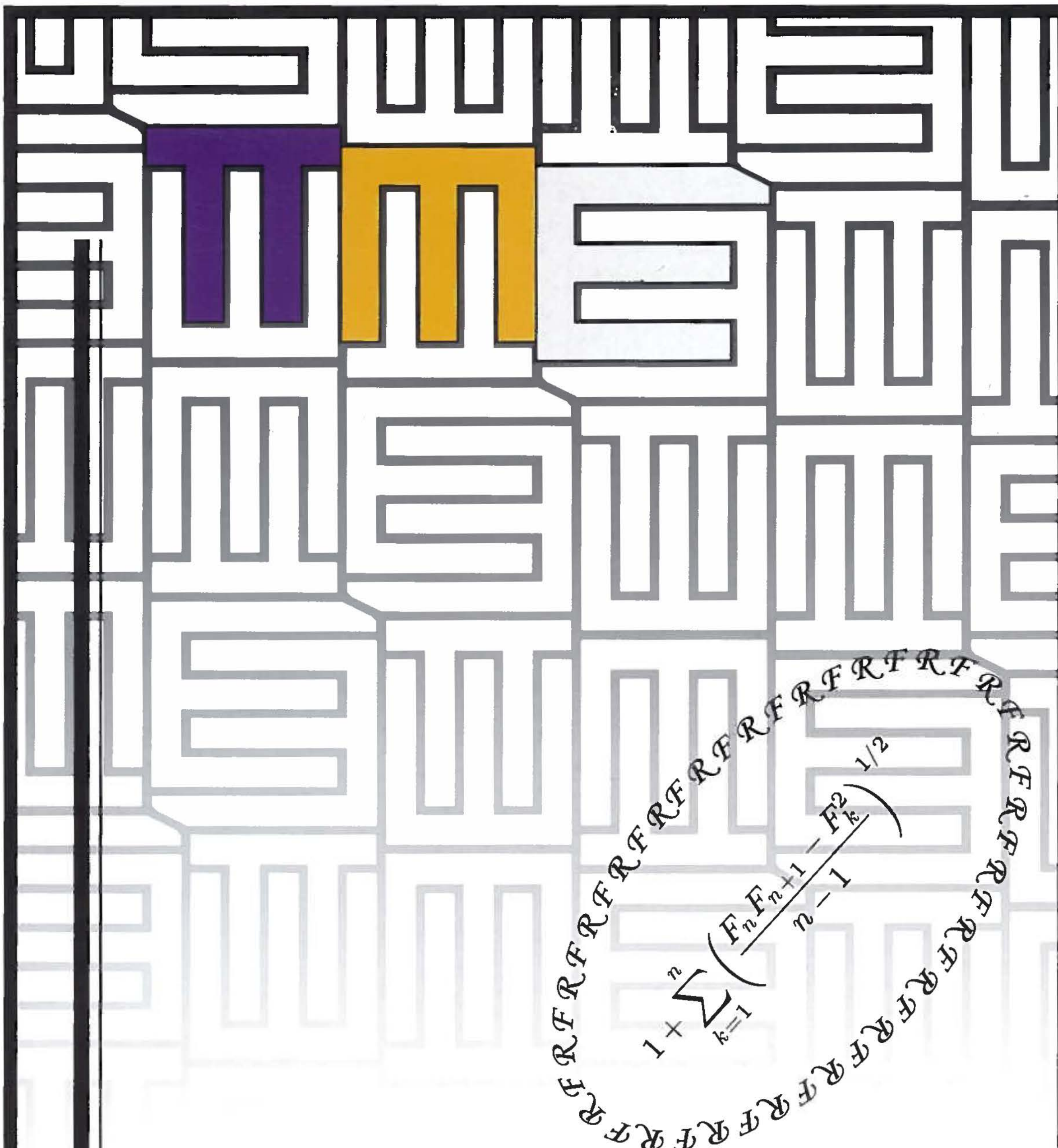

Can you estimate this Fibonacci sum?
Problem 1073

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ON THE EQUATION $Y^X = X^Y$.

ALI R. AMIR-MOEZ AND ROBERT E. BYERLY*

In this note we consider the deceptively simple equation $y^x = x^y$, $x > 0$, $y > 0$. A few moments algebraic work leaves one discouraged of ever finding a closed form expression for y in terms of x. On the other hand, there is an obvious solution curve, namely, $y = x$. Are all the solutions of this simple form? Actually no, since $2^4 = 16 = 4^2$. So what are other solutions?

Starting simply, consider $3^x = x^3$. These are two curves familiar from calculus, see Figure 1a. We know that the exponential curve and the cubic will intersect at

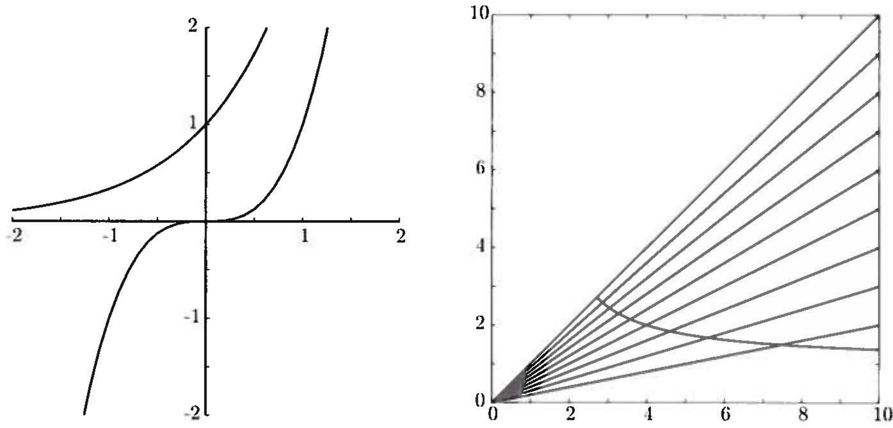

FIG. 1.

$x = 3$, when $y = 27$. Examining the derivatives at $x = 3$ we have $3 \cdot 3^2 < 3^3 \ln 3$, so the exponential crosses over the cubic at $(3, 27)$. Since $3^1 > 1^3$, there must be a value x_0 between 1 and 3 where the curves cross a second time, a non-trivial solution, $3^{x_0} = x_0^3$.

The exact same argument shows that, for any $a > e$, there are two solutions for $x^a = a^x$, with the non-trivial solution solution lying between 1 and a. For $a = e$, the curves intersect tangentially. The reader can check what happens when $0 < a < e$ – be careful of $a = 1$.

Since the open ray $y = x$, $x > 0$ lies entirely along the trivial portion of the solution set of $y^x = x^y$, it is natural to consider how other open rays intersect the solution set, since any intersection will be non-trivial. So consider the system

$$\begin{aligned} y^x &= x^y, \qquad & x > 0, y > 0; \\ y &= mx \qquad & 0 < m < 1 \end{aligned}$$

which is depicted in Figure 1b.

At each intersection point we have $(mx)^x = x^{mx}$, that is $x \log mx = mx \log x$, which implies $mx = x^m$, or $m = x^{m-1}$, giving us, with surprisingly little effort, a

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parametric representation for the section curve $y^x = x^y$ lying below the line $y = x$:

$$x = m^{\frac{1}{m-1}},$$
$$y = m^{\frac{m}{m-1}} \qquad 0 < m < 1$$

Using the parametric representation, it is easy for get a computer generated sketch.

Our representation applies to that portion of the curve below $y = x$. Since the curve is symmetrical with respect to $y = x$, the entire curve is now described, see Figure 2.

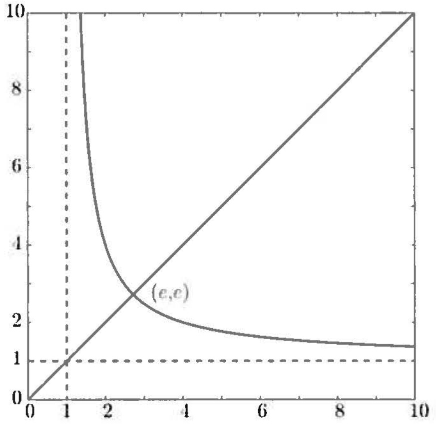

FIG. 2.

We already suspect from our opening remarks that the intersection of the trivial and non-trivial portions of the curves occurs at (e, e). This may be demonstrated by letting m approach one:

$$\lim_{m\to 1} \log m^{\frac{1}{1-m}} = \lim_{m\to 1} \frac{\log m}{m-1} = \lim_{m\to 1} \log x = \lim_{m\to 1} \frac{m^{-1}}{1} = 1.$$

by l'Hopital's rule, so

$$\lim_{m\to 1} x = e.$$

The reader may also readily verify that the non-trivial part of the curve is always decreasing and concave upwards.

We have also seen that the non-trivial solutions below $y = x$ occur with $1 < y < x$. Our computer sketch seems to indicate the $y = 1$ is a horizontal asymptote. Our parametric representations allows us to verify this as well.

$$\lim_{m\to 0} \frac{m \ln m}{m-1} = 0$$

by l'Hopital's Rule, so $y = 1$ is indeed a horizontal asymptote, and, by symmetry, so is $x = 1$.

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INTEGER POINTS ON CUBIC TWISTS OF ELLIPTIC CURVES

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1. Introduction. The problem of finding out whether a given integer may be written as the sum of two rational cubes is one of the oldest problems in number theory. Its history goes back nearly two thousand years, and it has been the motivation for hundreds of papers. More commonly, we express the problem as that of finding integral solutions to $x^3 + y^3 = Az^3$, for some integer A.

We briefly discuss the work of some of the notables (see Dickson [1]) for a comprehensive summary of early work on this problem). Euler showed that there is only the trivial solution ($x = z$, $y = 0$) if $A = 1$, only the trivial solution ($x = y = z$) if $A = 2$, and no solutions if $A = 4$. Dirichlet improved on Euler by showing the impossiblity of $x^3 \pm y^3 = 2^n z^3$ for $n > 2$. J.J. Sylvester showed that there are no (non-trivial) solutions if $A = 2$ or 3. Legendre showed that if $A \equiv \pm 3$ or $\pm 4 \pmod 9$, then z is divisible by 3. Fermat showed how to obtain infinitely many solutions, if one is has one solution with which to start. Many other special cases were investigated by C.A. Laisant, Moret-Blanc, T. Pepin, Sylvester, C. Henry, L. Varchon, et.al. E. Lucas noted that integral solutions will exist only if A is of the form $ab(a + b)/c^3$, with a, b, c integers.

Probably the most well-known incident in the history of this problem is that of the visit of Hardy to the sickbed of Ramanujan. "I remember once going to to see him when he was lying ill at Putney. I had ridden in taxi-cab no. 1729, and remarked that the number $(7 \cdot 13 \cdot 19)$ seemed to me rather a dull one, and I hoped that it was not an unfavorable omen. 'No,' he replied, 'it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways' " [7].

Recently, most of the attention focused on the family of curves

$$E_D : X^3 + Y^3 = D$$

has fallen into two camps.

First, researchers in transcendental number theory, analytic number theory and automorphic forms have tried to study the "average" behavior as D varies over some natural family. One expects that in any family which does not have predetermined behavior for trivial considerations (for example, congruence conditions), one will have a positive proportion of D which do in fact have a rational solution and a positive proportion of D which do *not* have a rational solution. The family of curves E_D is the family of cubic twists of the fixed elliptic curve E_1, and so the rich theory of elliptic curves may be brought to bear on this problem. Although considerable machinery has been developed to try to prove each of these expectations (cf. [4, 5, 6, 9, 10]), they both seem far beyond current technology.

Second, researchers in diophantine geometry have tried to study the distribution of the integral solutions, as part of a larger theory of integral solutions to more general diophantine equations. In particular, the question of the frequency for which a given D may be expressed in two different ways as a sum of two integral cubes has received

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a great deal of attention, as a special case of the question of how often an integer may be expressed in multiple ways as a sum of two k^{th} powers. Hooley [3] has proved that the number of integers $D < X$ such that D may be written in at least two distinct ways as a sum of relatively prime cubes is $\ll X^{5/9+\epsilon}$.

In this paper, we conduct an experimental investigation into the frequency of D which admit two distinct relatively prime solutions. We also evaluate this data against a hypothesis which does afford an explanation for why D which admit three relatively prime solutions are quite rare, and why no D has been found which may be expressed in four different ways as a sum of relatively prime cubes.

2. Heuristics. In order to predict the likelihood of an integral solution to a specific equation E_D, we consider the total number of non-trivial integral solutions to the family of curves E_D, $1 \leq D \leq X$. Writing $[x]$ for the greatest integer less than x, we see that the number of integral ordered pairs (not necessarily relatively prime) which are points on these curves is equal to the number of ordered pairs

$$\begin{gathered}(1,1),(1,2),\ldots,(1,[(X-1)^{1/3}])\\(2,1),(2,2),\ldots,(2,[(X-8)^{1/3}])\\\vdots\\([X^{1/3}],1),\ldots,([X^{1/3}],[(X-[X^{1/3}]^3)^{1/3}]).\end{gathered}$$

Roughly speaking, the number of ordered pairs is thus approximately

$$\int_0^{X^{1/3}} (X-x^3)^{1/3}\,dx.$$

If one expands the integrand as a power series and then integrates, one sees that the leading term upon evaluating the integral is $c_0X^{2/3}$. In [2], Dickson attributes both Cesàro and Sylvester with the result that the probability that any two integers randomly chosen from $1,\ldots,n$ are relatively prime is asymptotic to $6/\pi^2$. So we expect approximately $c_1X^{2/3}$ relatively prime integral solutions among the first X curves where $c_1 = \frac{6}{\pi^2}c_0$. We proceed to consider only relatively prime solutions; that is, solutions (x,y) where $\gcd(x,y)=1$. Considering only such co-prime solutions is sufficient, since any non-coprime solution occurs only if D is not cube-free and then is obtained from a co-prime solution by scaling.

We now make a simplification. We assume that the distribution function $f(D)$ of the expected number of relatively prime integral points on E_D is a continuous distribution. (Of course, this is false, since the number of such points is a discrete distribution.) Under this assumption, we can calculate $f(D)$. In fact, in light of our estimates above, we have

$$\int_1^X f(x)\,dx = c_1X^{2/3}.$$

By the Fundamental Theorem of Calculus, we see that $f(x) = c_2x^{-1/3}$. Thus we expect a random curve E_D to have roughly $c_2D^{-1/3}$ points, or in other words, we should find a constant number of points on each collection of curves E_D, $X \leq D \leq X^3 + X$. We remark that this expectation has been upheld in our experiments.

But what is the probability of two or more relatively prime solutions on a single curve? Experimentally, we have found that $c_2D^{-1/3} < 1$ and that the proportion of curves having multiple relatively prime pair solutions is very small indeed. In fact, over 99% of the curves we found having one relatively prime solution had exactly one relatively prime solution. Therefore, we assume that the probability that a curve has a single relatively prime solution is $c_2D^{-1/3}$. Now we make the hypothesis that the relatively prime integral points on cubic twists of elliptic curves are independently distributed. Under these assumptions, we predict the distribution of pairs, triples, etc., of solutions.

If we remove the "relatively prime" condition, we have exactly the same distributions, and will obtain the same predictions, but with different constants. These non-relatively prime predictions are easily seen to be quite different from observed data in even extremely small samplings. Thus the real question in our paper could be phrased as "non-relatively prime points do not seem to be independently distributed; are relatively prime points independently distributed?" It is this question we investigate in the remainder of this paper.

We proceed now under the independence hypothesis. In this case, the probability that a curve has k relatively prime solutions is given by $(c_2D^{-1/3})^k = c_3D^{-k/3}$. Then if we look at the first X curves, we find that the expected number of k-tuples of relatively prime solutions should be approximated by

$$\int_1^X c_3x^{-k/3}\,dx = \begin{cases} c_4X^{1/3} & k=2;\\ c_5\log(X) & k=3;\\ c_6X^{(3-k)/3} & k>3.\end{cases}$$

This is somewhat surprising - it says that we expect pairs of relatively prime solutions to be infrequent, but not too rare, triples of relatively prime solutions be much rarer, and k-tuples with $k \geq 4$ should only occur at most finitely many times for a given k.

It is impossible for us to test this expectation even for triples. Nonetheless, in the remainder of the paper we present data on the number of integral points on the curves E_D, where $1 \leq D \leq 65{,}000{,}000$.

3. Data. The data in the table below shows the number of integral points S_X on the curves E_D, $1 \leq D \leq X$, along with the ratio $S_X/(X^{2/3})$; the number of relatively prime solutions R_X and the ratio $R_X/(X^{2/3})$; and the number of relatively prime pairs of solutions P_X and the ratio $P_X/(X^{1/3})$.

X	S_X	$c_0 \approx S_X/(X^{2/3})$	R_X	$c_1 \approx R_X/(X^{2/3})$	c_1/c_0	P_X	$c_4 \approx P_X/(X^{1/3})$
5,000,000	13052	.446	7999	.274	.614	17	.0994
10,000,000	20377	.439	12429	.268	.610	24	.111
20,000,000	32523	.441	19782	.268	.608	33	.122
30,000,000	42653	.442	25901	.268	.606	37	.119
40,000,000	51638	.441	30946	.265	.601	41	.120
50,000,000	59979	.442	35991	.265	.600	50	.136
60,000,000	67747	.442	40818	.266	.602	55	.141
65,000,000	71467	.442	43068	.266	.602	57	.142

4. Conclusion. While we cannot predict values for each of the entries in the above table (we do not have exact values for the constants c_0 and c_1), we can use our data to test if the value of c_1/c_0 is asymptotic to $6/\pi^2 \approx .608$ and if the value of c_4 is constant (or at least approaching a bound) as predicted.

Were the hypothesis of an independent distribution true, we would expect the values for c_4 to be roughly constant. While it is possible that the values for c_4 are approaching some asymptotic bound, such a conclusion is not supported by our data. Causing further concern is the fact that the values of c_1/c_0 may not be aymptotically approaching the the predicted value of $6/\pi^2$. While it is possible that these values may be approaching $6/\pi^2$, we can not conclude this from our data. Thus it appears from the data that our hypothesis that relatively prime pair solutions on these elliptic curves are independently distributed is false.

Perhaps one should not find these results surprising as we note that rational points on elliptic curves are distributed according to a fairly complicated and subtle distribution, namely the group law. Two points on an elliptic curve may be "added" to give a third point on the curve (here we must include a point at infinity). The points of an elliptic curve, together with this "addition" operation form a group and the set of rational points forms a subgroup. (For an introduction to the group law for "adding" points on elliptic curves the reader is referred to [8].) Thus, perhaps, one should not necessarily expect relatively prime integral points on these elliptic curves to obey an independent distribution. We hope that the hypothesis, while apparently false, might in fact lend some insight into the frequency of how often one should expect triples, quadruples, etc., of relatively prime integral points on elliptic curves.

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The student coauthors were undergraduate or post-baccalaureate students. These students participated in a summer research program at the University of Missouri, Columbia. In this program, the participants studied the structure of elliptic curves and conducted experiments on the number of rational points on cubic twists of elliptic curves using the PARI computer package. PARI is a computer algebra system designed for fast computations in number theory including factorizations, algebraic number theory, and elliptic curves.

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THE TAU, SIGMA, RHO FUNCTIONS, AND SOME RELATED NUMBERS

WILLIAM CHAU*

Abstract. This article revisits the tau (number-of-positive-divisors) function, the sigma (sum-of-positive-divisors) function and some related numbers, namely perfect numbers, deficient numbers, abundant numbers, and amicable pairs. The theme is to study the rho (product-of-positive-divisors) function, and some related numbers, namely multiplicatively perfect numbers, multiplicatively deficient numbers, multiplicatively abundant numbers, and multiplicatively amicable pairs. A cross-classification of all positive integers in one direction as perfect, deficient, or abundant numbers, and another direction as multiplicatively perfect, multiplicatively deficient, or multiplicatively abundant is partially done. The asymptotic behavior of the tau, sigma, rho functions and some of the aforementioned numbers are covered. Some initial open questions encountered are posed to stimulate further research.

