n!} = e^u A(t+u). \quad (7)$$

In [7] there are applications of Euler–Seidel matrix for hyperharmonic and r –Stirling numbers. Also authors introduced "symmetric infinite matrix" and give some applications in [3].

In [5] the generalized second order recurrence sequence $\{W_n(a, b; p, q)\}$ is defined as for $n \geq 0$

$$W_{n+2} = pW_{n+1} - qW_n \quad (8)$$

with initial conditions

$$W_0 = a, \quad W_1 = b,$$

where $p^2 - 4q > 0$. Let the roots of the equation $t^2 - pt + q = 0$ be $\alpha = \frac{p+\sqrt{p^2-4q}}{2}$ and $\beta = \frac{p-\sqrt{p^2-4q}}{2}$. Then W_n can be written in the form

$$W_n = A\alpha^n + B\beta^n, \quad (9)$$

where $A = \frac{b-a\beta}{\alpha-\beta}$ and $B = \frac{a\alpha-b}{\alpha-\beta}$. The following generating functions of $\{W_n\}$ are given in [6, 8] as:

$$\sum_{n=0}^{\infty} W_n t^n = \frac{a + (b - pa)t}{1 - pt + qt^2} \quad (10)$$

and

$$\sum_{n=0}^{\infty} W_n \frac{t^n}{n!} = Ae^{\alpha t} + Be^{\beta t}. \quad (11)$$

Mező gave the generating functions of the general second-order recurrence relations in [8]. Here, we get some relation and generating functions of the general second-order recurrence relations by using generalized Euler–Seidel matrices.

The special cases of $\{W_n(a, b; p, q)\}$ give Fibonacci numbers F_n (Oeis A000045), Lucas numbers L_n (Oeis A000032), Pell numbers (or Silver Fibonacci numbers) P_n (Oeis A000129), Pell–Lucas numbers Q_n (Oeis A002203), Jacobsthal numbers J_n (Oeis A001045), Jacobsthal–Lucas numbers j_n (Oeis A014551), Bronze Fibonacci numbers $\mathcal{B}_n$ (Oeis A006190), Signed Fibonacci numbers $\mathcal{F}_n$ (Oeis A039834), Signed Pell numbers $\mathcal{P}_n$ (Oeis A215936).

Also we get the sequences; D_n : denominators of continued fraction convergents to $\sqrt{5}$ (Oeis A001076) and N_n : numerators of continued fraction convergents to $\sqrt{2}$ (Oeis A001333) as follows:

$$\begin{aligned} W_n(0, 1; 1, -1) &= F_n, & W_n(2, 1; 1, -1) &= L_n, \\ W_n(0, 1; 2, -1) &= P_n, & W_n(2, 2; 2, -1) &= Q_n, \\ W_n(0, 1; 1, -2) &= J_n, & W_n(2, 1; 1, -2) &= j_n, \\ W_n(0, 1; 3, -1) &= \mathcal{B}_n, & W_n(1, 1; -1, -1) &= \mathcal{F}_n, \\ W_n(0, 1; -2, -1) &= \mathcal{P}_n, & W_n(0, 1; 4, -1) &= D_n, \\ W_n(1, 1; 2, -1) &= N_n. \end{aligned}$$

2 Generalized Euler–Seidel matrices with two parameters

In this section, we consider the generalized Euler–Seidel matrix, which is given in [1] with parameters x, y . We obtain the connection between the generating functions of the initial sequence and the first column entries of the generalized Euler–Seidel matrices.

Let us consider a given sequence $(a_n)_{n \geq 0}$. Generalized Euler–Seidel matrix with parameters x and y (see [1]) corresponding to this sequence is recursively defined by the formulae

$$\begin{aligned} a_n^0 &= a_n \quad (n \geq 0) \\ a_n^k(x, y) &= xa_n^{k-1} + ya_{n+1}^{k-1} \quad (n \geq 0, k \geq 1 \text{ positive integers}). \end{aligned} \quad (12)$$

where a_n^k represents the k -th row and n -th column entry and x and y are nonzero real parameters; i.e;

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & a_n^{k-1} & a_{n+1}^{k-1} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & x \downarrow & \swarrow y & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & a_n^k & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

From now on for the sake of simplicity we represent $a_n^k(x, y)$ with a_n^k .

The following proposition gives the relation between the any entry of the matrix and the initial sequence.

Proposition 3. [1] We have

$$a_n^k = \sum_{i=0}^k \binom{k}{i} x^{k-i} y^i a_{n+i}^0. \quad (13)$$

Proof. By induction on $n + k$. □

The first row and column can be transformed into each other via the well known binomial inverse pair as follows.

Corollary 4.

$$a_0^n = x^n \sum_{i=0}^n \binom{n}{i} \left(\frac{y}{x}\right)^i a_i^0 \quad (14)$$

and

$$a_n^0 = \frac{1}{y^n} \sum_{i=0}^n \binom{n}{i} (-x)^{n-i} a_i^0. \quad (15)$$

Generating Functions. We give connections between the generating functions of the initial sequences and the first column entries.

Proposition 5. The recurrence (12) gives the following relation:

$$\overline{a_{x,y}}(t) = \frac{1}{1 - xt} a_{x,y} \left(\frac{yt}{1 - xt} \right) \quad (16)$$

where

$$\overline{a_{x,y}}(t) = \sum_{n=0}^{\infty} a_0^n t^n \quad \text{and} \quad a_{x,y}(t) = \sum_{n=0}^{\infty} a_n^0 t^n.$$

Proof. Considering (12) we write

$$\overline{a_{x,y}}(t) = \sum_{n=0}^{\infty} \left(\sum_{r=0}^n \binom{n}{r} x^{n-r} y^r a_r^0 \right) t^n.$$

By changing the order of the above sums and using Newton binomial sums formula we obtain

$$\begin{aligned}\overline{a_{x,y}}(t) &= \sum_{r=0}^{\infty} \left(\frac{y}{x}\right)^r a_r^0 \sum_{n=0}^{\infty} \binom{n+r}{r} (xt)^{n+r} \\ &= \frac{1}{1-xt} \sum_{r=0}^{\infty} a_r^0 \left(\frac{yt}{1-xt}\right)^r.\end{aligned}$$

This completes the proof. □

Now we give the generalization of the equation (7).

Proposition 6. *For the a_n^k entries of the Generalized Euler–Seidel Matrices we have:*

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n^k \frac{u^k}{k!} \frac{t^n}{n!} = e^{xu} A_{x,y}(t + yu)$$

where

$$A_{x,y}(t) = \sum_{n=0}^{\infty} a_n^0 \frac{t^n}{n!}.$$

Proof. Using (13) we have

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \left(\sum_{i=0}^k \binom{k}{i} x^{k-i} y^i a_{n+i}^0 \right) \frac{u^k}{k!} \frac{t^n}{n!} = \sum_{k=0}^{\infty} \sum_{i=0}^k \frac{x^{k-i} u^{k-i}}{(k-i)!} \sum_{n=0}^{\infty} a_{n+i}^0 \frac{t^n}{n!} \frac{(yu)^i}{i!}.$$

If we write RHS by means of Cauchy product we get:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} a_n^k \frac{u^k}{k!} \frac{t^n}{n!} = \sum_{k=0}^{\infty} \frac{(xu)^k}{k!} \sum_{n=0}^{\infty} \left(a_{n+k}^0 \frac{t^n}{n!} \right) \frac{(yu)^k}{k!}.$$

We can equally well write the last sum in the form $A_{x,y}(t + yu)$, which completes the proof. □

The following corollary also provides the connection between the exponential generating functions of the initial sequence and the first column entries.

Corollary 7. *[1] The following relation holds:*

$$\overline{A}_{x,y}(t) = e^{xt} A_{x,y}(yt) \tag{17}$$

where

$$\overline{A}_{x,y}(t) = \sum_{n=0}^{\infty} a_n^0 \frac{t^n}{n!} \quad \text{and} \quad A_{x,y}(t) = \sum_{n=0}^{\infty} a_n^0 \frac{t^n}{n!}.$$

3 Applications of generalized Euler–Seidel matrix

In this section, we show that the generalized Euler–Seidel method is useful to obtain some properties of the generalized second order recurrence relation.