1. Introduction. Arithmetic functions are functions defined for all positive integers. The number-of-positive-divisors arithmetic function (we shall call tau function), and the sum-of-positive-divisors arithmetic function (we shall call sigma function) and some related numbers, namely perfect numbers, deficient numbers, abundant numbers, and amicable pairs, are some of the widely exploited topics in the field of number theory. In this article, we revisit these topics and also exploit the product-of-positive-divisors arithmetic function (we shall call rho function), and some related numbers, namely multiplicatively perfect numbers, multiplicatively deficient numbers, multiplicatively abundant numbers, and multiplicatively amicable pairs.

The treatment of these topics here is not meant to be exhaustive, instead it is an attempt to present elementary and interesting aspects of the tau function, sigma function, rho function, and the aforementioned numbers. Materials that are well-known to most readers will be described in survey form. The results for the rho function and multiplicatively perfect numbers covered in here are probably not well-known, but they are not new. On the other hand, no treatment of multiplicatively deficient numbers, multiplicatively abundant numbers, and multiplicatively amicable pairs is found by the author anywhere in the literature. Therefore we will introduce them here.

2. Tau, Sigma, and Rho Functions.

DEFINITION 1. *Given a positive integer n, define $\tau(n)$, $\sigma(n)$, and $\rho(n)$[1] as the number, the sum, and the product of the positive divisors of n, respectively. We shall call $\tau(n)$, $\sigma(n)$, and $\rho(n)$ simply as the tau, sigma, and rho functions, respectively.*

We state the closed-form formulae for $\tau(n)$ and $\sigma(n)$ in the following two theorems, where the proofs can be easily found in many existing literature (see, ex., [8, pp. 188-191]).

THEOREM 2. *If $n = \prod_{k=1}^{k_n} p_k^{\alpha_k}$, where p_k are distinct primes, $\alpha_k > 0$, for $k = 1, \ldots, k_n$, then*

$$\tau(n) = \prod_{k=1}^{k_n} (1 + \alpha_k).$$

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[1]The selected choice of ρ is based on the fact that ρ, σ, and τ are consecutive letters in the Greek alphabet.

THEOREM 3. *If* $n = \prod_{k=1}^{k_n} p_k^{\alpha_k}$, *where* p_k *are distinct primes,* $\alpha_k > 0$, *for* $k = 1, \ldots, k_n$, *then*

$$\sigma(n) = \prod_{k=1}^{k_n} \frac{p_k^{\alpha_k+1} - 1}{p_k - 1}.$$

The multiplicative equivalent of Theorem 3 is the closed-form formula for $\rho(n)$. We supply a proof with an interesting presentation as follows (cf. [3, p. 114]).

THEOREM 4. $\rho^2(n) = n^{\tau(n)}$.

Proof. Let $q_1, q_2, \ldots, q_{\tau(n)}$ be the positive divisors of n listed in increasing order. We write $\rho(n)$ in two ways, one as a product of q_k in increasing order, another in decreasing order:

$$\begin{array}{lcccccc} \rho(n) & = & q_1 & \cdot & q_2 & \cdots & q_{\tau(n)} \\ \rho(n) & = & q_{\tau(n)} & \cdot & q_{\tau(n)-1} & \cdots & q_1 \end{array}$$

We multiply these equations column by column, noting that the product of divisors of each column is equal to n, to get $\rho^2(n) = n \cdot n \cdots n = n^{\tau(n)}$. Therefore the desired formula is valid. □

Note that the technique used in the proof is very similar to that employed by Gauss to sum a list of consecutive integers, being that it is not additive but multiplicative in nature.

DEFINITION 5. *An arithmetic function* f *is said to be multiplicative if*

$$f(mn) = f(m)f(n)$$

whenever $\gcd(m, n) = 1$.

It is easy to see that any multiplicative arithmetic function is completely determined once its values at prime-powers are known.

The following result is well-known, and its proof can be easily obtained (see, ex., [3, pp. 115-116]).

THEOREM 6. *The arithmetic functions* τ *and* σ *are both multiplicative.*

THEOREM 7. *If* $\gcd(m, n) = 1$, *then* $\rho(mn) = \rho^{\tau(n)}(m) \cdot \rho^{\tau(m)}(n)$.

Proof. If $\gcd(m, n) = 1$, then, by Theorem 4 and Theorem 6,

$$\begin{aligned} \rho^2(mn) &= (mn)^{\tau(mn)} \\ &= (mn)^{\tau(m)\tau(n)} \\ &= \left(m^{\tau(m)}\right)^{\tau(n)} \cdot \left(n^{\tau(n)}\right)^{\tau(m)} \\ &= \rho^{2\tau(n)}(m) \cdot \rho^{2\tau(m)}(n). \end{aligned}$$

So the theorem follows. □

It is easy to see that ρ is not multiplicative following from the last result.

3. Perfect Numbers and Multiplicatively Perfect Numbers.

DEFINITION 8. *A positive integer* n *is perfect if the sum of its proper divisors is equal to* n, *i.e.,* $\sigma(n) = 2n$. *A positive integer* n *is multiplicatively perfect if the product of its proper divisors is equal to* n, *i.e.,* $\rho(n) = n^2$.

THEOREM 9. *An integer* n *is an even perfect number iff* $n = 2^{p-1}(2^p - 1)$, *where both* p *and* $2^p - 1$ *are prime numbers.*

The standard proofs of sufficiency and necessity of Theorem 9 are originally due to, respectively, Euclid and Euler (see, ex., [3, pp. 220-221] and [6, pp. 58-59]). Although a form of even perfect numbers is uncovered in this result, it is difficult to use it to discover new even perfect numbers. For, it depends on the unsolved problem of discovering new prime numbers of the form $M_p = 2^p - 1$ where p is prime, so-called Mersenne primes. To avoid too much digression, we will not discuss Mersenne primes in this setting. Corresponding to $p = 2, 3, 5, 7, 13$, respectively, are the first five perfect numbers $n = 6, 28, 496, 8128, 33550336$, all of even parity, and the first five Mersenne primes $M_p = 3, 7, 31, 127, 8191$.

On the other hand, no one knows if there is any odd perfect number. Nevertheless, there are necessary conditions for their existence. One such condition (see, ex., [3, pp. 231-232]), due to Euler, is given in the next theorem.

THEOREM 10. *If* n *is an odd perfect number, then* n *is of the form* $n = p^\alpha m^2$, *where* p *is a prime,* $p \nmid m$, *and* $n \equiv p \equiv \alpha \equiv 1 \pmod 4$.

Nevertheless, lower bounds on the size of odd perfect numbers have been investigated. It is known that the lower bound is $n > 10^{300}$ ([10, OddPerfectNumber.html]).

Despite all the difficulties in discovering new perfect numbers, a simple necessary and sufficient condition, as shown in the next theorem (cf. the proof with [7, p. 19]), is used to look for all multiplicatively perfect numbers.

THEOREM 11. *A positive integer* n *is an multiplicatively perfect number iff* $n = 1$, $n = p^3$, *or* $n = pq$, *where* $p \neq q$ *are prime numbers.*

Proof. By definition, $n = 1$ is multiplicatively perfect.

Assume that $n > 1$ is multiplicatively perfect, so $\rho(n) = n^2$. By Theorem 4, $\rho(n) = n^{\tau(n)/2}$. Therefore $n^2 = n^{\tau(n)/2}$, or simply $\tau(n) = 4$. Since the only factorizations of 4 are $4 = 1 + 3$ and $2 \cdot 2 = (1 + 1) \cdot (1 + 1)$, so, by Theorem 2, the unique factorization of n is either $n = p^3$ or $n = pq$ for some primes $p \neq q$.

Assume $n = p^3$ or $n = pq$, where $p \neq q$ are primes. In the former case $\rho(n) = 1 \cdot p \cdot p^2 \cdot p^3 = n^2$, and in the latter case $\rho(n) = 1 \cdot p \cdot q \cdot pq = n^2$. So n is multiplicatively perfect. □

Using the result in Theorem 11, the first five multiplicatively perfect numbers are found to be 1, 6, 8, 10, and 14.

The next interesting result is an application of Theorem 11 – the reader can see [4] for the corresponding problem.

THEOREM 12. *Six is the only positive integer that is both perfect and multiplicatively perfect*

Proof. The integer one is ruled out since it is not perfect. By Theorem 11, if $n > 1$ is multiplicative perfect, then $n = p^3$ or $n = pq$, where $p < q$ are primes. Assume $n = p^3$. Since n is perfect, by Definition 8,

$$p^3 = 1 + p + p^2 = \frac{p^3 - 1}{p - 1} < p^3,$$

which is impossible. Similarly, if $n = pq$ with $p < q$, then $pq = 1 + p + q$, so

$$pq - p - q + 1 = (p - 1)(q - 1) = 2.$$

The only solution is $(p, q) = (2, 3)$. Hence $n = pq = 6$ is the only number that is both perfect and multiplicatively perfect. □

4. Deficient Numbers, Abundant Numbers and Amicable Pairs.

The next definition classifies positive integers that are not perfect.

DEFINITION 13. *A positive integer n is deficient if the sum of its proper divisors is less than n, i.e., $\sigma(n) < 2n$. A positive integer n is abundant if the sum of its proper divisors is greater than n, i.e., $\sigma(n) > 2n$.*

There is no complete formulation of deficient and abundant numbers. Nonetheless, two sufficient conditions for deficient numbers are given in Theorem 14, and another two sufficient conditions for abundant numbers are given in Theorem 15 (compiled from [3, p. 233] and [6, p. 62]). We leave it to the interested readers to supply the proofs for them. Another sufficient condition for deficient numbers, added by the author, is given in Theorem 16.

THEOREM 14. *If $n = p^\alpha$, where p is an odd prime and $\alpha \geq 1$, then n is deficient. Also, if $n = 2p(2p+1)$, where p and $2p+1$ are primes, and $p > 3$, then n is deficient.*

THEOREM 15. *If $n = 2^\alpha \cdot 3$, where $\alpha > 1$, then n is abundant. Also, if $n = 945 \cdot m$, where m is any integer not divisible by 2, 3, 5, or 7, then n is abundant.*

THEOREM 16. *If $n = \prod_{k=1}^{k_n} p_k^{\alpha_k}$, where $p_k > 2^{1/k_n}/(2^{1/k_n}-1)$ are distinct primes, $\alpha_k > 0$, for $k = 1, \ldots, k_n$, then n is deficient.*

Proof. Assume $n = \prod_{k=1}^{k_n} p_k^{\alpha_k}$, where p_k are distinct primes, $\alpha_k > 0$ for $k = 1, \ldots, k_n$. If $p_k > 2^{1/k_n}/(2^{1/k_n} - 1)$, then

$$\begin{aligned} 2^{1/k_n} &< (2^{1/k_n} - 1)p_k, \\ 2^{1/k_n} - \frac{1}{p_k^{\alpha_k}} &< 2^{1/k_n}p_k - p_k, \\ p_k - \frac{1}{p_k^{\alpha_k}} &< 2^{1/k_n}p_k - 2^{1/k_n}, \\ \frac{p_k - 1/p_k^{\alpha_k}}{p_k - 1} &< 2^{1/k_n}. \end{aligned}$$

Then, by Theorem 3,

$$\begin{aligned} \frac{\sigma(n)}{n} &= \left(\prod_{k=1}^{k_n} \frac{p_k^{\alpha_k+1} - 1}{p_k - 1}\right) \cdot \frac{1}{\prod_{k=1}^{k_n} p_k^{\alpha_k}} = \prod_{k=1}^{k_n} \left(\frac{p_k^{\alpha_k+1} - 1}{p_k - 1} \cdot \frac{1}{p_k^{\alpha_k}}\right) = \prod_{k=1}^{k_n} \frac{p_k - 1/p_k^{\alpha_k}}{p_k - 1}, \\ &< \prod_{k=1}^{k_n} 2^{1/k_n} = 2. \end{aligned}$$

Therefore n is deficient by Definition 13. □

The last set of numbers we discuss in this section that is defined in terms of the sigma function is the set of amicable pairs.

DEFINITION 17. *A pair of distinct positive integers m and n is called an amicable pair if the sum of proper divisors of one number equals the other, i.e., $\sigma(m) = \sigma(n) = m + n$.*

The following theorem classifies the members of a given amicable pair.

THEOREM 18. *If m and n, $m < n$, are members of an amicable pair, then m and n are, respectively, abundant and deficient.*

Proof. Let m and n, $m < n$, be members of an amicable pair. By Definition 17, $\sigma(m) = m + n > m + m = 2m$, and similarly, $\sigma(n) = m + n < n + n = 2n$. Thus m and n are, respectively, abundant and deficient. □

Again, there is no complete formulation of amicable pairs. A sufficient condition ([2, p. 27]) for a pair of numbers to be an amicable pair, due to an Arabian mathematician Thabit ben Korrah, is given next. Any interested reader can easily validate it.

THEOREM 19. *If $p = 3 \cdot 2^\alpha - 1$, $q = 3 \cdot 2^{\alpha-1} - 1$, and $r = 3^2 \cdot 2^{2\alpha-1} - 1$ are primes, where $\alpha > 1$, then $2^\alpha pq$ and $2^\alpha r$ are amicable pair.*

Unfortunately, the last result can be used to produce only three amicable pairs corresponding to the cases where $\alpha = 2, 4, 7$
([10, ThabitibnKurrahRule.html]).

It is easy to check, using Definition 13, that the first five deficient numbers are 1, 2, 3, 4, and 5. Similarly, we can check that the first five abundant numbers are 12, 18, 20, 24, and 30. The first five amicable pairs, given by [10, AmicablePair.html], are $\{220, 284\}$, $\{1184, 1210\}$, $\{2620, 2924\}$, $\{5020, 5564\}$, and $\{6232, 6368\}$.

5. Multiplicatively Deficient and Abundant Numbers and Multiplicatively Amicable Pairs. In the last section, we revisited the deficient numbers, abundant numbers, and amicable pairs, which are all defined in terms of the sigma function. As mentioned in the introduction, the author could not find any information for their multiplicative counterparts, which are defined instead using the rho function, anywhere in the current literature. For the sake of completeness, we provide both the definitions and the characterizations of these interesting numbers.

DEFINITION 20. *A positive integer n is multiplicatively deficient if the product of its proper divisors is less than n, i.e., $\rho(n) < n^2$. A positive integer n is multiplicatively abundant if the product of its proper divisors is greater than n, i.e., $\rho(n) > n^2$.*

THEOREM 21. *A positive integer n is multiplicatively deficient iff $n = p^\alpha$ for some prime p and $\alpha = 1, 2$.*

Proof. Let $n = \prod_{k=1}^{k_n} p_k^{\alpha_k}$, where p_k are distinct primes and $\alpha_k > 0$. Proceeding in a similar manner as in the proof of Theorem 11, one can show that $\tau(n) < 4$ iff n is multiplicatively deficient. Since $\tau(n) = \prod_{k=1}^{k_n}(1 + \alpha_k)$ (by Theorem 2), n is multiplicatively deficient iff $k_n = 1$ and $\alpha_1 < 3$. Therefore $n = p^\alpha$, where $\alpha = 1, 2$.
□

THEOREM 22. *A positive integer n is multiplicatively abundant iff $n = \prod_{k=1}^{k_n} p_k^{\alpha_k}$, where all p_k are distinct primes, and k_n, α_k satisfy one of the following:*

1. *$k_n = 1$ and $\alpha_1 > 3$.*
2. *$k_n = 2$, $\alpha_1, \alpha_2 > 0$, and $\alpha_1 + \alpha_2 > 2$*
3. *$k_n > 2$ and all $\alpha_k > 0$.*

The proof of Theorem 22 is omitted since it is very similar to that of Theorem 21.

Using the results of Theorem 21 and Theorem 22, the first five multiplicatively deficient numbers are found to be 2, 3, 4, 5, and 7. Similarly, the first five multiplicatively abundant numbers are found to be 12, 16, 18, 20, and 24.

DEFINITION 23. *A pair of distinct positive integers m and n is called a multiplicatively amicable pair if the product of proper divisors of one number equals the other, i.e., $\rho(m) = \rho(n) = mn$.*

THEOREM 24. *$m = n$ iff $\rho(m) = \rho(n)$.*

Proof. The proof of necessity is trivial. We only need to show the proof of sufficiency. Let $m = \prod_{i=1}^{k_m} p_i^{\alpha_i}$ and $n = \prod_{j=1}^{k_n} q_j^{\beta_j}$, where p_i, q_j are in increasing orders of i, j, and all $\alpha_i, \beta_j > 0$. Assume $\rho(m) = \rho(n)$, so by using Theorem 2 and Theorem 4, $m^{\tau(m)/2} = n^{\tau(n)/2}$, where $\tau(m) = \prod_{k=1}^{k_m}(1+\alpha_k)$ and $\tau(n) = \prod_{k=1}^{k_n}(1+\beta_k)$. Therefore

$$\left(m^{\tau(m)/2}\right)^2 = \left(n^{\tau(n)/2}\right)^2, \quad m^{\tau(m)} = n^{\tau(n)}, \text{ hence}$$

$$\prod_{i=1}^{k_m} p_i^{\alpha_i \prod_{k=1}^{k_m}(1+\alpha_k)} = \prod_{j=1}^{k_n} q_j^{\beta_j \prod_{k=1}^{k_n}(1+\beta_k)}.$$

This implies that $p_i = q_i$, $k_m = k_n$, and

$$\begin{aligned}\alpha_1(1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_{k_m}) &= \beta_1(1+\beta_1)(1+\beta_2)\cdots(1+\beta_{k_m}),\\ \alpha_2(1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_{k_m}) &= \beta_2(1+\beta_1)(1+\beta_2)\cdots(1+\beta_{k_m}), \qquad (1)\\ &\vdots\\ \alpha_{k_m}(1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_{k_m}) &= \beta_{k_m}(1+\beta_1)(1+\beta_2)\cdots(1+\beta_{k_m}).\end{aligned}$$

Divide the first equation into successive equations of (1) to get

$$\alpha_k = \alpha_1 \cdot \frac{\beta_k}{\beta_1}, \qquad (2)$$

for $2 \le k < k_m$. Using (2), the first equation in (1) can be expressed as

$$\alpha_1(1+\alpha_1)\left(1+\alpha_1\cdot\frac{\beta_2}{\beta_1}\right)\cdots\left(1+\alpha_1\cdot\frac{\beta_{k_m}}{\beta_1}\right) = \beta_1(1+\beta_1)(1+\beta_2)\cdots(1+\beta_{k_m}).$$

Multiplying both sides by $\beta_1^{k_m-1}$, we have

$$\alpha_1(1+\alpha_1)(\beta_1+\alpha_1\beta_2)\cdots(\beta_1+\alpha_1\beta_{k_m}) = \beta_1(1+\beta_1)(\beta_1+\beta_1\beta_2)\cdots(\beta_1+\beta_1\beta_{k_m}).$$

If $\alpha_1 \lesseqgtr \beta_1$, then the expression on the left side $\lesseqgtr$ that on the right side. Thus $\alpha_1 = \beta_1$, which when put into (2), gives $\alpha_k = \beta_k$ for $2 < k < k_m$. It follows that $m = n$. □

The next result follows immediately from Theorem 24.

COROLLARY 25. *There is no multiplicatively amicable pair.*

6. Cross-Classification of Positive Integers. According to some of the definitions given in sections 3, 4, and 5, every positive integer is classified as either perfect, deficient, or abundant. Furthermore, each positive integer can be classified in a completely different way as either multiplicatively perfect, multiplicatively deficient, or multiplicatively abundant. It is natural to do a cross-classification of all positive integers based on these classifications. Table 6.1 is an attempt to capture such cross-classification. We will explain how to populate this table next.

TABLE 6.1
Cross-classification of all positive integers

	Multiplicatively Perfect	Multiplicatively Deficient	Multiplicatively Abundant
Perfect	$\{6\}$		$\mathcal{A}$
Deficient	$\{1, p^3, pq \neq 6 \mid \text{primes } p \neq q\}$	$\{p, p^2 \mid \text{prime } p\}$	$\mathcal{B}$
Abundant			$\mathcal{C}$

The result of Theorem 12 classifies that six is the only number that is both perfect and multiplicatively perfect, so we fill up the entry in the upper left corner of Table 6.1 with $\{6\}$. It is then an easy exercise to apply the results of Theorem 11 and Theorem 21 to fill up the remaining entries in the first two columns of the table. Note that there exists no number that is both perfect and multiplicatively deficient, both abundant and multiplicatively perfect, or both abundant and multiplicatively deficient.