Proposition 8.

$$W_{n+2k} = \sum_{i=0}^k \binom{k}{i} (-q)^{k-i} p^i W_{n+i}. \quad (18)$$

Proof. By setting $x = -q$ and $y = p$ in (12), we obtain

$$a_n^k = -q a_n^{k-1} + p a_{n+1}^{k-1}. \quad (19)$$

For $a_n^0 = W_n$, $n \geq 0$. We can write $a_n^1 = W_{n+2}$. By induction on k and using equation (19), we obtain $a_n^k = W_{n+2k}$. Now considering equation (13) for $x = -q$ and $y = p$, we have

$$a_n^k = \sum_{i=0}^k \binom{k}{i} (-q)^{k-i} p^i a_{n+i}^0.$$

Then we obtain

$$W_{n+2k} = \sum_{i=0}^k \binom{k}{i} (-q)^{k-i} p^i W_{n+i}.$$

This completes the proof. $\square$

Using (18), we get the following identities of the Fibonacci numbers F_n , Lucas numbers L_n , Pell numbers P_n , Pell–Lucas numbers Q_n , Jacobsthal numbers J_n , Jacobsthal–Lucas numbers j_n , Bronze Fibonacci numbers $\mathcal{B}_n$, Signed Fibonacci numbers $\mathcal{F}_n$, Signed Pell numbers $\mathcal{P}_n$, and also D_n and N_n numbers

$$\begin{aligned} F_{n+2k} &= \sum_{i=0}^k \binom{k}{i} F_{n+i}, & L_{n+2k} &= \sum_{i=0}^k \binom{k}{i} L_{n+i}, \\ P_{n+2k} &= \sum_{i=0}^k \binom{k}{i} 2^i P_{n+i}, & Q_{n+2k} &= \sum_{i=0}^k \binom{k}{i} 2^i Q_{n+i}, \\ J_{n+2k} &= \sum_{i=0}^k \binom{k}{i} 2^{k-i} J_{n+i}, & j_{n+2k} &= \sum_{i=0}^k \binom{k}{i} 2^{k-i} j_{n+i}, \\ \mathcal{B}_{n+2k} &= \sum_{i=0}^k \binom{k}{i} 3^i \mathcal{B}_{n+i}, & \mathcal{F}_{n+2k} &= \sum_{i=0}^k \binom{k}{i} (-1)^i \mathcal{F}_{n+i}, \\ \mathcal{P}_{n+2k} &= \sum_{i=0}^k \binom{k}{i} (-2)^i \mathcal{P}_{n+i}, & D_{n+2k} &= \sum_{i=0}^k \binom{k}{i} 4^i D_{n+i}, \\ N_{n+2k} &= \sum_{i=0}^k \binom{k}{i} 2^i N_{n+i}. \end{aligned}$$

Corollary 9.

$$W_{2n} = \sum_{i=0}^n \binom{n}{i} (-q)^{n-i} p^i W_i, \quad (20)$$

$$W_n = \frac{1}{p^n} \sum_{i=0}^n \binom{n}{i} (q)^{n-i} W_{2i} \quad (21)$$

and

$$W_{2n+1} = \sum_{i=0}^n \binom{n}{i} (-q)^{n-i} p^i W_{i+1}, \quad (22)$$

$$W_n = \frac{1}{p^{n-1}} \sum_{i=1}^n \binom{n-1}{i-1} (q)^{n-i} W_{2i-1}. \quad (23)$$

From (20), we obtain some formulas for these well-known sequences by the new method.

$$\begin{aligned}
F_{2n} &= \sum_{i=0}^n \binom{n}{i} F_i, & L_{2n} &= \sum_{i=0}^n \binom{n}{i} L_i, \\
P_{2n} &= \sum_{i=0}^n \binom{n}{i} 2^i P_i, & Q_{2n} &= \sum_{i=0}^n \binom{n}{i} 2^i Q_i, \\
J_{2n} &= \sum_{i=0}^n \binom{n}{i} 2^{n-i} J_i, & j_{2n} &= \sum_{i=0}^n \binom{n}{i} 2^{n-i} j_i, \\
\mathcal{B}_{2n} &= \sum_{i=0}^n \binom{n}{i} 3^i \mathcal{B}_i, & \mathcal{F}_{2n} &= \sum_{i=0}^n \binom{n}{i} (-1)^i \mathcal{F}_i, \\
\mathcal{P}_{2n} &= \sum_{i=0}^n \binom{n}{i} (-2)^i \mathcal{P}_i, & D_{2n} &= \sum_{i=0}^n \binom{n}{i} 4^i D_i, \\
N_{2n} &= \sum_{i=0}^n \binom{n}{i} 2^i N_i.
\end{aligned}$$