The symbols $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ in the last column of Table 6.1 are unknown sets yet to be determined. Each perfect number not equal to six belongs to either

$$\mathcal{A}_1 = \left\{2^{p-1}(2^p-1) \neq 6 \mid p \text{ and } 2^p-1 \text{ are primes}\right\},$$

or

$$\mathcal{A}_2 = \left\{p^\alpha m^2 \mid p \text{ is a prime, } p \nmid m, \text{ and } n \equiv \alpha \equiv 1 (\text{mod } 4)\right\},$$

by Theorem 9 and Theorem 10. It is clear that each number in $\mathcal{A}_1$ or $\mathcal{A}_2$ is also of the second or third form of multiplicatively abundant numbers described in Theorem 22, so $\mathcal{A} \subset \mathcal{A}_1 \bigcup \mathcal{A}_2$. Next, let

$$\mathcal{B}_1 = \{p^\alpha \mid p \text{ is an odd prime, } \alpha > 3\},$$

$$\mathcal{B}_2 = \{2p(2p+1) \mid p \text{ and } 2p+1 \text{ are primes, } p > 3\}$$

and

$$\mathcal{B}_3 = \left\{\prod_{k=1}^{k_n} p_k^{\alpha_k} \mid \text{distinct primes } p_k > \frac{2^{1/k_n}}{2^{1/k_n}-1},\ \alpha_k > 0, \text{ for } k = 1, \ldots, k_n > 2\right\}.$$

Using the results of Theorem 14, Theorem 16, and Theorem 22, one can see that each number in any one of the sets $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$ is both deficient and multiplicatively abundant. Therefore $\mathcal{B} \supset \mathcal{B}_1 \bigcup \mathcal{B}_2 \bigcup \mathcal{B}_3$. At last, let

$$\mathcal{C}_1 = \{2^\alpha \cdot 3 \mid \alpha > 1\}$$

and

$$\mathcal{C}_2 = \{945 \cdot m \mid m \text{ is not divisible by 2, 3, 5, or 7}\}.$$

By the results of Theorem 15 and Theorem 22, it follows immediate that each number in $\mathcal{C}_1$ or $\mathcal{C}_2$ is both abundant and multiplicatively abundant. Thus $\mathcal{C} \supset \mathcal{C}_1 \bigcup \mathcal{C}_2$.

The sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ described above are only partially determined, so more related research is needed to completely determine them.

7. Average Order of Tau, Sigma, and Rho Functions. In this section, we will examine the asymptotic behavior of the tau, sigma, and rho functions. All of these functions fluctuate rapidly infinitely often. For example, each takes on relatively small values at prime numbers and relatively large values at numbers with many divisors. To smooth out the fluctuation effect, it is more convenient to study some average value for each of these functions. Specifically, we will consider the functions

$$\tau(x) = \frac{1}{x}\sum_{k \le x}\tau(k),\ \sigma(x) = \frac{1}{x}\sum_{k \le x}\sigma(k), \text{ and } \rho(x) = \left[\prod_{k \le x}\rho(k)\right]^{1/x},$$

defined by the (arithmetic or geometric) mean of the front-end values of the tau, sigma, and rho functions, respectively.

The big-oh notation is helpful for studying these functions, see [1, p. 53]:

DEFINITION 26. *Let $f(x)$ and $g(x)$ be functions on $\mathbb{R}$, and $g(x) > 0$ for all $x \geq a$. If there exists a constant $M > 0$ such that $|f(x)| \leq Mg(x)$ for all $x \geq a$, then we say that $f(x)$ is big oh of $g(x)$, and we write $f(x) = O(g(x))$. If $h(x)$ is another function on $\mathbb{R}$, then $f(x) = h(x) + O(g(x))$ means that $f(x) - h(x) = O(g(x))$.*

The next two results ([1, pp. 57-60]) give the average order of the tau and sigma functions.

THEOREM 27. *If $x \geq 1$, then*

$$\bar{\tau}(x) = \ln x + 2\gamma - 1 + O(\frac{1}{\sqrt{x}}),$$

where $\gamma \approx 0.5772156649$ is called Euler's constant.

THEOREM 28. *If $x \geq 1$, then*

$$\bar{\sigma}(x) = \frac{\pi^2}{12}x + O(\ln x).$$

The next theorem gives the Euler's summation formula which is often used to approximate partial sums, see [1, p. 54]. We will use it to establish Lemma 30 (3) later.

THEOREM 29 (Euler's Summation Formula). *If f has a continuous derivative f' on the interval $[a, b]$, where $0 < a < b$, then*

$$\sum_{a<k\leq b} f(k) = \int_a^b f(t)dt + \int_a^b (t - [t])f'(t)dt + f(b)([b] - b) - f(a)([a] - a).$$

LEMMA 30. *If $x \geq 1$, then*

$$\ln \lfloor x \rfloor! = \sum_{k\leq x} \ln k = x \ln x - x + O(\ln x).$$

$$\sum_{k\leq x} \frac{1}{k} = \ln x + \gamma + O\left(\frac{1}{x}\right).$$

$$\sum_{k\leq x} \frac{\ln k}{k} = \frac{1}{2}\ln^2 x + A + O\left(\frac{\ln x}{x}\right),$$

where A satisfies $-1 \leq A \leq 1$ and is given by

$$A = \int_1^\infty (t - \lfloor t \rfloor) \frac{1 - \ln t}{t^2} dt. \tag{3}$$

Proof. The first two estimations are proved in [1, pp. 67-68, 55-56]; and the last estimation, with neither the value of the constant A nor its bounds specified, is stated without proof in [1, Exercise 1 (a), p. 70]. Henceforth, we will only prove the last estimation, and establish a definite integral (3) for A and its bounds $-1 \leq A \leq 1$.[2]

Let $f(t) = \ln(t)/t$, $a = 1$, and $b = x$ in the Euler's summation formula. Then

$$\begin{aligned}\sum_{k\leq x} \frac{\ln k}{k} &= \int_1^x \frac{\ln t}{t} dt + \int_1^x (t - \lfloor t \rfloor) \frac{1 - \ln t}{t^2} dt + \frac{\ln x}{x}(\lfloor x \rfloor - x) \\ &= \frac{1}{2}\ln^2 x + \int_1^\infty (t - \lfloor t \rfloor) \frac{1 - \ln t}{t^2} dt - \int_x^\infty (t - \lfloor t \rfloor) \frac{1 - \ln t}{t^2} dt + O\left(\frac{\ln x}{x}\right).\end{aligned}$$

[2] The referee computed A to be approximately -0.07278 using *Mathematica*.

First, we need the integrals

$$\int_x^\infty \frac{1}{t^2} dt = \frac{1}{x} \quad \text{and} \quad \int_x^\infty \frac{\ln t}{t^2} dt = \frac{1 + \ln x}{x},$$

which can be evaluated easily. Next

$$0 \leq \int_x^\infty (t - \lfloor t \rfloor) \frac{1}{t^2} dt \leq \int_x^\infty \frac{1}{t^2} dt = \frac{1}{x},$$

and

$$0 \leq \int_x^\infty (t - \lfloor t \rfloor) \frac{\ln t}{t^2} dt \leq \int_x^\infty \frac{\ln t}{t^2} dt = \frac{1 + \ln x}{x}.$$

It follows that

$$-\frac{1 + \ln x}{x} \leq \int_x^\infty (t - \lfloor t \rfloor) \frac{1 - \ln t}{t^2} dt = \int_x^\infty (t - \lfloor t \rfloor) \frac{1}{t^2} dt - \int_x^\infty (t - \lfloor t \rfloor) \frac{\ln t}{t^2} dt \leq \frac{1}{x},$$

and, after setting $x = 1$,

$$-1 \leq \int_1^\infty (t - \lfloor t \rfloor) \frac{1 - \ln t}{t^2} dt \leq 1.$$

Finally, (4) reduces to

$$\sum_{k\leq x} \frac{\ln k}{k} = \frac{1}{2}\ln^2 x + A + O\left(\frac{\ln x}{x}\right),$$

where A is given in (3) and $-1 \leq A \leq 1$. □

Finally, we come to the important result of this section:

THEOREM 31. *If $x \geq 1$, then*

$$\ln \bar{\rho}(x) = \frac{1}{2}\ln^2 x - (1 - \gamma)\ln x - \gamma - A + O\left(\frac{\ln x}{x}\right). \tag{4}$$

Proof: Since

$$\bar{\rho}(x) = \left(\prod_{k\leq x} \rho(k)\right)^{1/x} = \left(\prod_{k\leq x}\prod_{d|k} d\right)^{1/x} = \left(\prod_{dq\leq x} d\right)^{1/x} = \left(\prod_{d\leq x}\prod_{q\leq x/d} q\right)^{1/x},$$

we have

$$\ln \bar{\rho}(x) = \frac{1}{x}\sum_{d\leq x}\sum_{q\leq x/d} \ln q.$$

By Lemma 30,

$$\begin{aligned}\ln \bar{\rho}(x) &= \frac{1}{x}\left[\sum_{d\leq x} \frac{x}{d}(\ln x - \ln d - 1) + O(\ln x - \ln d)\right] \\ &= (\ln x - 1)\sum_{d\leq x} \frac{1}{d} - \sum_{d\leq x} \frac{\ln d}{d} + O\left(\frac{\ln x}{x}\right) \\ &= (\ln x - 1)\left[\ln x + \gamma + O\left(\frac{1}{x}\right)\right] - \frac{1}{2}\ln^2 x - A + O\left(\frac{\ln x}{x}\right) \\ &= \frac{1}{2}\ln^2 x - (1 - \gamma)\ln x - \gamma - A + O\left(\frac{\ln x}{x}\right). \quad \square\end{aligned}$$

Ignoring the constant and the error terms, we can estimate $\ln \bar{\rho}(x)$ by

$$\ln \bar{\bar{\rho}}(x) = \frac{1}{2}\ln^2 x - (1-\gamma)\ln x. \tag{5}$$

Figure 1 (generated using MATLAB[3]) shows in the first subplot the graph of the function $\bar{\rho}(x)$ and its estimate function $\bar{\bar{\rho}}(x)$ given by (4) and (5), respectively, for a range of values of x. To ease checking visually how good the estimate function is, we also include a second plot to show the relative error of the estimate function for the same values of x. The subplot of relative error of $\bar{\bar{\rho}}(x)$ seems to suggest that the error vanishes when x is sufficiently large. Note that the x-axes of both subplots are in logarithmic scales.

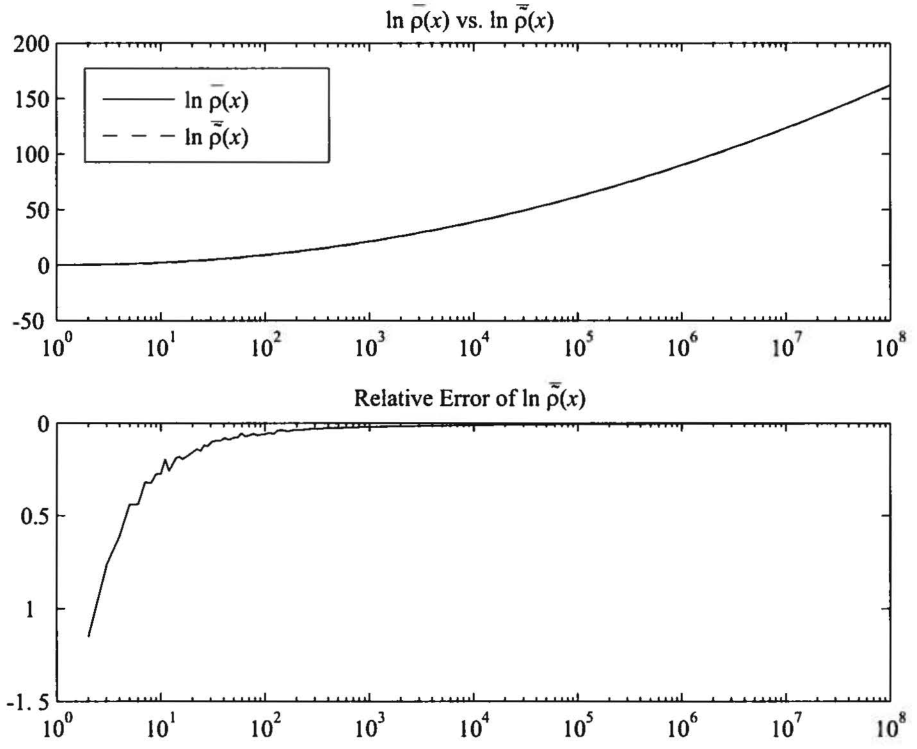

FIG. 1. *Comparison of the function* $\ln \bar{\rho}(x)$ *and its estimate function* $\ln \bar{\bar{\rho}}(x)$.

8. Asymptotic Behavior of Abundant, Multiplicatively Deficient, Multiplicatively Perfect, and Multiplicatively Abundant Numbers. In Table 6.1, we can see that there are infinitely many deficient, abundant, multiplicatively perfect, multiplicatively deficient, and multiplicatively abundant numbers. In spite of this, the distribution of each of these sets of numbers is still unclear to us. To study the distribution of infinite subsets of $\mathbb{N}$ or more generally $\mathbb{R}$, the terminology provided in the next two definitions are useful, see [1, p. 53] and [8, p. 473].

[3]The corresponding MATLAB M-files are available upon request by sending to the author's email address.

DEFINITION 32. *Let $f(x)$ and $g(x)$ be functions on $\mathbb{R}$. If $\lim_{x\to\infty} f(x)/g(x) = 1$, then we say that $f(x)$ is asymptotic to $g(x)$, and we write $f(x) \sim g(x)$.*

DEFINITION 33. *Let $S \subset \mathbb{R}$, and $S(x)$ be the number of positive integers in S that are not exceeding x. If the sequence $S(x)/x$ has a limit, then we say that S has a natural density.*

Let S and $S(x)$ be as described in Definition 33, it is easy to see that $S(x) \sim Dx$ if S has a natural density D.

Deléglise, in [5], established bounds on some complicated sums and products, then performed computations on them to arrive at some tight bounds for the natural density of the set of abundant numbers. We state the result in the next theorem.

THEOREM 34. *The natural density of the set of abundant numbers is bounded by 0.2474 and 0.2480.*

To examine the asymptotic behavior of the multiplicatively perfect numbers, we need to borrow the result from the prime number theorem ([1, pp. 82-84, 278-291]) and two estimations ([1, pp. 89, 90]) from analytic number theory as shown in the next theorem and lemma.

THEOREM 35 (Prime Number Theorem). *If $\pi(x)$ denotes the number of primes not exceeding x, then*

$$\pi(x) \sim \frac{x}{\ln x}.$$

LEMMA 36. *For all $x \geq 2$,*

$$\sum_{p\leq x}\frac{1}{p} = \ln\ln x + O(1) \quad \text{and} \quad \sum_{p\leq x}\frac{\ln p}{p} = \ln x + O(1),$$

where the sums are extended over all primes $p \leq x$.

THEOREM 37. *If $P(x)$ denotes the number of multiplicatively perfect numbers not exceeding x, then*

$$P(x) \sim \frac{x\ln\ln x}{\ln x}. \tag{6}$$

Proof. By Theorem 11, $P(x) = P_1(x) + P_2(x)$, where $P_1(x) = \pi(x^{1/3})$, and $P_2(x) = \sum_{p<q, pq\leq x} 1$ where the sum is extended over all numbers that are products of two distinct primes p, q such that $p<q$ and $pq \leq x$. We now count $\sum_{p<q,pq\leq x} 1$. Note that if $p<q$ and $pq \leq x$, then $p \leq \sqrt{x}$. So for each fixed $p \leq \sqrt{x}$, we can count all prime numbers q such that $q \leq x/p$, and then sum over all $p \leq \sqrt{x}$. Therefore

$$P_2(x) = \sum_{p\leq\sqrt{x}}\sum_{q\leq x/p} 1 = \sum_{p\leq\sqrt{x}} \pi\left(\frac{x}{p}\right).$$

It follows that

$$P(x) = \pi(x^{1/3}) + \sum_{p\leq\sqrt{x}} \pi\left(\frac{x}{p}\right). \tag{7}$$

By using the prime number theorem,

$$P_1(x) \sim \frac{3x^{1/3}}{\ln x} \tag{8}$$

and

$$P_2(x) \sim \sum_{p \le \sqrt{x}} \frac{x}{p \ln(x/p)}. \tag{9}$$

Now we estimate

$$\begin{aligned}\sum_{p\le\sqrt{x}} \frac{x}{p\ln(x/p)} &= \frac{x}{\ln x}\sum_{p\le\sqrt{x}} \frac{1}{p}\frac{\ln x}{\ln x - \ln p} \\ &= \frac{x}{\ln x}\sum_{p\le\sqrt{x}} \frac{1}{p}\frac{1}{1-(\ln p/\ln x)} \\ &= \frac{x}{\ln x}\sum_{p\le\sqrt{x}} \frac{1}{p}\left[1 + \frac{\ln p}{\ln x} + \sum_{k=2}^{\infty}\left(\frac{\ln p}{\ln x}\right)^k\right] \\ &= \frac{x}{\ln x}\left[\sum_{p\le\sqrt{x}}\frac{1}{p} + \frac{1}{\ln x}\sum_{p\le\sqrt{x}}\frac{\ln p}{p} + \sum_{p\le\sqrt{x}}\frac{1}{p}\frac{\ln^2 p}{\ln^2 x}\sum_{k=0}^{\infty}\left(\frac{\ln p}{\ln x}\right)^k\right] \quad (10)\end{aligned}$$

Since $0 < p \le \sqrt{x}$ implies $\ln p \le \ln x/2$, so $1/(\ln x - \ln p) \le 1/\ln p$ and consequently

$$\begin{aligned}\sum_{p\le\sqrt{x}}\frac{1}{p}\frac{\ln^2 p}{\ln^2 x}\sum_{k=0}^{\infty}\left(\frac{\ln p}{\ln x}\right)^k &= \sum_{p\le\sqrt{x}}\frac{1}{p}\frac{\ln^2 p}{\ln^2 x}\frac{1}{1-(\ln p/\ln x)} = \sum_{p\le\sqrt{x}}\frac{1}{p}\frac{\ln^2 p}{\ln x(\ln x - \ln p)} \\ &\le \sum_{p\le\sqrt{x}}\frac{1}{p}\frac{\ln^2 p}{\ln x(\ln p)} = \sum_{p\le\sqrt{x}}\frac{1}{p}\frac{\ln p}{\ln x},\end{aligned}$$

hence

$$\sum_{p\le\sqrt{x}}\frac{1}{p}\frac{\ln^2 p}{\ln^2 x}\sum_{k=0}^{\infty}\left(\frac{\ln p}{\ln x}\right)^k = O\left(\frac{1}{\ln x}\sum_{p\le\sqrt{x}}\frac{\ln p}{p}\right).$$

Therefore (10) simplifies to

$$\sum_{p\le\sqrt{x}}\frac{x}{p\ln(x/p)} = \frac{x}{\ln x}\left[\sum_{p\le\sqrt{x}}\frac{1}{p} + \frac{1}{\ln x}\sum_{p\le\sqrt{x}}\frac{\ln p}{p} + O\left(\frac{1}{\ln x}\sum_{p\le\sqrt{x}}\frac{\ln p}{p}\right)\right].$$

Substitute the partial-sums with the expressions in Lemma 36, we have

$$\begin{aligned}\sum_{p\le\sqrt{x}}\frac{x}{p\ln(x/p)} &= \frac{x}{\ln x}\left[\ln\ln\sqrt{x} + O(1) + \frac{\ln\sqrt{x} + O(1)}{\ln x} + O\left(\frac{\ln\sqrt{x}}{\ln x}\right)\right] \\ &= \frac{x\ln\ln x}{\ln x} + O\left(\frac{x}{\ln x}\right).\end{aligned}$$

Thus (9) can be rewritten as

$$P_2(x) \sim \frac{x\ln\ln x}{\ln x}.$$

Equation (8) shows that $P_1(x) = O(x/\ln x)$ which is dominated by $P_2(x)$, so $P(x) \sim P_2(x)$ and hence (6) is established. □

With the help of the Prime Number Theorem, it is an easy matter to estimate the asymptotic behavior of the set of multiplicatively deficient numbers:

THEOREM 38. *If $D(x)$ denotes the number of multiplicatively deficient numbers not exceeding x, then*

$$D(x) \sim \frac{x + 2x^{1/2}}{\ln x}. \tag{11}$$

Proof. By Theorem 21,

$$D(x) = \pi(x) + \pi(x^{1/2}). \tag{12}$$

Therefore, the prime number theorem gives

$$D(x) \sim \frac{x}{\ln x} + \frac{x^{1/2}}{\ln x^{1/2}},$$

which is (11). □

THEOREM 39. *If $A(x)$ denotes the number of multiplicatively abundant numbers not exceeding x, then*

$$A(x) \sim \frac{x(\ln x - \ln\ln x)}{\ln x}. \tag{13}$$

Proof. Since each positive integer is exactly one of multiplicatively deficient, multiplicatively perfect, or multiplicatively abundant,

$$A(x) = x - D(x) - P(x). \tag{14}$$

It is easy to see from (11) and (6) that $P(x)$ dominates $D(x)$. Therefore $A(x) \sim x - P(x)$ which is (13). □

Estimation functions for the number of multiplicatively deficient, multiplicatively perfect, and multiplicatively abundant numbers, not exceeding x, based on (6), (11), and (13), are given by

$$\tilde{P}(x) = \frac{x\ln\ln x}{\ln x}, \tag{15}$$

$$\tilde{D}(x) = \frac{x + 2x^{1/2}}{\ln x}, \tag{16}$$

and

$$\tilde{A}(x) = \frac{x(\ln x - \ln\ln x)}{\ln x}, \tag{17}$$

respectively. Figures 2, 3, 4 (generated by MATLAB[4]) show, in their first subplots, the graph of functions described by (7), (12), (14), and the corresponding estimate functions described by (15), (16), (17), respectively, for different values of x. The second subplot of each figure shows the relative error of each estimate function for the same values of x. The subplots of relative errors of $\tilde{P}(x)$, $\tilde{D}(x)$, and $\tilde{A}(x)$ seem to suggest that these errors tend to 0 as x tends to infinity. Note that both axes of the first subplot and the x-axis of the second subplot are in logarithmic scales.

[4]Again, the associated MATLAB M-files are available upon request by emailing the author.

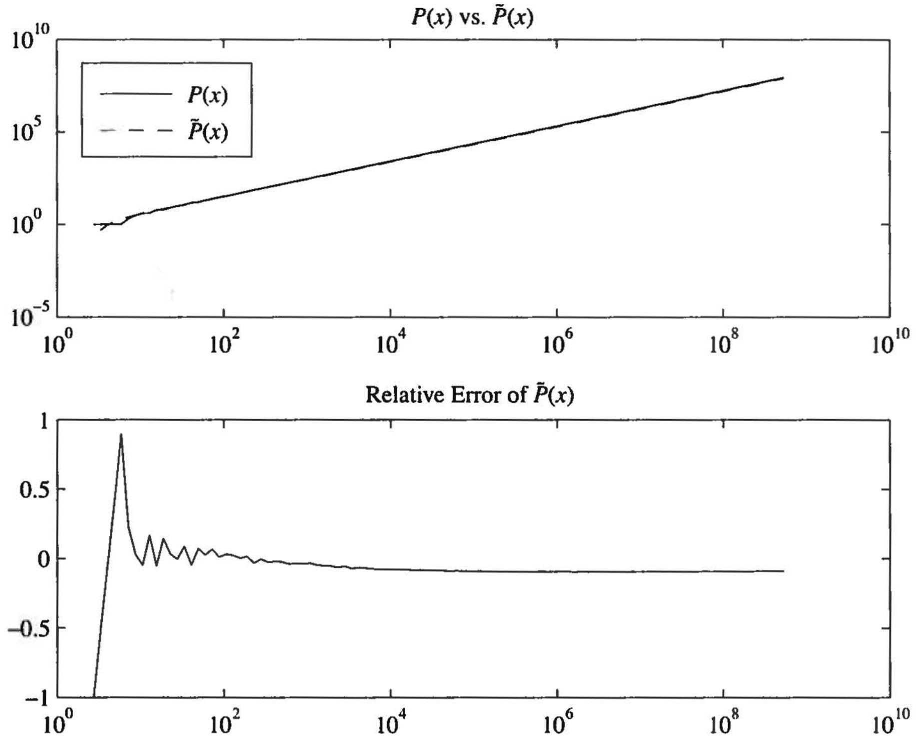

FIG. 2. *Comparison of the function $P(x)$ and its estimate function $\tilde{P}(x)$ of number of multiplicatively perfect numbers*

9. Open Questions. In the previous sections, we have provided an elementary treatment of the topics relating to the tau, sigma, and rho functions, and some numbers that are defined using them. There are still many open questions remaining to be answered. Here is a list to start from:

1. Is there any odd perfect number?
2. Are there infinitely many even perfect numbers, or equivalently, are there infinitely many Mersenne primes?
3. Can even perfect numbers be described in a different form than that in Theorem 9 which can be used to discover new elements in the set easily? The answer to this question might also help answering question 2.
4. There are sufficient conditions for a number to be deficient (or abundant) (Theorem 14, Theorem 15, and Theorem 16). Are there necessary conditions for a number to be deficient (or abundant)?
5. We know that there is no multiplicatively amicable pair (Corollary 25). Are there infinitely many amicable pairs?
6. The sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ described in section 6, that are, respectively, the set of numbers that are both perfect and multiplicatively abundant, the set of numbers that are both deficient and multiplicatively abundant, and the set of numbers that are both abundant and multiplicatively abundant, are only partially determined in that section. Precisely, we have necessary (sufficient) conditions for numbers to be elements of $\mathcal{A}$ ($\mathcal{B}$, $\mathcal{C}$). Are there any sufficient (necessary) conditions for numbers to be in $\mathcal{A}$, ($\mathcal{B}$, $\mathcal{C}$)?

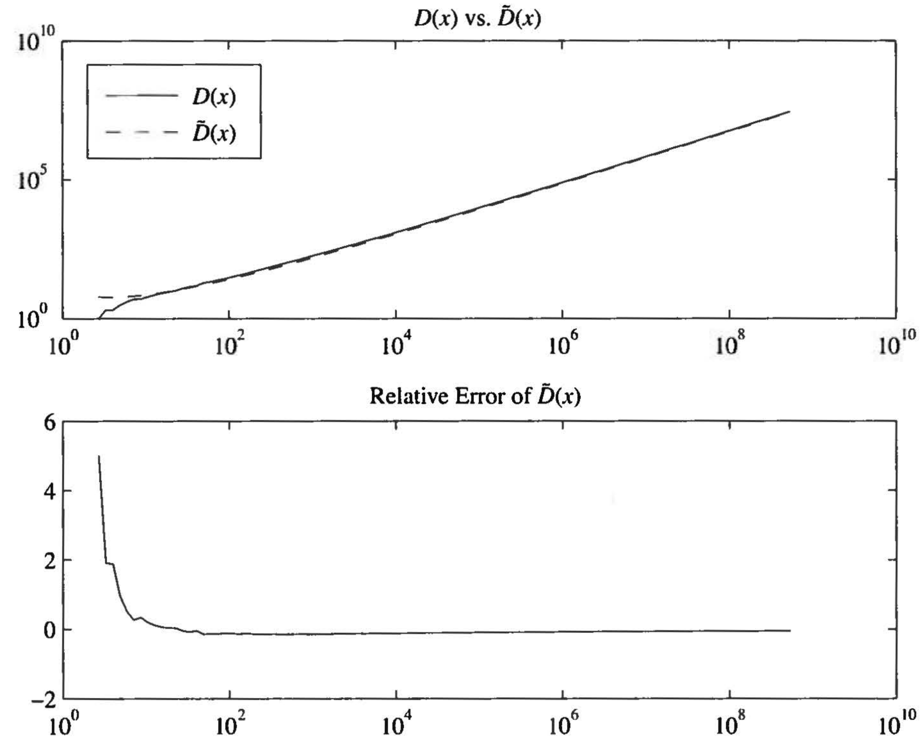

FIG. 3. *Comparison of the function $D(x)$ and its estimate function $\tilde{D}(x)$ of number of multiplicatively deficient numbers*

7. Some progress has been made in estimating the asymptotic behavior of abundant, Theorem 34, multiplicatively perfect, Theorem 37, multiplicatively deficient, Theorem 38, and multiplicatively abundant numbers, Theorem 39. Are there good estimates on the asymptotic behavior of deficient numbers?

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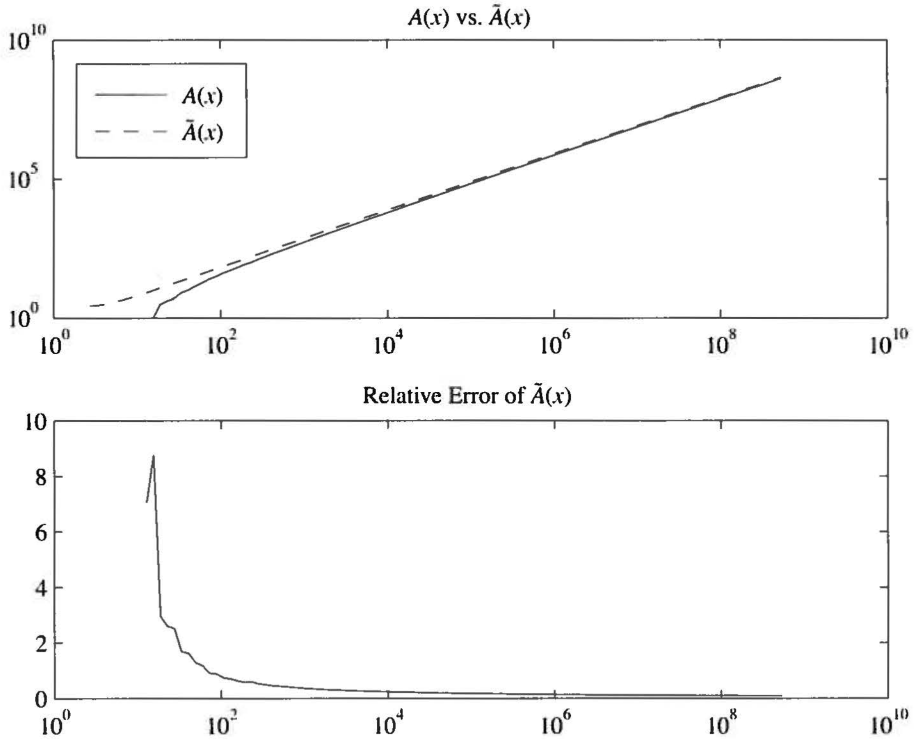

FIG. 4. *Comparison of the function $A(x)$ and its estimate function $\tilde{A}(x)$ of number of multiplicatively abundant numbers*

[10] ERIC WEISSTEIN, Eric Weisstein's World of Mathematics, http://mathworld.wolfram.com/

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EXTREMAL INTERSECTION SIZES

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1. Introduction. Suppose we are investigating a finite collection of finite sets, and suppose we know the size of each set and the size of their union. What, if anything, can we say about the size of their intersection? We write n for the number of sets, and A_i for the sets. If there are only two sets, then the exact size of the intersection is given by

$$|A_1 \cap A_2| = |A_1| + |A_2| - |A_1 \cup A_2|. \tag{1}$$

However, when $n > 2$ we don't have enough information to determine the size of the intersection. In this note we will generalize the two-set case to get tight upper and lower bounds on the size of the intersection. The inspiration for this note is a problem that we received at MathNerds.com [2]; we solve this problem in Section 2 as an application of the results in this note. A similar problem was presented by Lewis Carroll in [3, Knot X, §1, pp. 142–144], and we discuss his solutions in Section 4.

THEOREM 1. *With the notation above,*

$$\sum_i |A_i| - (n-1)\left|\bigcup_i A_i\right| \le \left|\bigcap_i A_i\right| \le \frac{1}{n-1}\left(\sum_i |A_i| - \left|\bigcup_i A_i\right|\right).$$

For $n = 2$, the upper and lower bounds are the same and we get equation (1). Note that for $n > 2$, the bounds given by Theorem 1 may be worse than the trivial bounds $0 \le |\cap_i A_i| \le \min_i |A_i|$. The next theorem shows that Theorem 1 combined with the trivial bounds gives the best-possible bounds. If c is in the range specified by (3) and (4) below, then there is a collection having intersection size c, and if c is not in that range then Theorem 1 and the trivial bounds show that no collection can have intersection size c.

THEOREM 2. *Suppose each of $u, a_1, \ldots, a_n$ is a positive integer, and suppose*

$$\max_i a_i \le u \le \sum_i a_i. \tag{2}$$

Suppose c is an integer satisfying

$$0 \le c \le \min_i a_i \tag{3}$$

and

$$\sum_i a_i - (n-1)u \le c \le \frac{1}{n-1}\left(\sum_i a_i - u\right). \tag{4}$$

Then there is a collection of sets $\{A_i\}$ having sizes $|A_i| = a_i$, union size $|\cup_i A_i| = u$, and intersection size $|\cap_i A_i| = c$.

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2. Example. A health club has 600 members. Of these, 80% are male, 70% play tennis, and 60% swim. What are the lowest and highest possible percentages of males who swim and play tennis?

Let A_1, A_2, A_3 be the sets of the 480 members who are male, the 420 who play tennis, and the 360 who swim, respectively. The intersection of these sets is the set of males who swim and play tennis. The size of the union of these three sets is at most 600, but we don't know the exact size because there may be some women who neither swim nor play tennis. Therefore Theorems 1 and 2 do not apply directly, but we do have estimates for the union size and we can use these in the theorems. Write $u = |\cup_i A_i|$ and $c = |\cap_i A_i|$. Theorem 1 tells us $480 + 420 + 360 - 2u \leq c \leq \frac{1}{2}(480 + 420 + 360 - u)$. From the sizes of A_i and the total membership we know $480 \leq u \leq 600$. Using these upper and lower bounds for u we get $60 \leq c \leq 390$. The upper bound is worse than the trivial bound 360. Our conclusion is that, for any value of u, we have $60 \leq |\cap_i A_i| \leq 360$.

We don't know yet that these extremal values can be met for a total membership of 600, but we will show this by construction. The idea is to put aside all members of the intersection in one group, and then spread out the properties of the remaining members as much as possible, such that none of the remaining members are in all the sets. (We'll use the same idea in general form to prove Theorem 2.)

For the lower bound, first assign 60 of the members to be male, play tennis, and swim. Then we have 540 members remaining, from which we will assign properties such that 420 are male, 360 are tennis players, and 300 are swimmers, and such that no person is all three. Imagine arranging the remaining 540 members in a big circle, and numbering them counterclockwise 1–540. Go around the circle assigning properties as follows: Members 1–420 are male, members 421–540 and 1–240 play tennis, and members 241–540 swim. Then none of these 540 is male, plays tennis, and swims.

The upper bound construction is similar. First assign 360 of the members to be male, play tennis, and swim. Then we have 240 members remaining, from which we need 120 males, 60 tennis players, and 0 swimmers, such that no person is all three. Arrange the remaining members in a circle and number them 1–240, and assign their properties as follows: Members 1–120 are male, members 121–180 play tennis, and no members swim. Then none of these 240 is male, plays tennis, and swims.

3. Proofs. Proof of Theorem 1: We use a counting argument. Write $U = \cup_i A_i$ and write C_k for the set of elements of U that are in exactly k of the A_i. Then $\sum_k |C_k| = |U|$ and $C_n = \cap_i A_i$. The sum $\sum_i |A_i|$ counts each element of C_k exactly k times, so $\sum_i |A_i| = \sum_k k|C_k|$. We can now estimate

$$\sum_i |A_i| = \sum_{k=1}^{n-1} k|C_k| + n|C_n| \leq \sum_{k=1}^{n-1}(n-1)|C_k| + n|C_n| = (n-1)|U| + |C_n|$$

and solving the inequality gives $|\cap_i A_i| = |C_n| \geq \sum_i |A_i| - (n-1)|U|$.

A similar argument gives the upper bound:

$$\sum_i |A_i| = \sum_{k=1}^{n-1} k|C_k| + n|C_n| \geq \sum_{k=1}^{n-1} 1 \cdot |C_k| + n|C_n| = |U| + (n-1)|C_n|$$

and therefore $|C_n| \leq (1/(n-1))(\sum_i |A_i| - |U|)$. □

Proof of Theorem 2: We construct the sets A_i by drawing two circles in the plane, cutting them into arcs of unit length, and defining the A_i as certain collections of these arcs. The size of a set is both the number of arcs in the set and their total length, and we use this duality in the proof.

If $c = 0$, let $C = \emptyset$, otherwise draw a circle of circumference c and divide it into c arcs of unit length. From inequalities (2) and (3) we have $u \geq \max_i a_i \geq \min_i a_i \geq c$. If $u = c$ then we can take all $A_i = C$, because then $|\cup_i A_i| = |\cap_i A_i| = |C| = c = u$. Therefore assume $u > c$ in the following.

Draw a second circle, disjoint from the first, and of circumference $u - c$. Divide the circle into $u - c$ arcs of unit length. Define the set B_1 as the set of the first $a_1 - c$ arcs, B_2 as the set of the next $a_2 - c$ arcs, and so on (some B_i may be empty). From inequality (2) we have that $a_i - c \leq u - c$, and so no B_i wraps around to contain the same arc twice. Finally define $A_i = B_i \cup C$.

Because the circles are disjoint, they have no arcs in common and it is clear that $|A_i| = (a_i - c) + c = a_i$. We need to show that $|\cup_i A_i| = u$ and that $|\cap_i A_i| = c$. Because C is disjoint from each B_i, these are equivalent to showing $|\cup_i B_i| = u - c$ and that $\cap_i B_i = \emptyset$.

To show that $|\cup_i B_i| = u - c$, observe from inequality (4) that $\sum_i a_i \geq (n-1)c + u$ and so $\sum_i (a_i - c) = \sum_i a_i - nc \geq (n-1)c + u - nc = u - c$. Therefore the sum of the lengths of the arcs in the B_i is at least $u - c$, the circumference of the circle. The arcs in the B_i are contiguous, so $\cup_i B_i$ contains all arcs of the circle.

To show that $\cap_i B_i = \emptyset$, from the left-hand inequality of (4) we have that $\sum_i a_i \leq c + (n-1)u$ and therefore $\sum_i (a_i - c) = \sum_i a_i - nc \leq c + (n-1)u - nc = (n-1)(u - c)$. Therefore the B_i may wrap around the circle, but not more than $n - 1$ times, and so there are no arcs that are in more than $n - 1$ of the B_i. Therefore $\cap_i B_i = \emptyset$. □

4. Related results. We believe the results in this note must be already known, but we were not able to find them anywhere in the literature. The branch of combinatorics that deals with this kind of problem is Extremal Set Theory; see for example [1, Chapter 6].

A related result is the Inclusion-Exclusion Principle that is discussed in most books on combinatorics, for example, [1, Chapter 10]. It states in our notation that if $V \supseteq U$ then

$$|V \setminus U| = |V| - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \cdots + (-1)^n |\cap_i A_i|.$$

This is another generalization of equation (1), because we can take $V = U$ and $n = 2$ to get equation (1). The Inclusion-Exclusion Principle is proved by another counting argument. An element of $V \setminus U$ is counted exactly once by the right-hand side because it appears only in V, and an element of U that is in exactly k of the A_i is counted

$$1 - \binom{k}{1} + \binom{k}{2} - \binom{k}{3} + \cdots + (-1)^k \binom{k}{k} = (1-1)^k = 0$$

times.

The Inclusion-Exclusion Principle is used in many kinds of combinatorial problems and is also the basis of sieve methods in number theory, but it requires that you know or can estimate the sizes of all possible combinations of intersections of the A_i, unlike Theorem 1 that only requires knowledge of the size of the sets and of their union.