Here with help of equation (21), we have following identities:

$$\begin{aligned}
F_n &= \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} F_{2i}, & L_n &= \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} L_{2i}, \\
P_n &= \frac{1}{2^n} \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} P_{2i}, & Q_n &= \frac{1}{2^n} \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} Q_{2i}, \\
J_n &= \sum_{i=0}^n \binom{n}{i} (-2)^{n-i} J_{2i}, & j_n &= \sum_{i=0}^n \binom{n}{i} (-2)^{n-i} j_{2i}, \\
\mathcal{B}_n &= \frac{1}{3^n} \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} \mathcal{B}_{2i}, & \mathcal{F}_n &= \sum_{i=0}^n \binom{n}{i} (-1)^i \mathcal{F}_{2i}, \\
\mathcal{P}_n &= \frac{1}{2^n} \sum_{i=0}^n \binom{n}{i} (-1)^i \mathcal{P}_{2i}, & D_n &= \frac{1}{4^n} \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} D_{2i}, \\
N_n &= \frac{1}{2^n} \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} N_{2i}.
\end{aligned}$$

We show from (22)

$$\begin{aligned}
F_{2n+1} &= \sum_{i=0}^n \binom{n}{i} F_{i+1}, & L_{2n+1} &= \sum_{i=0}^n \binom{n}{i} L_{i+1}, \\
P_{2n+1} &= \sum_{i=0}^n \binom{n}{i} 2^i P_{i+1}, & Q_{2n+1} &= \sum_{i=0}^n \binom{n}{i} 2^i Q_{i+1}, \\
J_{2n+1} &= \sum_{i=0}^n \binom{n}{i} 2^{n-i} J_{i+1}, & j_{2n+1} &= \sum_{i=0}^n \binom{n}{i} 2^{n-i} j_{i+1}, \\
\mathcal{B}_{2n+1} &= \sum_{i=0}^n \binom{n}{i} 3^i \mathcal{B}_{i+1}, & \mathcal{F}_{2n+1} &= \sum_{i=0}^n \binom{n}{i} (-1)^i \mathcal{F}_{i+1}, \\
\mathcal{P}_{2n+1} &= \sum_{i=0}^n \binom{n}{i} (-2)^i \mathcal{P}_{i+1}, & D_{2n+1} &= \sum_{i=0}^n \binom{n}{i} 4^i D_{i+1}, \\
N_{2n+1} &= \sum_{i=0}^n \binom{n}{i} 2^i N_{i+1}.
\end{aligned}$$

The similar results obtained from equation (23):

$$\begin{aligned}
F_n &= \sum_{i=1}^n \binom{n-1}{i-1} (-1)^{n-i} F_{2i-1}, & L_n &= \sum_{i=1}^n \binom{n-1}{i-1} (-1)^{n-i} L_{2i-1}, \\
P_n &= \frac{1}{2^{n-1}} \sum_{i=1}^n \binom{n-1}{i-1} (-1)^{n-i} P_{2i-1}, & Q_n &= \frac{1}{2^{n-1}} \sum_{i=1}^n \binom{n-1}{i-1} (-1)^{n-i} Q_{2i-1}, \\
J_n &= \sum_{i=1}^n \binom{n-1}{i-1} (-2)^{n-i} J_{2i-1}, & j_n &= \sum_{i=1}^n \binom{n-1}{i-1} (-2)^{n-i} j_{2i-1}, \\
\mathcal{B}_n &= \frac{1}{3^{n-1}} \sum_{i=1}^n \binom{n-1}{i-1} (-1)^{n-i} \mathcal{B}_{2i-1}, & \mathcal{F}_n &= \sum_{i=1}^n \binom{n-1}{i-1} (-1)^{1-i} \mathcal{F}_{2i-1}, \\
\mathcal{P}_n &= \frac{1}{2^{n-1}} \sum_{i=1}^n \binom{n-1}{i-1} (-1)^{1-i} \mathcal{P}_{2i-1}, & D_n &= \frac{1}{4^{n-1}} \sum_{i=1}^n \binom{n-1}{i-1} (-1)^{n-i} D_{2i-1}, \\
N_n &= \frac{1}{2^{n-1}} \sum_{i=1}^n \binom{n-1}{i-1} (-1)^{n-i} N_{2i-1}.
\end{aligned}$$

4 Some results on generating functions

4.1 Results on ordinary generating functions

Proposition 10. *Generating function of the even W_n numbers is*

$$\sum_{n=0}^{\infty} W_{2n} t^n = \frac{a(1+qt) + (b-pa)pt}{(1+qt)^2 - p^2 t}. \quad (24)$$

Proof. Firstly we realize that by setting $a_n^0 = W_n$ in *GES* we get $a_0^n = W_{2n}$ (see Eq. (19)). Here by considering (16) we have

$$\overline{a_{-q,p}}(t) = \sum_{n=0}^{\infty} W_{2n} t^n = \frac{1}{1+qt} a_{-q,p} \left(\frac{pt}{1+qt} \right).$$

Also we know from equation (10)

$$a_{-q,p}(t) = \sum_{n=0}^{\infty} W_n t^n = \frac{a + (b-pa)t}{1-pt+qt^2}$$

which completes the proof. $\square$

Using (24), we obtain the generating functions of the Fibonacci numbers F_n , Lucas numbers L_n , Pell numbers P_n , Pell–Lucas numbers Q_n , Jacobsthal numbers J_n , Jacobsthal–Lucas numbers j_n , Bronze Fibonacci numbers $\mathcal{B}_n$, Signed Fibonacci numbers $\mathcal{F}_n$, Signed Pell numbers $\mathcal{P}_n$, and also D_n and N_n numbers, respectively.

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{2n} t^n &= \frac{t}{1-3t+t^2}, & \sum_{n=0}^{\infty} L_{2n} t^n &= \frac{2-3t}{1-3t+t^2}, \\
\sum_{n=0}^{\infty} P_{2n} t^n &= \frac{2t}{1-6t+t^2}, & \sum_{n=0}^{\infty} Q_{2n} t^n &= \frac{2-6t}{1-6t+t^2},
\end{aligned}$$

$$\begin{aligned}
\sum_{n=0}^{\infty} J_{2n} t^n &= \frac{t}{1-5t+4t^2}, & \sum_{n=0}^{\infty} j_{2n} t^n &= \frac{2-5t}{1-5t+4t^2}, \\
\sum_{n=0}^{\infty} \mathcal{B}_{2n} t^n &= \frac{3t}{1-11t+t^2}, & \sum_{n=0}^{\infty} \mathcal{F}_{2n} t^n &= \frac{1-3t}{1-3t+t^2}, \\
\sum_{n=0}^{\infty} \mathcal{P}_{2n} t^n &= \frac{-2t}{1-6t+t^2}, & \sum_{n=0}^{\infty} D_{2n} t^n &= \frac{4t}{1-18t+t^2}, \\
\sum_{n=0}^{\infty} N_{2n} t^n &= \frac{1-3t}{1-6t+t^2}.
\end{aligned}$$

Proposition 11. *Generating function of the odd W_n numbers is*

$$\sum_{n=0}^{\infty} W_{2n+1} t^n = \frac{(b-pa)(1+qt) + ap}{(1+qt)^2 - p^2 t}. \quad (25)$$

Proof. In view of the recurrence (8) we have,

$$\sum_{n=0}^{\infty} W_{2n+1} t^n = \frac{1}{p} \left[\sum_{n=0}^{\infty} W_{2n+2} t^n + q \sum_{n=0}^{\infty} W_{2n} t^n \right].$$

Employing (24) on the right in the above equation we obtain (25). $\square$

From (25), we get the generating functions for odd indexed of these well-known sequences.

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{2n+1} t^n &= \frac{1-t}{1-3t+t^2}, & \sum_{n=0}^{\infty} L_{2n+1} t^n &= \frac{1+t}{1-3t+t^2}, \\
\sum_{n=0}^{\infty} P_{2n+1} t^n &= \frac{1-t}{1-6t+t^2}, & \sum_{n=0}^{\infty} Q_{2n+1} t^n &= \frac{2+2t}{1-6t+t^2}, \\
\sum_{n=0}^{\infty} J_{2n+1} t^n &= \frac{1-2t}{1-5t+4t^2}, & \sum_{n=0}^{\infty} j_{2n+1} t^n &= \frac{1+2t}{1-5t+4t^2}, \\
\sum_{n=0}^{\infty} \mathcal{B}_{2n+1} t^n &= \frac{1-t}{1-11t+t^2}, & \sum_{n=0}^{\infty} \mathcal{F}_{2n+1} t^n &= \frac{1-2t}{1-3t+t^2}, \\
\sum_{n=0}^{\infty} \mathcal{P}_{2n+1} t^n &= \frac{1-t}{1-6t+t^2}, & \sum_{n=0}^{\infty} D_{2n+1} t^n &= \frac{1-t}{1-18t+t^2}, \\
\sum_{n=0}^{\infty} N_{2n+1} t^n &= \frac{1+t}{1-6t+t^2}.
\end{aligned}$$