We received the Example problem (lower bound only) at MathNerds.com [2], a web site offering free help to students. The problem was submitted by Ms. Regis R. Park and solved by MathNerds volunteer Esther Fontova by a "worst case" analysis, pushing as many of the swimmers and tennis players as possible into the female category, then pushing the remaining swimmers and tennis players as far apart as possible.

In [3, Knot X, §1, pp. 142–144], Lewis Carroll states a problem about the Chelsea Pensioners (war veterans):

> If 70 per cent have lost a eye, 75 per cent an ear, 80 per cent an arm, and 85 per cent a leg: what percentage, *at least*, must have lost all four?

The problems in [3] were published as a serial in a monthly magazine, and Carroll published and discussed in later issues the solutions sent in by the readers. The solution to the Chelsea Pensioners that he judged best used a version of the Pigeonhole Principle: Suppose we have 100 men, and add up all the injuries, so that we have 70 lost eyes and so on, for a total of 70 + 75 + 80 + 85 = 310 injuries among 100 men, so at least 10 men must have all four injuries. Carroll also presents briefly a second solution, which is essentially to spread out the injuries as much as possible among the 100 men, leading to the least overlap.

Carroll's solutions are discussed in more generality and in more mathematical terms in Herstein & Kaplansky [4, pp. 12–14]. They treat the first solution as an example of De Morgan's law. We seek the size of the intersection of the four classes of men, and by De Morgan the complement of this set is the union of the complements. The size of the union is bounded above by the sum of the sizes of the complements, or 30 + 25 + 20 + 15 = 90, so the size of the original intersection is bounded below by 10. They also treat Carroll's second solution by three applications of (1).

Both Carroll and Herstein & Kaplansky gloss over the problem of whether the lower bound is actually reached (although in this example it is easy to show that it is). Neither discussion attempts to investigate the upper bound (which would be the trivial upper bound 70 in this example) or to generalize the problem.

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HEROIC TETRAHEDRA AND SIMPLEXES

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The famous formula, named after Hero (or Heron) of Alexandria and sometimes attributed to Archimedes, for the area Δ of a triangle, is

$$\Delta^2 = s(s-a)(s-b)(s-c), \tag{1}$$

where the lengths of the sides are denoted by a, b, c and $s = \frac{1}{2}(a+b+c)$. Hersh (2002) gave elegant methods for deriving (1) from the assumption that Δ^2 is a polynomial in the lengths of the edges (and therefore has to be a quartic). Without a proof, we don't know that Δ^2 isn't, for example, the square root of a polynomial of the eighth degree. One way to prove that Δ^2 is indeed a polynomial is to use Theorem 4036 in Carr (1970). He gives a simple geometrical proof of a formula for Δ in terms of the coordinates $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ of the vertices. If we take the third vertex as the origin O, then that formula reduces to one half of the absolute value of $x_1y_2 - x_2y_1$ and this also is a well-known formula. It can be written, if preferred, in terms of a 2 by 2 determinant

$$2\Delta = \pm \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}. \tag{2}$$

On multiplying this determinant by its transpose, we get the symmetric determinant

$$4\Delta^2 = \begin{vmatrix} x_1^2 + y_1^2 & x_1x_2 + y_1y_2 \\ x_1x_2 + y_1y_2 & x_2^2 + y_2^2 \end{vmatrix} = \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} \end{vmatrix} \tag{3}$$

The elements of the right-hand determinant, which *no longer depend on the location of the origin*, are inner or scalar products, where the bold **a** and **b** denote the sides, a and b, expressed more fully as vectors (that is, with their *directions* represented). To obtain a formula in terms of the lengths of the sides alone write $\mathbf{a}.\mathbf{b}$ as $1/2(a^2+b^2-c^2)$ (even when $a = b$, in which case $c = 0$), and we obtain

$$16\Delta^2 = \begin{vmatrix} 2a^2 & a^2+b^2-c^2 \\ a^2+b^2-c^2 & 2b^2 \end{vmatrix}. \tag{4}$$

This is a polynomial in a, b, and c, as is required to convert Hersh's derivation of Hero's formula into a proof. But in a sense we have proved too much, for the determinant equals (by the factorization formula for the "difference of two squares" applied thrice)

$$\begin{aligned} 4a^2b^2 &- (a^2+b^2-c^2)^2 && (5)\\ &= [2ab + (a^2+b^2-c^2)][2ab - (a^2+b^2-c^2)] \\ &= [(a+b)^2 - c^2][c^3 - (a-b)^2] \\ &= (a+b+c)(a+b-c)(a-b+c)(-a+b+c) && (6) \end{aligned}$$

and this proves Hero's formula (1).

Thus formula (4) leads to a proof of Hero's formula by algebra even more elementary than what Hersh used in his 'derivations'. But I like the didactic aspects of Hersh's methods.

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The argument thus far could have been expressed without mentioning determinants or scalar products, but their use can be generalized to higher dimensions. The three-dimensional form of (2) is given, for example, by Sommerville (1934, top of page 35) and presumably dates back at least to the 19^{th} century. The volume of a p-dimensional simplex or "p-simplex" is given by the natural extension of (2) with the factor 2 replaced by $p!$.

Given that a, b, and c are (strictly) positive numbers, a necessary and sufficient condition for the expression (6) to be positive is that the three triangle inequalities $b+c > a, c+a > b, a+b > c$ should be satisfied. This condition is obviously sufficient; it is also necessary because it is algebraically impossible for two of the factors in (6) to be negative. The reader will be able to see this in a minute's thought.

The argument thus far has been very elementary. It will now become slightly more advanced.

The natural generalization of formula (4) to a p-simplex (in notations discussed below) is

$$(p!)^2 2^p \Delta^2 = \begin{vmatrix} 2a_1^2 & a_1^2 + a_2^2 - a_{12}^2 & \cdots & a_1^2 + a_p^2 - a_{1p}^2 \\ a_2^2 + a_1^2 - a_{21}^2 & 2a_2^2 & \cdots & a_2^2 + a_p^2 - a_{2p}^2 \\ \vdots & \vdots & \vdots & \vdots \\ a_p^2 + a_1^2 - a_{p1}^2 & \cdots & a_p^2 + a_2^2 - a_{p2}^2 & 2a_p^2 \end{vmatrix} \qquad (7)$$

which is formula (17) of Good and Gaskins (1971,p. 349) (we shall cite that paper as G^2). Sommerville (1934, top of page 36) gives a formula for Δ^2, in terms of the lengths of the edges, as a 5 by 5 determinant. Or see Misner et al (1973, p. 7), who cite Hero, Tartaglia, and Blumenthal (from various centuries).

(7) can be obtained easily from what was called above "the natural extension of (2)". (It is sometimes convenient to let the subscripts run from 0 to $p-1$ instead of from 1 to p). Here a_{rs} (meaning $a_{r,s}$) denotes the length of the third side of the triangle whose other two sides are the r^{th} and s^{th} edges incident with $O(a_{rs} = 0$ when $r = s)$. Of course a_{rs} doesn't denote an element of the determinant. The three-term quadratic in position (r, s) of the determinant can be written as twice the inner product of the two vectors corresponding to the edges of lengths a_r and a_s, just as in the case $p = 2$. Like Hero's formula the determinant depends only on the lengths of the edges.

The determinant is equal to the product of the eigenvalues (characteristic roots) of the corresponding matrix. An example is provided by a p-simplex sitting on the origin with its p legs astride, each of unit length and along an axis of rectangular Cartesian coordinates. Thus each pair of feet are separated by a distance of $\sqrt{2}$. Then the determinant becomes diagonal and we onfirm at once that $\Delta = 1/p!$

Let us write D for the determinant even when the $p(p+1)/2$ 'variables', or 'arguments', a_r and a_{rs} (r and s running from 1 to p) are arbitrary real numbers. The corresponding matrix is real and symmetric so its eigenvalues are all real (for example, Bellman 1970, p. 35) but are not necessarily positive though their sum must be (it is the trace of the matrix whose diagonal elements are positive). The number of negative eigenvalues must be even when $D > 0$ because their product is equal to D. That D is proportional to the square of Δ partially explains why the

square root of a determinant occurs under the integral sign in tensor calculus (see, for example, Misner et al 1972, p. 222).

The sign or orientation of a p-dimensional simplex is positive or negative according to that of the familiar determinant, implicitly mentioned above, from which (7) was derived. This orientation is not conveyed by the determinant in (7). In fact, if $D < 0$, the $p(p+1)/2$ variables cannot correspond to a real p-dimensional simplex, only to an imaginary one. In theoretical physics imaginary volumes might occur since negative energy states occur in Dirac's theory of elementary particles; see, for example, Coughlan and Dodd (1991, p. 26).

While briefly on the topic of theoretical physics, observe that four-dimensional volumes in flat space-time are invariant under the Lorentz transformation although three-dimensional volumes are not. An elementary proof, based as above on a simplicial dissection of a region of space-time into small simplexes, is given, for example, in Endnote 4 in Good (2001) where further references are provided. Let us return now to pure mathematics.

For the case $p = 3$, the lengths of the edges incident with O will be denoted by a, b, and c, and the lengths of the other three edges will be denoted by α, β, and γ, where, for example, the edge of length α is opposite the edge of length a (opposite edges have no vertex in common). Then

$$\begin{aligned} 144\Delta^2 = \; & (a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2)(a^2 + b^2 + c^2 + \alpha^2 + \beta^2 + \gamma^2) \\ & -2[(a^2+\alpha^2)a^2\alpha^2 + (b^2+\beta^2)b^2\beta^2 + (c^2+\gamma^2)c^2\gamma^2] \\ & -(b^2c^2\alpha^2 + c^2a^2\beta^2 + a^2b^2\gamma^2) - \alpha^2\beta^2\gamma^2. \end{aligned} \qquad (8)$$

This is formula (8) of G^2 except for the slip there of omitting the last term $\alpha^2\beta^2\gamma^2$. But a somewhat simpler form of (8), derived more easily from (7), is

$$144\Delta^2 = 4a^2b^2c^2 + (S-X)(S-Y)(S-Z) - a^2(S-X)^2 - b^2(S-Y)^2 - c^2(S-Z)^2 \quad (9)$$

where $S = a^2 + b^2 + c^2, X = a^2 + \alpha^2, Y = b^2 + \beta^2$, and $Z = c^2 + \gamma^2$. Neither (8) nor (9) is expressed explicitly as a product of linear factors, and in that sense neither of them resembles Hero's formula (1). Later, when considering "isosceles" pyramids and simplexes, we shall come across formulae more like Hero's. (Formula (9) can, however, be conveniently programmed for some hand-held calculators such as the HP15C).

The formula for the volume of a general tetrahedron or simplex was incidental to the main theme of G^2. That article discussed the "centroid method" of integration in p dimensions ($p = 1, 2, 3, \ldots$). The centroid method is the generalization to several dimensions of the (one-dimensional) mid-point method (which is at least as simple and somewhat more accurate than the trapezoidal method). For integration over a region, one can insert many points in the region and join adjacent points, or vertices, to obtain a simplicial dissection into small or 'infinitesimal' simplexes. See, for example, Misner et al, p. 309. But we shall not be further concerned with methods of integration in the present article.

On page 349 of G^2, we said "It is natural to ask whether there is a generalization of Hero's formula to more than two dimensions. The answer seems to be that the formula cannot be of similar form [in general] but a formula can be given for the p-dimensional 'volume' of a simplex in terms of the lengths of the edges." That comment led to the discussion of the volume or 'content' of a general simplex. (I will use the symbol Δ in this general sense, although the simplex isn't a triangle if $p > 2$). *I will call a formula for D heroic when it is expressed as a product of p real factors each* *linear in* a_r *and* $a_{rs}(r, s = 0, 1, \ldots, p-1)$. The expression of D as the product of its p eigenvalues or characteristic roots is not necessarily of this form because these roots might be the roots of cubics, quartics, etc.

Even when $a_r > 0$ and $| a_{rs} | < a_r a_s$ for all r and s, a simplex can be 'degenerate', in that its set of vertices can belong to fewer than p dimensions. When this happens, the expressions (8) and (9) vanish. Now recall that D is *defined* as the value of the determinant (not as the left-hand side of (7)). This is a function of the $p(p+1)/2$ arguments (not a symmetric function) even if these arguments are not the lengths of the sides of a real simplex. Sometimes D is negative, as mentioned before, and then D isn't equal to $(p!)^2 2^p \Delta^2$ for any real simplex.

For a simplex to be real, and not degenerate, it is *necessary* that the corresponding determinant D be positive. To determine whether the condition is *sufficient* is a most challenging problem. We saw above that the condition is sufficient when $p = 2$ and I suspect it is sufficient for $p = 3$, having failed to find a counterexample.

Special cases. Some special cases of formulae (8) and (9) will now be mentioned. For example, the square of the 'volume' of an equilateral simplex in p dimensions, in which each side has unit length, is $(p+1)2^{-p}(p!)^{-2}$ (multiply by a^p if the sides have length a). If $p = 1, 2, 3, 4, 5, 6, 7, 8$, then $(p+1)$ times the reciprocals of the squared volumes are

The simplex is imaginary when $\alpha/a > \sqrt{[2p/(p-1)]}$ and is degenerate when its 'height' is zero, that is, when its apex coincides with the center of its base. This happens when $D = 0$, that is when $\alpha/a = \sqrt{[2p/p-1)]}$. In the spirit of number theory we note that the "degeneration ratio" α/a is rational only if $2p$ is a square and $p-1$ is also a square. This happens for an infinite sequence of values of the dimension p beginning with

$$2, 50, 1682, 57122, 1940450, 65918162, \ldots \tag{13}$$

These are twice the squares of the denominators of the $1^{st}, 3^{rd}, 5^{th}, 7^{th}, \ldots$, convergents u_n/v_n of the simple continued fraction for $\sqrt{2}$. Here

$$u_n = (k^n + k^{-n})/2, v_n = [k^n + (-k)^{-n}]/(2\sqrt{2}), \tag{14}$$

where $k = \sqrt{2} + 1, k^{-1} = \sqrt{2} - 1$. (See, for example, Hardy and Wright 1938, pp. 145-146). Some values of u_n and v_n are shown in the following table:

n	1	2	3	4	5	6	7	8	9
u_n	1	3	7	17	41	99	239	577	1393
v_n	1	2	5	12	29	70	169	408	985

Slips in G^2. Three slips in G^2, not affecting the present article, were mentioned by Good and Tideman (1975, page 367). Another one is that, two lines above formula (17) in G^2, the expression $p-1$ should be p. We mentioned above that there was a term missing in (18), corrected in (8).

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[Proofs of several of the formulae are on pages 4659 to 4669 of my bound papers filed in the VA Tech main library and in the Statistic Department [illegible]

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AN ALTERNATING SERIES EXPANSION FOR $(\ln 2)^2$

STEPHEN W. HRUSKA, REBEKAH B. JOHNSON, AND RYAN L. YATES*

Abstract. We develop an alternating series for $(\ln 2)^2$ where terms are related to the partial sums of the harmonic series.

1. Introduction. Napier's definition of logarithm paved the way for Mercator's further work with logarithms with base e (he was the first to call them "natural logs"). In his 1668 work *Logarithmotechnia*, Mercator published the series $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ [1]. This series gives the well known result that the sum of the alternating harmonic series is $\ln 2$. Here we investigate an alternating series expansion of $(\ln 2)^2$ and its connection to the harmonic series.

2. Finding a Series for $[\ln(1+x)]^2$. Following the Taylor series method for finding an alternating series expansion for $\ln 2$ we begin by finding the Maclaurin series for $[\ln(1+x)]^2$. Let

$$[\ln(1+x)]^2 = \sum_{i=0}^{\infty} a_i x^i. \tag{1}$$

Integrating term by term,

it follows that

$$-2\ln(1+x)=\sum_{i=1}^{\infty}\frac{2(-1)^i}{i}x^i=-2x+\sum_{i=2}^{\infty}\frac{2(-1)^i}{i}x^i$$

and

$$-2x\ln(1+x)=\sum_{i=1}^{\infty}\frac{2(-1)^i}{i}x^{i+1}=\sum_{i=2}^{\infty}\frac{2(-1)^{i+1}}{i-1}x^i,$$

so the right hand side of (2) has series representation

$$\begin{aligned}2x-2(1+x)\ln(1+x)&=\sum_{i=2}^{\infty}\left[\frac{2(-1)^i}{i}+\frac{2(-1)^{i+1}}{i-1}\right]x^i\\&=\sum_{i=2}^{\infty}\frac{2(-1)^{i+1}}{i(i-1)}x^i.\end{aligned} \tag{4}$$

Expanding the left hand side of (2) using (2) and the observation that

$$x\,[\ln(1+x)]^2=\sum_{i=0}^{\infty}a_ix^{i+1}=\sum_{i=1}^{\infty}a_{i-1}x^i$$

leads to the series representation

$$\begin{aligned}(1+x)\,[\ln(1+x)]^2&=a_0+\sum_{i=1}^{\infty}(a_i+a_{i-1})x^i\\ \int[\ln(1+x)]^2\,dx-(1+x)\,[\ln(1+x)]^2&=c+\left[\sum_{i=1}^{\infty}\frac{a_{i-1}x^i}{i}\right]\\&\quad-\left[a_0+\sum_{i=1}^{\infty}(a_i+a_{i-1})x^i\right].\end{aligned} \tag{5}$$

Substituting (4) and (5) into equation (2) gives

$$\begin{aligned}c+\left[\sum_{i=1}^{\infty}\frac{a_{i-1}x^i}{i}\right]-\left[a_0+\sum_{i=1}^{\infty}(a_i+a_{i-1})x^i\right]&=\sum_{i=2}^{\infty}\frac{2(-1)^{i+1}}{i(i-1)}x^i\\ c-a_0+\sum_{i=1}^{\infty}\left[\frac{a_{i-1}}{i}-a_i-a_{i-1}\right]x^i&=\sum_{i=2}^{\infty}\frac{2(-1)^{i+1}}{i(i-1)}x^i.\end{aligned}$$

Matching the constant terms we have $c-a_0=0$. Since $a_0=f(0)=(\ln 1)^2=0$, it follows that c must also be zero. Matching the coefficients of x^1 we have $a_0-a_1-a_0=0$ implying that a_1 is also zero.

More generally, matching of the coefficients of x^i $(i>1)$ yields the recurrence relation

$$\begin{aligned}\left(\frac{1}{i}-1\right)a_{i-1}-a_i&=\frac{2(-1)^{i+1}}{i(i-1)},\ \text{or}\\ a_i&=\frac{1-i}{i}a_{i-1}+\frac{2(-1)^i}{i(i-1)},\ \text{where } a_1=0.\end{aligned} \tag{6}$$

3. Relating the Terms in the Expansion of $(\ln 2)^2$ to the Harmonic Series. By setting $x=1$ in (1), we obtain

$$(\ln 2)^2=a_0+a_1+a_2+a_3+\cdots=0+0+a_2+a_3+\cdots,$$

where the terms in the expansion are obtained using (6). By squaring the Maclaurin series for $\ln(1+x)$, we have

$$\begin{aligned}[\ln(1+x)]^2&=(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots)^2\\&=(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots)(x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots)\\&=x^2-(\frac{1}{1\cdot2}+\frac{1}{2\cdot1})x^3+(\frac{1}{1\cdot3}+\frac{1}{2\cdot2}+\frac{1}{3\cdot1})x^4\\&\quad-(\frac{1}{1\cdot4}+\frac{1}{2\cdot3}+\frac{1}{3\cdot2}+\frac{1}{4\cdot1})x^5+\cdots\\&=b_1x^2+b_2x^3+b_3x^4+b_4x^5+\cdots.\end{aligned}$$

Substituting $x=1$ gives

$$(\ln 2)^2=b_1+b_2+b_3+b_4+\cdots.$$

The relationship between b_i and a_i is

$$b_i=a_{i+1},\quad i=1,2,\ldots.$$

Using (6) we obtain a recurrence relation for b_i

$$\begin{aligned}b_{i+1}&=\frac{1-(i+2)}{i+2}a_{i+1}+\frac{2(-1)^{i+2}}{(i+1)(i+2)}\\&=-\frac{i+1}{i+2}b_i+\frac{2(-1)^i}{(i+1)(i+2)}.\end{aligned}$$

Multiplying by $i+2$ and rearranging, we obtain

$$(i+2)b_{i+1}+(i+1)b_i=\frac{2(-1)^i}{i+1}. \tag{7}$$

Now let

$$S_i=\frac{(-1)^{i+1}}{2}(i+1)b_i, \tag{8}$$

so that $S_1=1$. By substituting (8) into (7), we obtain

$$\begin{aligned}2S_{i+1}(-1)^i+2S_i(-1)^{i+1}&=\frac{2(-1)^i}{i+1}\\ S_{i+1}-S_i&=\frac{1}{i+1}\\ S_i&=S_{i-1}+\frac{1}{i}.\end{aligned} \tag{9}$$

Knowing that $S_1 = 1$, combined with (9), we can conclude that S_i is the i^{th} partial sum of the harmonic series. Solving (8) for b_i in terms of S_i yields

$$b_i = \frac{2S_i(-1)^{i+1}}{i+1},$$

where $S_i = 1 + \frac{1}{2} + \cdots + \frac{1}{i}$ is the i^{th} partial sum of the harmonic series.

Therefore, the explicitly defined series for $(\ln 2)^2$ is

$$(\ln 2)^2 = \sum_{i=1}^{\infty} b_i = \sum_{i=1}^{\infty} \frac{2S_i(-1)^{i+1}}{i+1}.$$

Note that the latter implies that

$$\lim_{i\to\infty} \frac{S_i}{i+1} = 0,$$

a limit that the reader is challenged to prove by some other method.

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MAGIC STARS

MARIÀN TRENKLER*

You have probably already met with the following brain-twisters whose author is the English mathematician Henry Dudeney, [1].

Problem 1. Into the circlets of the star $\mathbb{S}_5$ in Figure 1 write ten different numbers from the set $\{1, 2, 3, \dots, 12\}$ in such a way that the four numbers on each line sum to twenty-four.

Problem 2. Into the circlets of the star $\mathbb{T}_8$ in Figure 1 write numbers $1, 2, \dots, 16$ so that the four numbers on each line sum to thirty.

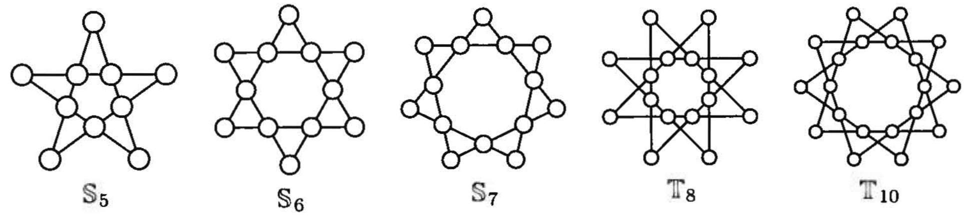

FIG. 1.

In this paper we will solve both problems and describe a generalization.

In Figure 1 five n-sided (n-vertex) stars of two types are depicted. The first three stars are denoted $\mathbb{S}_n$, $n = 5, 6, 7$, and the two others $\mathbb{T}_n$, $n = 8, 10$. Each star contains $2n$ circlets such that there are four on each line. Both types of stars arise from a regular n-gon by placing n circlets $V_1, V_2, \dots, V_n$ centered at the vertices of the n-gon. In the star $\mathbb{S}_n$, which is defined for $n \geq 5$, the circlets $U_1, U_2, \dots, U_n$ are situated in the intersection of lines $V_{i-2}V_{i-1}$, V_iV_{i+1}, for $i = 1, 2, 3, \dots, n$. (subscript being taken modulo n.) In the star $\mathbb{T}_n$, $n \geq 7$, the U_i's are situated at the intersection of lines $V_{i-2}V_{i-1}$, $V_{i+1}V_{i+2}$, for $i = 1, 2, 3, \dots, n$. See Figure 2.

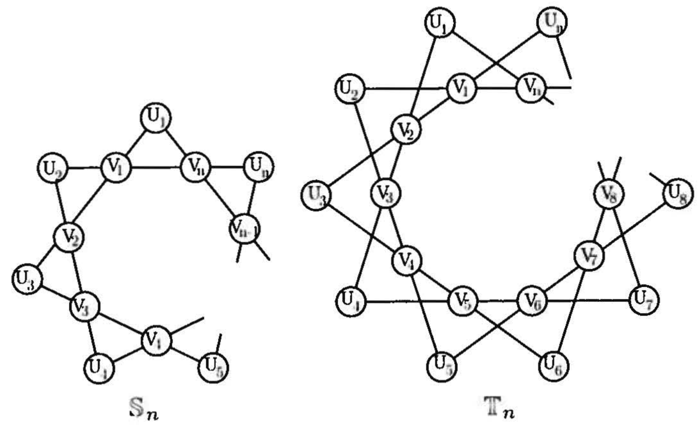

FIG. 2.

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An n-sided star $\mathbb{S}_n$ (or $\mathbb{T}_n$) is called a *magic star* and labelled $\mathbb{S}_n^M$ (or $\mathbb{T}_n^M$) if the numbers $1, 2, 3, \ldots 2n$ can be placed in its circlets so that the sums of numbers on each line are the same. These stars are sometimes referred in literature as *magic stars of David.* Observe that the numbers on each line sum to twice the sum of all numbers divided by n, that is $4n + 2$.

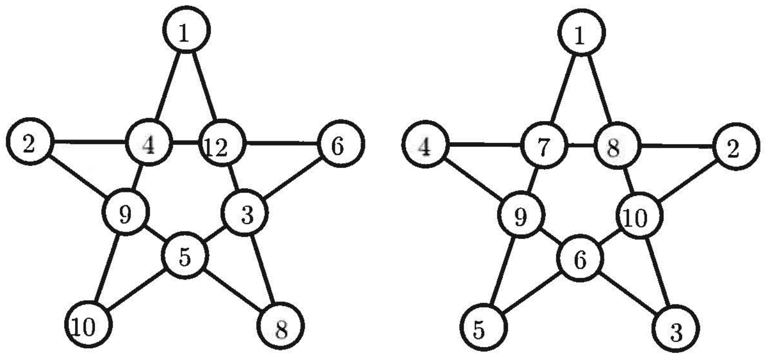
We call a star $\mathbb{S}_n$ (or $\mathbb{T}_n$) a *weakly-magic* star and denote it $\mathbb{S}_n^W$ (or $\mathbb{T}_n^W$), if distinct integers can be placed in its circlets so that the sum on each line is the same. Each magic star is also weakly-magic, however the reverse implication is not true.

If there are numbers $1, 2, \ldots 2n$ located in the circlets of a star $\mathbb{S}_n$ (or $\mathbb{T}_n$) so that the sum on $n-2$ of the lines is $4n+2$, and the numbers on the remaining two lines sum to $4n+1$ and $4n+3$ respectively, we call it an *almost magic star*, and we denote it $\mathbb{S}_n^A$ (or $\mathbb{T}_n^A$.)

Let's go back to Problem 1. Its solution is the content of the following theorem:

THEOREM 1. *The star $\mathbb{S}_5$ is weakly-magic and almost-magic, but it is not magic.*

Proof. In Figure 3 we see a weakly-magic star $\mathbb{S}_5^W$ and an almost-magic star $\mathbb{S}_5^A$. The sums on the lines containing the number 2 are 21 and 23 respectively.



$\mathbb{S}_5^W$ $\mathbb{S}_5^A$

FIG. 3.

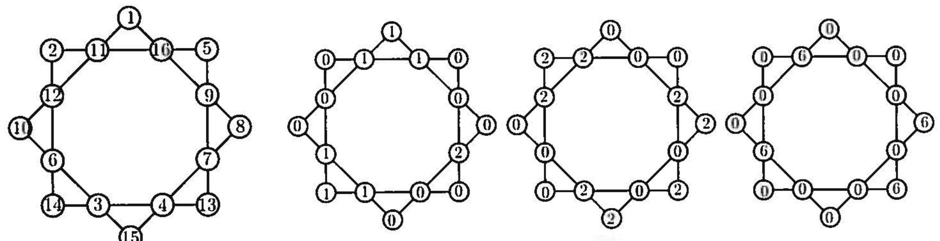
Suppose that a magic star $\mathbb{S}_5^M$ exists. On each line there are exactly four different numbers whose sum is 22. Each number is situated on exactly two lines. On individual lines there can be only the following quadruples of numbers:

$$\begin{array}{cccccc}
(10,9,2,1) & (10,8,3,1) & (10,7,4,1) & (10,7,3,2) & (10,6,5,1) & (10,6,4,2) \\
(10,5,4,3) & (9,8,4,1) & (9,8,3,2) & (9,7,5,1) & (9,7,4,2) & (9,6,5,2) \\
(9,6,4,3) & (8,7,6,1) & (8,7,5,2) & (8,7,4,3) & (8,6,5,3) & (7,6,5,4)
\end{array}$$

In the magic star $\mathbb{S}_5^M$ two lines must contain 10. Let's suppose that on one of these lines contains the quadruple (10,9,2,1). On the other one can be only the quadruple (10,5,4,3) as all the other quadruples containing 10 also contain 1 or 2. Similarly number 9 is also situated on two lines. Let's consider the second line containing 9. It can not contain numbers 2 and 1, therefore only the quadruple (9,6,4,3) comes into consideration, but it contains numbers 3 and 4 which are situated on the line containing number 10.

If one line contains (10,8,3,1) or (10,7,3,2) the other must contain (10,6,4,2) or (10,6,5,1), respectively. As before, we can argue that none of these options work. No other quadruples of numbers containing 10 and not containing any other common numbers do exist. □

We will now provide a solution to Problem 2. By a *basic n-sided magic star* we understand star $\mathbb{S}_n$ or $\mathbb{T}_n$, with numbers 0, k or 0, k, $2k$ written in its circlets so that the sum of the numbers on each line is the same.



$\mathbb{S}_8^M$ $\mathbb{E}_8$

FIG. 4.

Figure 4 depicts a magic star $\mathbb{S}_8^M$ and three basic 8-sided stars. We got this $\mathbb{S}_8^M$ by summing over the nine basic stars, whose valuation is shown in the rows of Table 1. The basic stars from Figure 4 are in the 3'rd and 5'th and 9'th row.

U_1	U_2	V_1	V_8	U_8	V_2	V_7	U_3	U_7	V_3	V_6	U_4	V_4	V_5	U_6	U_5
1	0	0	0	1	0	0	0	0	0	0	1	0	0	0	1
1	0	1	1	0	0	0	0	0	1	2	1	1	0	0	0
0	0	2	0	0	0	2	0	0	0	0	2	0	0	0	2
0	2	2	0	0	2	2	0	2	0	0	0	2	0	2	2
0	0	0	5	0	0	0	5	0	0	5	5	0	0	0	0
0	0	0	5	0	5	0	0	0	0	0	0	0	0	5	5
0	0	0	5	5	5	5	5	0	0	0	5	0	5	0	5
0	0	6	0	0	0	0	0	6	6	0	0	0	0	6	0
-1	0	0	0	-1	0	0	0	0	-1	0	0	0	-1	0	0
1	2	11	16	5	12	9	10	8	6	7	14	3	4	13	15

TABLE 1
Construction of $\mathbb{S}_n^M$

Figure 4 shows magic stars $\mathbb{S}_n^M$ for $n = 6, 7, 9$. Values for $\mathbb{S}_n^M$ for $n = 9, 10, 11$ are in Table 4. These were obtained by computer experimentation. The natural question for which n a magic star exists has been answered for some n, but as of yet, no general answer is known.

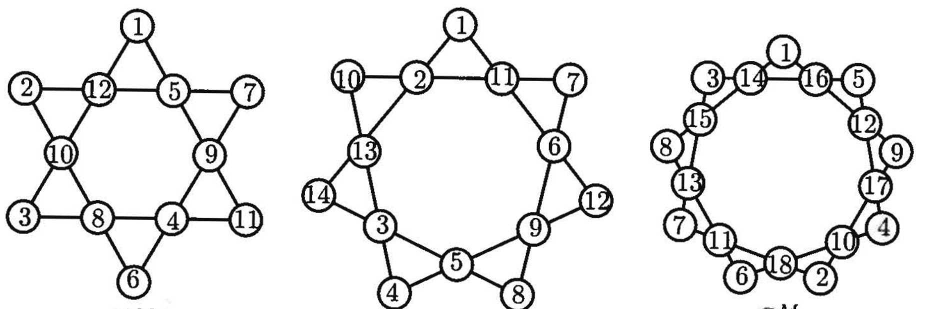
For stars $\mathbb{T}_n$, however, we do have an algorithm for producing $\mathbb{T}_n^M$ for every even integer $n \geq 8$.

THEOREM 2. *A magic star $\mathbb{T}_n$ exists for every even integer $n \geq 8$.*

Proof. We divide numbers $1, 2, \ldots, 2n$ into four rows as shown in Table 2.

1	2	3	4	$\ldots$	$n/2$
n	$n-1$	$n-2$	$n-3$	$\ldots$	$n/2+1$
$n+1$	$n+1$	$n+2$	$n+3$	$\ldots$	$3n/2$
$2n$	$2n-1$	$2n-2$	$2n-3$	$\ldots$	$3n/2+2$

TABLE 2



$\mathbb{S}_6^M$ $\mathbb{S}_7^M$ $\mathbb{S}_9^M$

FIG. 5.

The following relations are true:

$$i+(n-i+1)=n+1,\ (n+i)+(2n-i+1)=3n+1,\ i=1,2,\ldots,\frac{n}{2},$$

$$(i+1)+(n-i+1)=n+2,\qquad (n+i)+(2n-i)=3n,\qquad i=1,2,\ldots,\frac{n-2}{2},$$

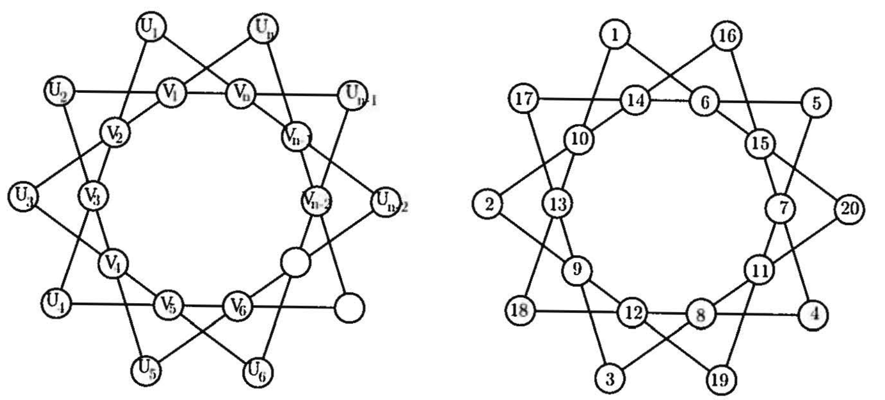


FIG. 6. *Stars* $\mathbb{T}_n$ *and* $\mathbb{T}_{10}^M$

Into the circlet corresponding to the entry in the i-th row and j-th column of Table 3 we inscribe the number from i-th row and j-th column of Table 2.

U_1	U_3	U_5	U_7	$\ldots$	U_{n-3}	U_{n-1}
V_2	V_4	V_6	V_8	$\ldots$	V_{n-2}	V_n
V_{n-3}	V_{n-5}	V_{n-7}	V_{n-9}	$\ldots$	V_1	V_{n-1}
U_{n-2}	U_{n-4}	U_{n-6}	U_{n-8}	$\ldots$	U_2	U_n

TABLE 3

On each line, except of $V_{n-1}V_n$ of the star $\mathbb{T}_n$ there are two pairs of circlets whose sums are $[(n+1)+(3n+1)]$ or $[(n+2)+3n]$. On the line $V_{n-1}V_n$ are numbers $(n/2+1), 1, 3n/2, 2n$, whose sum is $4n+2$. □

For weakly magic stars we also have a constructive existence theorem. Let U and V be two circlets of a star $\mathbb{S}_n$ (or $\mathbb{T}_n$) and let $\mathbb{E}_n$ be a basic star. We say that $\mathbb{E}_n$ *separates the circlets* U *and* V if it assigns to them different numbers.

THEOREM 3. *A weakly-magic star* $\mathbb{S}_n^W$ *(or* $\mathbb{T}_n^W$*) exists for every integer* $n \geq 7$.

Proof. We inscribe number 1 into each circlet of a star. We get a weakly-magic valuation of the star by multiple use of the following construction:

If the numbers in the circlets U and V are the same we choose a basic star $\mathbb{E}_n$ which separates them. By adding the valuation $\mathbb{E}_n$ to the original valuation we will get a new valuation in which different numbers will be assigned to circlets U and V. If we choose for k a number which is bigger than all the numbers in the previous valuation every two circlets which have different numbers will have different numbers also in the new valuation.

We repeat this construction as long as two different circlets with the same values exist. After a finite number of repetitions we will get a weakly-magic star. □

Note. The above shown construction leads to relatively big numbers in stars $\mathbb{S}_n^W$ and $\mathbb{T}_n^W$. If we choose basic stars and values of k appropriately we will get a magic valuation (if it exists), or as the case may be a weakly-magic star with small numbers.

	U_1	V_1	U_2	V_2	U_3	V_3	U_4	V_4	U_5	V_5	U_6	V_6	U_7	V_7	U_8	V_8	U_9	V_9	U_{10}	V_{10}	U_{11}	V_{11}
$\mathbb{S}_6^M$	1	9	5	12	4	3	6	11	8	7	2	10										
$\mathbb{S}_6^M$	29	31	53	71	73	43	37	41	47	67	59	61										
$\mathbb{S}_9^M$	1	14	3	15	8	13	7	11	6	18	2	10	4	17	9	12	5	16				
$\mathbb{S}_{10}^M$	1	3	7	20	18	4	11	14	6	8	9	16	12	15	2	10	5	17	13	19		
$\mathbb{S}_{11}^M$	1	19	7	16	10	18	5	15	3	17	9	22	4	13	2	21	8	12	11	20	6	14
$\mathbb{T}_7^A$	1	8	2	14	3	11	4	10	7	12	6	9	5	13								
$\mathbb{T}_8^M$	1	11	14	8	2	10	15	7	3	9	16	6	4	12	13	5						

TABLE 4
Some stars

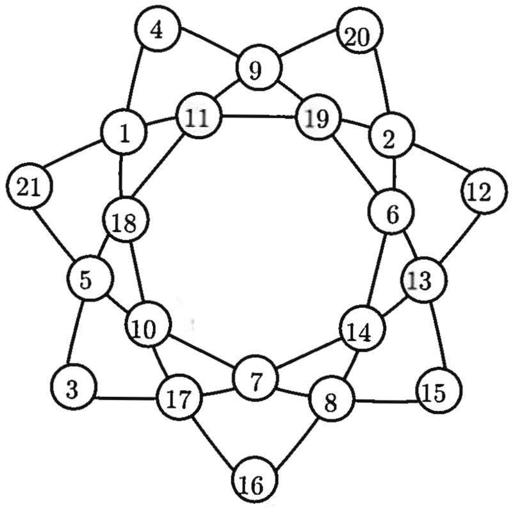
Some interesting stars are contained in Table 4. In the first row a weakly-magic star $\mathbb{S}_6^M$ for which the sum of numbers in circlets $U_1, U_2, \ldots, U_6$ is the same as on individual lines. In the second row is the star S_6^W containing only prime numbers. (Such a star is in literature often referred as *prime-number magic star of David*). In the nearly-magic star $\mathbb{T}_7^A$ (6th row) the lines containing number 6 have sums different from the others.

We may ask similar questions about magic and weakly-magic valuations on different kinds of stars, for example the stars which we denote by $\mathbb{S}_7^T$ and $\mathbb{S}_9^T$, see Figure 7. In the first we placed the numbers $1, 2, \ldots, 21$ so that the sum of all six numbers on every line is sixty-six. We now pose a problem for you to solve.