4.2 Results on exponential generating functions

Proposition 12. *Exponential generating function of the W_{2n} numbers is*

$$\sum_{n=0}^{\infty} W_{2n} \frac{t^n}{n!} = A e^{(\alpha p - q)t} + B e^{(\beta p - q)t}. \quad (26)$$

Proof. For $a_n^0 = W_n$ in GES we get $a_n^0 = W_{2n}$ (see Eq. (19)). Using equation (11) we get

$$\bar{A}_{-q,p}(t) = \sum_{n=0}^{\infty} W_{2n} \frac{t^n}{n!} = e^{-qt} (A e^{\alpha p t} + B e^{\beta p t}),$$

which completes the proof. $\square$

From (26)

$$\begin{aligned}
\sum_{n=0}^{\infty} F_{2n} \frac{t^n}{n!} &= \frac{e^{\left(\frac{3+\sqrt{5}}{2}\right)t} - e^{\left(\frac{3-\sqrt{5}}{2}\right)t}}{\sqrt{5}}, \\
\sum_{n=0}^{\infty} L_{2n} \frac{t^n}{n!} &= e^{\left(\frac{3+\sqrt{5}}{2}\right)t} + e^{\left(\frac{3-\sqrt{5}}{2}\right)t}, \\
\sum_{n=0}^{\infty} P_{2n} \frac{t^n}{n!} &= \frac{e^{(3+2\sqrt{2})t} - e^{(3-2\sqrt{2})t}}{2\sqrt{2}}, \\
\sum_{n=0}^{\infty} Q_{2n} \frac{t^n}{n!} &= e^{(3+2\sqrt{2})t} + e^{(3-2\sqrt{2})t}, \\
\sum_{n=0}^{\infty} J_{2n} \frac{t^n}{n!} &= \frac{e^{4t} - e^t}{3}, \\
\sum_{n=0}^{\infty} j_{2n} \frac{t^n}{n!} &= e^{4t} + e^t, \\
\sum_{n=0}^{\infty} \mathcal{B}_{2n} \frac{t^n}{n!} &= \frac{e^{\left(\frac{11+3\sqrt{13}}{2}\right)t} - e^{\left(\frac{11-3\sqrt{13}}{2}\right)t}}{\sqrt{13}}, \\
\sum_{n=0}^{\infty} \mathcal{F}_{2n} \frac{t^n}{n!} &= \frac{(\sqrt{5}+3)e^{\left(\frac{3-\sqrt{5}}{2}\right)t} + (\sqrt{5}-3)e^{\left(\frac{3+\sqrt{5}}{2}\right)t}}{2\sqrt{5}}, \\
\sum_{n=0}^{\infty} \mathcal{P}_{2n} \frac{t^n}{n!} &= \frac{e^{(3-2\sqrt{2})t} - e^{(3+2\sqrt{2})t}}{2\sqrt{2}}, \\
\sum_{n=0}^{\infty} D_{2n} \frac{t^n}{n!} &= \frac{e^{(9+4\sqrt{5})t} - e^{(9-4\sqrt{5})t}}{2\sqrt{5}}, \\
\sum_{n=0}^{\infty} N_{2n} \frac{t^n}{n!} &= \frac{e^{(3-2\sqrt{2})t} + e^{(3+2\sqrt{2})t}}{2}.
\end{aligned}$$

Proposition 13. Exponential generating function of the W_{2n+1} numbers is

$$\sum_{n=0}^{\infty} W_{2n+1} \frac{t^n}{n!} = A \left(p - \frac{q}{\alpha} \right) e^{(\alpha p - q)t} + B \left(p - \frac{q}{\beta} \right) e^{(\beta p - q)t}. \quad (27)$$

Remark 14. For the sake of simplicity we use the following representation in the proof:

$$W_e(t) = \sum_{n=0}^{\infty} W_{2n} \frac{t^n}{n!} \quad \text{and} \quad W_o(t) = \sum_{n=0}^{\infty} W_{2n+1} \frac{t^n}{n!}.$$