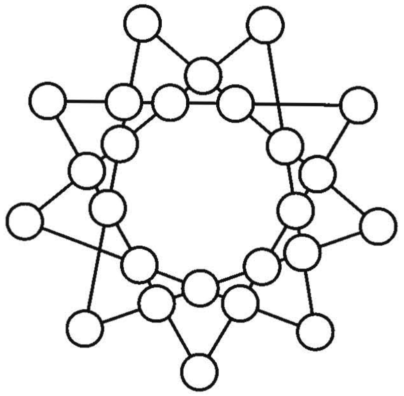
Problem 3. Into the circlets of the star $\mathbb{S}_9^T$ (Fig. 7) inscribe 27 different natural numbers so that the sum of the 6 numbers on each of the lines is the same.

If you succeed in finding a magic star $\mathbb{S}_n^M$ for $n \geq 12$ or for $\mathbb{T}_n^M$ for odd $n \geq 7$, please, send me an information about it.

Finally, Figure 8 depicts a special magic star, whose properties might inspire you to formulate and investigate a wealth of similar problems.



$\mathbb{ST}_7^M$



$\mathbb{ST}_9$

FIG. 7.

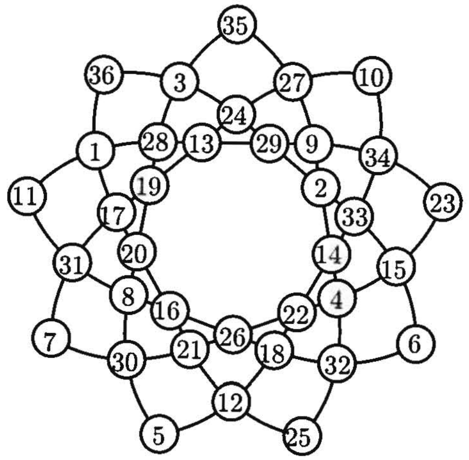


FIG. 8.

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[1] H. E. DUDENEY, "536 Puzzles and Curious Problems," Scribner's 1967.

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Mariàn Trenkler teaches at Safarik University in Kosice, Slovakia. His areas of research are Convex Polytopes, Graph Theory, and Magic Squares and Cubes.

The 2003 National Pi Mu Epsilon Meeting

The Annual Meeting of the Pi Mu Epsilon National Honorary Mathematics Society was held in Boulder Colorado from July 30 through August 1, 2003. The meeting was held in conjunction with the national meeting of the Mathematical Association of America's Student Sections.

The J. Sutherland Frame Lecturer was **Robert Devaney** from Boston University who spoke on "Chaos Games and Fractal Images"

Student Presentations. The following student papers were presented at the meeting. An asterisk(*) after the name of the presenter indicates that the speaker received a best paper award.

Andrew Anschutz Ashland University - Ohio Rho
Partitions of $\{1, 2, \ldots, n\}$ *into subsets of equal cardinalities and equal sums*

Sarah Bendall Randolph-Macon College - Virginia Iota
Centers of n*-fold tensor products of graphs*

Vincent Berardi St. Peter's College - New Jersey Epsilon
Chaos

Angela Brown Sam Houston State University - Texas Epsilon
An Overview Of Molecular And Crystallographic Symmetries

Paul Carmany Ashland University - Ohio Rho
A survey of $(8, 8, 8, 1)$ *relative difference sets*

Adam Christman St. Norbert College - Wisconsin Delta
Does Your Vote Count?

Nicholas Ciotola Benedictine University - Illinois Iota
Space Group Investigations With Magma

Christopher Coffin Mount Union College - Ohio Omicron
Math, springs, and squishy things

David Gohlke Youngstown State University - Ohio Xi
Making use of geiger counters

Gina M. Grisola Mount Union College - Ohio Omicron
There Are No Subgroups Of A_4 *Of Order Six*

Sarah Grove Youngstown State University - Ohio Xi
How To Parallel Park Using Multiple Processors

Brian Hahn St. Norbert College - Wisconsin Delta
Honey, Where Should We Sit?

Tristan Hauser St. Michael's College - Vermont Alpha
Relative Consistency Of Geometries

Hai He Hunter College - New York Beta
Infinite products

Peter Horn Hendrix College - Arkansas Beta
Bipolar plots

James Jessup Seton Hall University - New Jersey Delta
The Isoperimetric Inequality

Chris Jones* Youngstown State University - Ohio Xi
Mathematical Freedom

Rebecca Jungman South Dakota State University - South Dakota Gamma
How Does It Do That?!

Emily King* Texas A & M University - Texas Eta
A Matricial Algorithm For Polynomial Refinement

Marta Kobiela* Texas A & M University - Texas Eta
Knots In The Cubic Lattice

Joel Lepak Youngstown State University - Ohio Xi
Difference sets without squares

Jennifer Novak* Texas A & M University - Texas Eta
"Nice surface!" and other ways to complement your knot

Luke Oeding, Franklin and Marshall College - Pennsylvania Eta
Why are there only five regular polyhedra?

Derek Pope* Seton Hall University - New Jersey Delta
Bifurcations Of The Henon Map

Sara M. Rogala Elmhurst College - Illinois Iota
Types Of Block Products

Brenda Russo* Salisbury University - Maryland Zeta
Algebraic Structures And The Long-Term Behavior Of Discrete Dynamical Systems

Maria Salcedo* Youngstown State University - Ohio Xi
Crystallographic Fractal Tilings

Barbara Sexton* Sam Houston State University - Texas Epsilon
Means of complex numbers

Steve Stanislav Youngstown State University - Ohio Xi
Pick'S Theorem

Holly Thomson St. Norbert College - Wisconsin
Fun With Triangles

Jennifer Webb Hood College - Maryland Delta
Gerbert: The Mathematician Who Sold His Soul To The Devil

Delilah Whittington Millsaps College - Mississippi Delta
Thinking Through The Fourth Dimension

Adam Winchester University of Nevada at Las Vegas - Nevada Beta
Busemann points of infinite graphs

Mathfest 2004

Call For Papers

The next ΠME meeting will take place at Mathfest 2004 in Providence, Rhode Island, August 12 – August 14. See the ΠME webpage (http://www.pme-math.org/) for application deadlines and forms. There will be mathematics talks and social events and don't forget, the ΠME banquet. See also the MAA webpage for details as well as for other activities in the Ocean State.

2003 2003

The Richard V. Andree Awards

First Prize
Patrick Hummel,
"Solid constructions using ellipses",
Pi Mu Epsilon Journal, Vol. 11, No. 8, Spring 2003.

Second Prize
John M. Zobitz,
"Pascal matrices and particular solutions to differential equations",
Pi Mu Epsilon Journal, Vol. 11, No. 8, Spring 2003.

Third Prize
Stephanie Libera and Paul Tlucek,
"Some Perfect Order Subset Groups",
Pi Mu Epsilon Journal, Vol. 11, No. 9, Fall 2003.

2003 2003

The Richard V. Andree Awards. The Richard V. Andree Awards are given annually to the authors of the papers, written by undergraduate students, that have been judged by the officers and councilors of Pi Mu Epsilon to be the best that have appeared in the Pi Mu Epsilon Journal in the past year.

The officers and councilors of the Society congratulate the 2003 winners on their achievements and wish them well for their futures.

From the Right Side

Try to play the few lines of music you see below and you might not think much of it. In fact, if you ask any listener for the composer, nobody would guess that it is Mozart. Copy the page, cut off the text surrounding the music, and place it on your coffee table. It is no longer clear which is the top of the page. You should try both ways and, indeed, as you expected, you find that you can play two different pieces of music of equal length – so why not get a friend and play them both simultaneously.

What an experience!

Mozart wrote several of these table canons, as he called them, with the intention that they be played by two people sitting at opposite sides of a table looking at the same sheet of music – in the absence of copiers or printers a particularly valuable idea. It is a mathematical idea involving several symmetries. A 180° rotation of the staff transforms a high note into a low note. The piece itself is not quite symmetric, neither in rhythm nor melody, but if played by two players the resulting sound is independent of the orientation of the sheet-music on the table.

Music notation, because of its symmetry, is well suited for such kinds of games, much better than our 26 letter alphabet, which contains only a few letters which yield other letters after a 180° rotation. The invention of an alphabet with the quality that every letter also has an upside down meaning could revolutionize literature and maybe enable us to interpret a bill as a love letter.

Check out http://sunsite.univie.ac.at/Mozart/dice/ for more of Mozart's mathematical ideas.

The ΠME Journal invites those of you who paint, draw, compose, or otherwise use the other side of your brains to submit your mathematically inspired compositions.

ΠME Journal, Vol. 11, No. 10, pp 559–571, 2004.

PROBLEM DEPARTMENT

EDITED BY MICHAEL MCCONNELL, AND JON A. BEAL

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All correspondence should be addressed to Jon Beal, 840 Wood Street, Mathematics Department, Clarion University, Clarion, PA 16214, or sent by email to jbeal@clarion.edu. Electronic submissions using LaTeX are encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by December 1, 2004. Solutions identified as by students are given preference.

The editors apologize to Paul S. Bruckman and Kenneth B. Davenport for failing to acknowledge their problem solutions for problems from Fall 2002 in the Fall 2003 issue.

Problems for Solution.

1071. *Proposed by Peter A. Lindstrom, Batavia, NY.*
In a beginning calculus course, one finds two Mean Value Theorems:

MVT for the Derivative: If a function $f \in C[a,b]$ and $f \in D(a,b)$, then there exists at least one point $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

MVT for the Definite Integral: If a function $f \in C[a,b]$, then there exists at least one point $d \in (a,b)$ such that $\int_a^b f(x)\,dx = f(d)(b-a)$.

Does there exist a non-constant function f such that $c = d$ in these two theorems for any given a and b, where $a < b$?

1072. *Proposed by Andrew Cusumano, Great Neck, NY.*

Let P, N be nonnegative integers. If $(P+N+3)$ is prime then prove that

$$[P! + (P+1)!][N! + (N+1)!] + (-1)^{P+1}$$

is divisible by $(P+N+3)$.

1073. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

Let n be a positive integer greater than or equal to 2. Prove that

$$1 + \sum_{k=1}^{n} \left(\frac{F_n F_{n+1} - F_k^2}{n-1} \right)^{1/2} \geq F_{n+2}$$

where F_n is the n^{th} Fibonacci number. That is, $F_0 = 0$, $F_1 = 1$ and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.

1074. *Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College, Abingtion PA.*

Use the limit,

$$\lim_{n\to 0} \frac{(1+x)^n - 1}{n}$$

to show that

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots, \qquad -1 < x \le 1$$

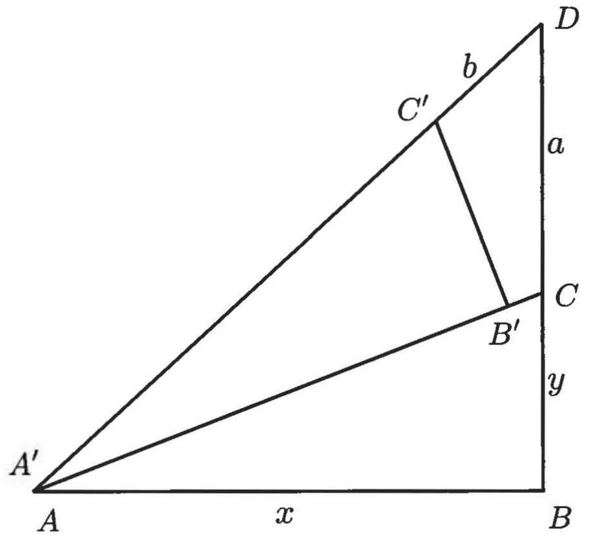
1075. *Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI.*

Let a and b be two positive real numbers. Evaluate

$$\int_{-\infty}^{\infty} \frac{\ln(1+e^{ax})}{1+e^{bx}}\, dx$$

1076. *Proposed by Peter A. Lindstrom, Batavia, NY.*

Consider the right triangle ABC, where $|AB| = x$, $|BC| = y$ so that $|AC| = \sqrt{x^2+y^2}$. Consider another right triangle $A'B'C'$ which is congruent to ABC, where $A = A'$ and B' is a point on AC. Extend BC and $A'C'$ so that they intersect at D. Let $|CD| = a$ and $|C'D| = b$. Express a and b in terms of x and y.



1077. *Proposed by Robert C. Gebhardt, Hopatcong, NJ. and the Editors*

Robert Gebhardt made the following proposal:

Consider a prime number p. Add its digits to get another number. If this sum has more than one digit, add them to get another number, and so on until you get a single digit (call it $R(p)$, the digital root.) For example, if $p = 199$ then $1+9+9 = 19$, $1+9 = 10$, and $1+0 = 1$, so $R(199) = 1$.

Obviously, $R(p)$ cannot equal 0. Is there a prime number p for which $R(p) = 6$ or 9? (Note: All other possible digital sums do occur.)

We have generalized the problem:

Consider a whole number $n > 2$. If x is any number, let $R_n(x)$ be the digital root of x in base n. (All numerical representations are in base n.) An example from base five: if $x = 1123_{\text{five}}$ then $1+1+2+3 = 12_{\text{five}}$, and $1+2 = 3_{\text{five}}$, so $R_5(1123_{\text{five}}) = 3$.

Suppose p is a prime number. Is it possible for $\gcd(R_n(p), n-1)$ to be larger than 1?

1078. *Proposed by Robert C. Gebhardt, Hopatcong, NJ.*

Student solutions especially solicited

Let a and b be positive real numbers. Consider the Fibonacci-type sequence $S_1, S_2, \dots, S_n, \dots$ where $S_{n+2} = aS_n + bS_{n+1}$ for $n \ge 1$. For any real numbers S_1 and S_2, both not zero, find

$$\lim_{n\to\infty} \frac{S_n}{S_{n+k}}$$

for $k = 1, 2, 3, \dots$

Solutions. 1052[Spring 2004] *Proposed by Peter A. Lindstrom, Batavia, NY.*
The previous Problem Editor, Clayton DODGE, was a GREAT EDITOR. Solve the following addition alphametic in base ten:

$$\begin{array}{r} DODGE \\ \underline{GREAT} \\ EDITOR \end{array}$$

Solution by **Jacob McMillen**, *undergraduate, SUNY at Fredonia.*

Clearly, E=1. Then from the 10^4 column we conclude that G=9 since D+G+(carry)=10+D. Now from the 10^0 and 10^2 columns we have that R=T+1=D+3 and from the 10^1 column we have A=O+1. So either R=8,T=7, and D=5, or R=7,T=6, and D=4, since we must have either A=3 and O=2 or A=4 and O=3. Suppose we have R=7,T=6, and D=4. This makes A=3 and O=2. So we would have:

$$\begin{array}{r} 42491 \\ \underline{97136} \\ 14I627 \end{array}$$

This would give us I=9, but we already have G=9. So we must have R=8,T=7, and D=5. Now suppose that A=4 and O=3. We would then have:

$$\begin{array}{r} 53591 \\ \underline{98147} \\ 15I738 \end{array}$$

This would give us I=1, but we already have E=1. Hence A=3 and O=2. After substituting in, we get 2+8=I-10 in the 10^3 column, so I=0. So we have

I=0, E=1, O=2, A=3, D=5, T=7, R=8, G=9

and the sum becomes:

$$\begin{array}{r} 52591 \\ \underline{98137} \\ 150728 \end{array}$$

Also solved by **Scott H. Brown**, Auburn Univeristy Montgomery, **Paul S. Bruckman**, Sointula, BC, **Pat Costello**, Eastern Kentucky University, Richmond, KY, **Dan Dence**, sophomore, Huron High School, Huron, OH, **Mark D. Evans**, Louisville, KY, **Robert C. Gebhardt**, Hopatcong, NJ, **Steve Gendler**, Clarion University, Clarion, PA, **Richard Hess**, Rancho Palos Verdes CA, **S.C. Locke**, Florida Atlantic University, Boca Raton, FL, **David E. Manes**, SUNY College at Oneonta, Oneonta, NY, **James Meadows**, student, Angelo State University, San Angelo, TX, **Yoshinobu Murayoshi**, Okinawa, Japan, **William H. Peirce**, Rangeley, ME, **J. Suck**, Essen, Germany, and by the **Proposer**

1053[Spring 2004] *Proposed by Robert C. Gebhardt, Hopatcong, NJ.*

Find exactly

$$\int_0^\infty \frac{u}{e^u+1}\,du \qquad \text{and} \qquad \int_0^\infty \frac{u^3}{e^u+1}\,du.$$

This series converges (CRC Tables) to

$$\frac{\pi^{2n}(2^{2n-1}-1)}{(2n)!}|B_{2n}|$$

where the $B_n's$ are the Bernoulli numbers

$$B_1 = \frac{-1}{2}, B_2 = \frac{1}{6}, B_4 = \frac{-1}{30}, \ldots$$

and so we obtain

$$\int_0^\infty \frac{u^{2n-1}}{e^u+1}\,du = \frac{\pi^{2n}(2^{2n-1}-1)}{(2n)}|B_{2n}|$$

Also solved by **Paul S. Bruckman**, Sointula, BC, **Charles Diminnie**, Angelo State University, San Angelo, TX, **Steve Gendler**, Clarion University, Clarion, PA, **Richard Hess**, Rancho Palos Verdes CA, **Joe Howard**, Portales, NM, **S.C. Locke**, Florida Atlantic University, Boca Raton, FL, **Gus Mavrigian**, Youngstown, OH, **Yoshinobu Murayoshi**, Okinawa, Japan, **J. Ernest Wilkins, Jr.**, Clark Atlanta University, Atlanta, GA, **Roger Zarnowski**, Angelo State University, San Angelo, TX, and by the **Proposer**

1054 [Spring 2004] *Proposed by Ronald Kopas, Clarion, PA* we see that the critical points for $f(x)$ in $[0, 2\pi]$ are $x = a$ and $x = 2\pi - a$. Also, since

$$f''(x) = -\sin(x)$$

and $a \in (\pi, \frac{3\pi}{2})$, we get

$$f''(a) > 0$$

and

$$f''(2\pi - a) < 0,$$

so that $f(x)$ has a relative minimum at $x = a$ and a relative maximum at $x = 2\pi - a$. Further, since $\tan(a) = a$,

$$f(a) = \sin(a) - \cos(a)a = 0$$

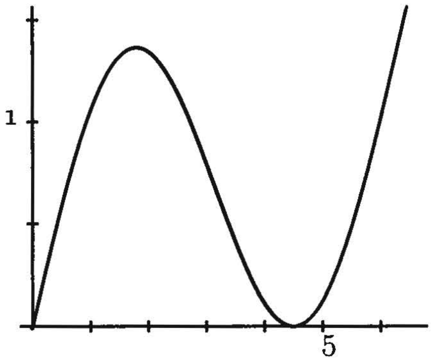
and

$$\begin{aligned} f(2\pi - a) &= \sin(2\pi - a) - (\cos(a))(2\pi - a) \\ &= -\sin(a) + a\cos(a) - 2\pi\cos(a) \\ &= -2\pi\cos(a). \end{aligned}$$

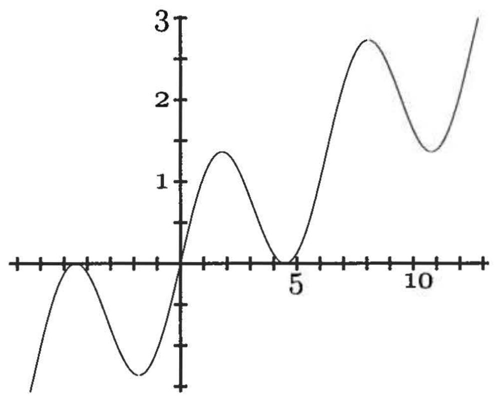
Noting that $\cos(a) < 0$ and

$$f(2\pi) = f(2\pi - a) = -2\pi\cos(a),$$

we obtain the following graph for $f(x)$ on $[0, 2\pi]$.



By (1), we conclude that on $[2\pi, 4\pi]$, $f(x)$ has a minimum value $-2\pi\cos(a)$ at $x = a + 2\pi$. Also, on $[-2\pi, 0]$, $f(x)$ has a maximum value 0 at $x = -a$. Hence, our graph may be extended as shown below.



Now, it is clear that if $0 \le k \le -2\pi\cos(a)$, then there are exactly three solutions for the equation

$$f(x) = k.$$

By (1), this is easily extended to all other values of k.