Proof. From equation (8) we have

$$W_o(t) - b = pW_e(t) - pa - q \int W_o(t) dt.$$

This, combined with (26) to gives

$$\frac{d}{dt} W_o(t) + qW_o(t) = p \frac{d}{dt} \{ A e^{(\alpha p - q)t} + B e^{(\beta p - q)t} \}.$$

Hence we have the following differential equation:

$$W_o'(t) + qW_o(t) = Ap(\alpha p - q)e^{(\alpha p - q)t} + Bp(\beta p - q)e^{(\beta p - q)t}.$$

The solution of this linear differential equation is:

$$W_o(t) = A\left(p - \frac{q}{\alpha}\right)e^{(\alpha p - q)t} + B\left(p - \frac{q}{\beta}\right)e^{(\beta p - q)t} + Ke^{-qt}.$$

Considering $W_o(0) = b$ we calculate the constant K as

$$K = b - A\left(p - \frac{q}{\alpha}\right) - B\left(p - \frac{q}{\beta}\right) = 0.$$

Combining these results and after some rearrangement we complete the proof. □

Using (26)

$$\sum_{n=0}^{\infty} F_{2n+1} \frac{t^n}{n!} = \frac{(1+\sqrt{5})e^{\left(\frac{3+\sqrt{5}}{2}\right)t} - (1-\sqrt{5})e^{\left(\frac{3-\sqrt{5}}{2}\right)t}}{2\sqrt{5}},$$

$$\sum_{n=0}^{\infty} L_{2n+1} \frac{t^n}{n!} = \frac{(1+\sqrt{5})e^{\left(\frac{3+\sqrt{5}}{2}\right)t} + (1-\sqrt{5})e^{\left(\frac{3-\sqrt{5}}{2}\right)t}}{2},$$

$$\sum_{n=0}^{\infty} P_{2n+1} \frac{t^n}{n!} = \frac{(1+\sqrt{2})e^{(3+2\sqrt{2})t} - (1-\sqrt{2})e^{(3-2\sqrt{2})t}}{2\sqrt{2}},$$

$$\sum_{n=0}^{\infty} Q_{2n+1} \frac{t^n}{n!} = (1+\sqrt{2})e^{(3+2\sqrt{2})t} + (1-\sqrt{2})e^{(3-2\sqrt{2})t},$$

$$\sum_{n=0}^{\infty} J_{2n+1} \frac{t^n}{n!} = \frac{2e^{4t} + e^t}{3},$$

$$\sum_{n=0}^{\infty} j_{2n+1} \frac{t^n}{n!} = 2e^{4t} - e^t,$$

$$\sum_{n=0}^{\infty} \mathcal{B}_{2n+1} \frac{t^n}{n!} = \frac{(3+\sqrt{13})e^{\left(\frac{11+3\sqrt{13}}{2}\right)t} - (3-\sqrt{13})e^{\left(\frac{11-3\sqrt{13}}{2}\right)t}}{2\sqrt{13}},$$

$$\sum_{n=0}^{\infty} \mathcal{F}_{2n+1} \frac{t^n}{n!} = \frac{(\sqrt{5}+1)e^{\left(\frac{3+\sqrt{5}}{2}\right)t} + (\sqrt{5}-1)e^{\left(\frac{3-\sqrt{5}}{2}\right)t}}{2\sqrt{5}},$$

$$\sum_{n=0}^{\infty} \mathcal{P}_{2n+1} \frac{t^n}{n!} = \frac{(\sqrt{2}-1)e^{(3-2\sqrt{2})t} - (\sqrt{2}+1)e^{(3+2\sqrt{2})t}}{2\sqrt{2}},$$

$$\sum_{n=0}^{\infty} D_{2n+1} \frac{t^n}{n!} = \frac{(2+\sqrt{5})e^{(9+4\sqrt{5})t} - (2-\sqrt{5})e^{(9-4\sqrt{5})t}}{2\sqrt{5}},$$

$$\sum_{n=0}^{\infty} N_{2n+1} \frac{t^n}{n!} = \frac{(1+\sqrt{2})e^{(3+2\sqrt{2})t} + (1-\sqrt{2})e^{(3-2\sqrt{2})t}}{2}.$$

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