Also solved by **Paul S. Bruckman**, Sointula, BC, **Richard Hess**, Rancho Palos Verdes CA, **S.C. Locke**, Florida Atlantic University, Boca Raton, FL, and by the **Proposer**

1056[Spring 2004] *Proposed by William Chau, SoftTechies Corp., East Brunswick, New Jersey*

Given a positive integer n. Take the sum of its digits to obtain a different number, then take the sum of the digits of the new number to obtain yet another number, and so on until the remaining number has only one digit. We call the one digit number the digital root of n. Taking the digital roots of the first five even perfect numbers 6, 28, 496, 8128, and 33550336, we found that they are 6, 1, 1, 1, and 1, respectively. Is it true that all even perfect numbers except 6 have digital root 1?

This problem brought in solutions using a variety of approaches. We would like to thank David E. Manes of SUNY Oneonta for noting that the result was shown by M. Brooke: "On the digital roots of perfect numbers", *Mathematics Magazine* 34, (1960/61), pp. 100, 124.

Solution by **Michael J. Coons**, *student, University of Montana, Missoula, MT*

We note that all even perfect numbers X have the form $X = 2^{p-1}(2^p - 1)$, where p and 2^{p-1} are both prime. We also note that the first perfect number, $X = 6$, used the prime $p = 2$. We consider the case $X > 6$, thus we know p is an odd prime. Since p is odd, $p - 1$ is even and

$$2^{p-1} \equiv 1 \pmod 3,$$

so we let

$$2^{p-1} = 3n + 1$$

for some $n \in \mathbb{Z}$. Multiplying the above equation by 2, then subtracting 1, we yield

$$2^p - 1 = 6n + 1,$$

and

$$2^{p-1}(2^p - 1) = (3n + 1)(6n + 1) = 18n^2 + 9n + 1 \equiv 1 \pmod 9,$$

thus

$$2^{p-1}(2^p - 1) - 1 \equiv 0 \pmod 9.$$

Since $2^{p-1}(2^p - 1) - 1 \equiv 0 \pmod 9$, we know that by the divisibility property of 9, the digital root of $2^{p-1}(2^p - 1) - 1$ is 9. Adding 1 to $2^{p-1}(2^p - 1) - 1$, we see that the digital sum of $2^{p-1}(2^p - 1)$ is 10, which yields the digital root of 1. Therefore it is true that all even perfect numbers except 6 have digital root 1.

Also solved by **Paul S. Bruckman**, Sointula, BC, **Pat Costello**, Eastern Kentucky University, Richmond, KY, **Charles R. Diminnie**, Angelo State University, San Angelo, TX, **Richard I. Hess**, Rancho Palos Verdes, CA, **Kathleen E. Lewis**, SUNY Oswego, Oswego, NY, **Peter A.**

Lindstrom, Batavia, NY, **S.C. Locke** Flordia Atlantic University, Boca Raton, FL, **Yoshimobu Murayoshi**, Okinawa, Japan, **James A. Sellers**, Pennsylvania State University, University Park, PA, **J. Ernest Wilkins, Jr.**, Clark Atlanta University, Atlanta, GA and by the **Proposer**.

1057[Spring 2004] *Proposed by Mark Snavely, Mathematics Department, Carthage College, Kenosha, WI*

Many books include exercises similar to the following example.

Prove using induction that 4 divides $5^n - 1$ for all $n \in \mathbb{N}$.

For natural numbers p, q and r, show that r divides $p^n - q$ for all $n \in \mathbb{N}$ if and only if r divides $q^n - p$ for all $n \in \mathbb{N}$. [Hint: As a first step, characterize all natural numbers p, q and r such that r divides $p^n - q$ for all $n \in \mathbb{N}$.]

Solution by **David E. Manes**, *SUNY at Oneonta.*

Assume that r divides $p^n - q$ for all $n \in \mathbb{N}$. Then $p^n \equiv q \pmod r$ for all n in $\mathbb{N}$. For $n = 1$, we have $p \equiv q \pmod r$ so that $p^n \equiv q^n \pmod r$ for all $n \in \mathbb{N}$. Therefore, if $n \in \mathbb{N}$, then

$$q^n - p \equiv p^n - p \equiv p^n - q \equiv 0 \pmod r$$

so that r divides $q^n - p$ for all $n \in \mathbb{N}$. The converse implication is clear.

More can be said from this problem:

Solution by **Andre Pruett**, *student, Millsaps College, Jackson, MS.*

There remains a further (and worthy) question of which p and q are "fixed" by particular values of r.

By substitution, we see that the problem of characterizing those p and q such that $p^n - q \equiv 0 \pmod r$ reduces to that of finding those p such that $p^n - p \equiv 0 \pmod r$.

PROPOSITION: 1. *Let $p \in \mathbb{N}$ with $d = \gcd(p, r)$. Then $p^n \equiv p \pmod r$ if and only if $p \equiv 1 \pmod{\frac{r}{d}}$.*

Proof. Since $d|p$ and $d|r$, write $p = dp'$ and $r = dr'$. [Note that an easy induction shows that $p^n \equiv p \pmod r$ for all $n \in \mathbb{N}$ if and only if it is true for $n = 2$.]

For the forward direction, we have

$$(dp')^2 \equiv (dp') \pmod{dr'}$$
$$dp'^2 \equiv p' \pmod{r'}$$
$$p'(d \cdot p' - 1) \equiv 0 \pmod{r'}$$

As $\gcd(p', r') = 1$, no power of p' is either zero or a zero divisor in $\mathbb{Z}_{r'}$, so

$$d \cdot p' \equiv 1 \pmod{r'}$$

But $dp' = p$, so $p \equiv 1 \pmod{\frac{r}{d}}$.

In order to demonstrate the necessity, suppose $p \equiv 1 \pmod{r'}$. Since $d|p$, $p \equiv 0 \pmod d$. Also, $p \equiv 1 \pmod{r'}$ implies $\gcd(p, r') = 1$. Since $d|p$, we also have $\gcd(d, r') = 1$, hence d divides r completely. Then the Chinese Remainder Theorem tells us that p is the unique solution, modulo r, to the equations $x \equiv 0 \pmod d$ and $x \equiv 1 \pmod{r'}$. However, for any $n \in \mathbb{N}$, p^n is also a solution, so $p^n \equiv p \pmod r$. □

Note: Several correspondents sent in a solution to a slightly different problem. They assumed that the natural numbers contained all non-negative integers, rather than all positive integers.

Also solved by **Paul S. Bruckman**, Sointula, BC, **Mark Evans**, Louisville, KY, **Steve Gendler**, Clarion University, Clarion, PA, **Gary R. Greenfield**, Univeristy of Richmond, Richmond VA, **Joe Howard**, Portales, NM, **Kathleen E. Lewis**, SUNY Oswego, Oswego, NY, **S.C. Locke**, Florida Atlantic University, Boca Raton, FL and by the **Proposer**.

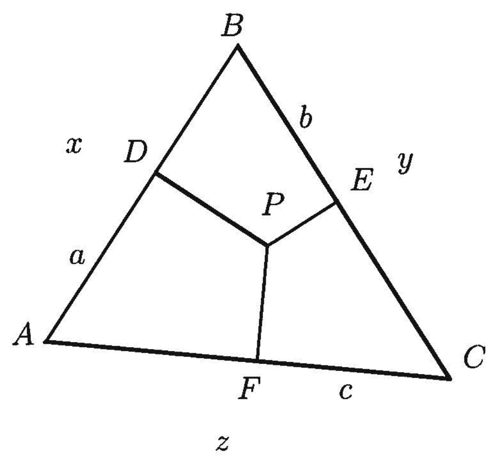
1058[Spring 2004] *Proposed by Peter A. Lindstrom, Batavia, NY*

Suppose that $\triangle ABC$ has an interior point P. Let D, E, and F be points on sides AB, BC, and CA, respectively, so that $PD \perp AB$, $PE \perp BC$, and $PF \perp CA$. Let $|AB| = x$, $|BC| = y$, $|CA| = z$, $|AD| = a$, $|BE| = b$ and $|CF| = c$.

1. Show that $(x - a)^2 + (y - b)^2 + (z - c)^2 = a^2 + b^2 + c^2$.
2. Show that if $\triangle ABC$ is an equilateral triangle, then $a + b + c = \frac{1}{2}$(perimeter of $\triangle ABC$).

Solution by **James Meadows**, *student, Angelo State University, TX.*

By drawing the triangle and its components as described, six right triangles can be formed by connecting P with the vertices A, B, and C (see the illustration).



From these triangles, the following conclusions can be made using the Pythagorean Theorem:

$$a^2 = PA^2 - PD^2 \qquad (x - a)^2 = PB^2 - PD^2$$

$$b^2 = PB^2 - PE^2 \qquad (y - b)^2 = PC^2 - PE^2$$

$$c^2 = PC^2 - PF^2 \qquad (z - c)^2 = PA^2 - PF^2.$$

Therefore

$$a^2 + b^2 + c^2 = PA^2 - PD^2 + PB^2 - PE^2 + PC^2 - PF^2$$
$$(x - a)^2 + (y - b)^2 + (z - c)^2 = PB^2 - PD^2 + PC^2 - PE^2 + PA^2 - PF^2.$$

From this, it can be concluded that

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = a^2 + b^2 + c^2.$$

Now, if $\triangle ABC$ is equilateral, then each side has the same length, x. Then the perimeter is

$$P = 3x.$$

Since $a^2 + b^2 + c^2 = (x-a)^2 + (y-b)^2 + (z-c)^2$, and $x = y = z$, then

$$a^2 + b^2 + c^2 = 3x^2 - 2ax - 2bx - 2cx + (a^2 + b^2 + c^2),$$

which reduces to

$$0 = x(3x - 2a - 2b - 2c).$$

Since $x \neq 0$,

$$3x - 2a - 2b - 2c = 0$$

and

$$a + b + c = \frac{1}{2}(3x)$$

Therefore $a + b + c$ is $\frac{1}{2}$ the perimeter of $\triangle ABC$.

Also solved by **Paul S. Bruckman**, Sointula, BC, **Mark Evans**, Louisville, KY, **Richard I. Hess**, Rancho Palos Verdes, CA, **Tracey M. Hagedorn**, student, Angelo State University, San Angelo, TX, **Joe Howard**, Portales, NM, **S.C. Locke** Flordia Atlantic University, Boca Raton, FL, **David E. Manes**, SUNY College at Oneonta, Oneonta, NY, **Gus Mavrigian**, Youngstown, OH, **Yoshimobu Murayoshi**, Okinawa, Japan, **William J. Peirce**, Rangeley, ME, and by the **Proposer**.

1059[Spring 2004] *Proposed by Peter A. Lindstrom, Batavia, NY*

Every even perfect number is of the form $2^{p-1}(2^p - 1)$ where both p and $2^p - 1$ are primes. If $X = 2^{p-1}(2^p - 1)$ is a perfect number, show that

$$\prod_{d|X} d = X^p.$$

Solution by **Tracey M. Hagedorn**, *student, Angelo State University, San Angelo, TX.*

Let $q = 2^p - 1$ and $n = p - 1$, so $X = 2^n p$. Finding the possible divisors gives

$$\begin{aligned}\prod_{d|X} d &= 1 \cdot 2 \cdot 2^2 \cdots 2^n \cdot q \cdot (2q) \cdot (2^2 q) \cdots (2^n q)\\ &= 2^{1+2+\ldots n} q^{n+1} 2^{1+2+\ldots n}.\end{aligned}$$

Using the formula for the sum of the first n integers,

$$\prod_{d|X} d = 2^{\frac{n(n+1)}{2}} q^{n+1} 2^{\frac{n(n+1)}{2}} = 2^{n(n+1)} q^{n+1} = (2^n q)^{n+1} = X^{n+1}.$$

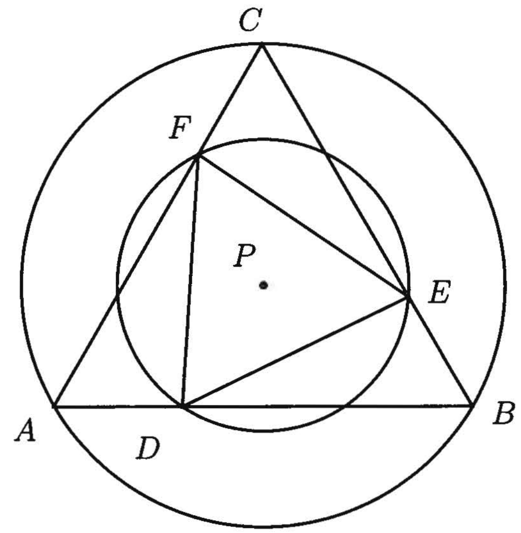
Substituting $n = p - 1$ gives

$$\prod_{d|X} d = X^{(p-1)+1} = X^p.$$

Also solved by **Paul S. Bruckman**, Sointula, BC, **Michael J. Coons**, student, University of Montana, Missoula, MT, **Richard Hess**, Rancho Palos Verdes CA, **Doo-Hyung Lee**, student, Eastern Kentucky University, Richmond, KY, **S.C. Locke**, Florida Atlantic University, Boca Raton, FL, **David E. Manes**, SUNY College at Oneonta, Oneonta, NY, **James Meadows**, student, Angelo State University, San Angelo, TX, **Yoshinobu Murayoshi**, Okinawa, Japan, **Mike Pinter**, Belmont University, Nashville, TN, **James A. Sellers**, Pennsylvania State University, University Park, PA and by the **Proposer**

1060[Spring 2004] *Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College, Abington PA*

Suppose $\triangle ABC$ is an equilateral triangle. The points D, E, and F are on AB, BC and CA respectively such that $|AD| = |BE| = |CF|$. Show that the circumcircles of $\triangle ABC$ and $\triangle DEF$ are concentric.



Some of the solutions dealt with points as complex numbers or vectors:

Solution by **William H. Peirce**, *Rangeley, ME.*

From the equality of AD, BE, and CF, it follows that $\triangle$'s ADF, BED, and CFE are congruent triangles from which it then follows that $\triangle DEF$ is equilateral. Since $\triangle ABC$ is equilateral, the equality of AD, BE, and CF allows us to write, for $0 < \lambda < 1$, $D = (1-\lambda)A + \lambda B$, $E = (1-\lambda)B + \lambda C$, and $F = (1-\lambda)C + \lambda A$.

The circumcenter of an equilateral triangle is the average of its vertices. The circumcenter of $\triangle ABC$ is $P = (A+B+C)/3$ and the circumcenter of $\triangle DEF$ is

$$\begin{aligned}(D+E+F)/3 &= [(1-\lambda)A + \lambda B + (1-\lambda)B + \lambda C + (1-\lambda)C + \lambda A]/3\\ &= (A+B+C)/3\\ &= P.\end{aligned}$$

Thus the circumcircles of $\triangle$'s ABC and DEF are concentric.

Another solution worked with transformations of the plane:

Solution by **S.C. Locke**, *Florida Atlantic University, Boca Raton, FL.*

Let P denote the circumcenter of $\triangle ABC$, that is, the center of the circumcircle of $\triangle ABC$. When $\triangle ABC$ is rotated 120 degrees about P, with A falling on B, B on

C, and C on A, the line segment AD ends up where BE was, BE ends up where CF was, and CF lands on AD. Therefore $|PD| = |PE| = |PF|$ and P is the center of the circumcircle of $\triangle DEF$.

Also solved by **Paul S. Bruckman**, Sointula, BC, **Mark Evans**, Louisville, KY, **Steve Gendler**, Clarion University, Clarion, PA, **David E. Manes**, SUNY College at Oneonta, Oneonta, NY, **James Meadows**, student, Angelo State University, San Angelo, TX, **Yoshimobu Murayoshi**, Okinawa, Japan, and by the **Proposer**.

1061[Spring 2004] *Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College, Abingtion PA*

Let

$$x = \sum_{k=0}^{n} 2^k \binom{2n+1}{2k+1}.$$

Then $\dfrac{x^2-1}{2}$ is the product of two consecutive whole numbers.

Solution by **Charles R. Diminnie**, *Angelo State University, San Angelo, TX*

If we let

$$y = \sum_{k=0}^{n} 2^k \binom{2n+1}{2k},$$

then by the Binomial Theorem,

$$\begin{aligned}(1+\sqrt{2})^{2n+1} &= \sum_{k=0}^{2n+1} \binom{2n+1}{k} (\sqrt{2})^k \\ &= \sum_{k=0}^{n} \binom{2n+1}{2k+1} (\sqrt{2})^{2k+1} + \sum_{k=0}^{n} \binom{2n+1}{2k} (\sqrt{2})^{2k} \\ &= \sqrt{2}\sum_{k=0}^{n} \binom{2n+1}{2k+1} (\sqrt{2})^{2k} + \sum_{k=0}^{n} \binom{2n+1}{2k} 2^k \\ &= \sqrt{2}x + y,\end{aligned}$$

and similarily,

$$(1-\sqrt{2})^{2n+1} = -\sqrt{2}x + y.$$

Hence,

$$x = \frac{(1+\sqrt{2})^{2n+1} - (1-\sqrt{2})^{2n+1}}{2\sqrt{2}}$$

and

$$y = \frac{(1+\sqrt{2})^{2n+1} + (1-\sqrt{2})^{2n+1}}{2}$$

From these, it is easily demonstrated the (x, y) is a solution of the Pell equation

(1) $$2x^2 - y^2 = 1.$$

Since y is clearly an odd positive integer, we may consider the consecutive positive integers

$$a = \frac{y-1}{2} \qquad \text{and} \qquad b = \frac{y+1}{2}.$$

Then, by (1),

$$ab = \frac{y^2-1}{4} = \frac{2x^2-2}{4} = \frac{x^2-1}{2}.$$

Remark: Since gcd(a,b)=1 and (1) implies that

$$a^2 + b^2 = \frac{(y-1)^2}{4} + \frac{(y+1)^2}{4} = \frac{2y^2+2}{4} = \frac{4x^2}{4} = x^2,$$

it follows that (a,b,x) is also a primitive Pythagorean triple whose legs are consecutive positive integers.

Also solved by **Paul S. Bruckman**, Sointula, BC, **Pat Costello**, Eastern Kentucky University, Richmond, KY, **Richard Hess**, Rancho Palos Verdes CA, **S.C. Locke**, Florida Atlantic University, Boca Raton, FL, **David E. Manes**, SUNY College at Oneonta, Oneonta, NY, **Yoshinobu Murayoshi**, Okinawa, Japan, **William H. Peirce**, Rangeley, ME.

MATH A CROSTIC

Dan Hurwitz, Skidmore College

a. English algebraist (1814–1897) taught at Johns Hopkins — 081 054 012 146 027 103 165 035 157

b. Maximal quotient (2 wds.) — 114 132 078 033 030 005 096 175 100 043 163 051

c. Following from the premises — 150 058 015 174 131 111 069 142 167

d. Place vacuous cases hold — 154 034 075 121 007 045 095

e. American Mathematician (1839–1903) worked in thermodynamics — 161 179 123 085 062

f. Statistical item — 106 125 013 073 048

g. Author of popular books on number theory and game theory (full name) — 067 098 119 020 127 102 159 171 031 052 003

h. What a positive discriminant does with respect to roots of a quadratic (3 wds.) — 139 155 004 055 168 117 092 039 068 182 080 025 148 124 049 101

i. They form a cyclic group under multiplication (4 wds.) — 105 082 134 029 091 180 116 143 042 173 153 016 061 120 071

j. They had an alphabetic number system — 017 097 079 156 023 172

k. Culture with a symbol for zero — 047 115 128 011 145

l. One more time — 057 130 110 065 141 076

m. He studied the five regular solids, friend of Plato — 063 002 135 022 086 056 107 152 126 164

n. Greek mathematician, wrote commentaries on Diophantus — 014 090 066 137 074 144 170

o. Topic studied in numerical analysis — 084 169 050 118 108

p. Function — 136 087 099 026 166 151 044

r. Sixth century Constantinople mathematician, studied the parabola — 028 041 060 037 077 176 010 088 113

r. This is used to check tautologies (3 wds.) — 089 032 104 001 021 178 147 072 158 038 138 122 053 109

s. A sense important to Brower — 149 181 036 162 059 133 018 094 008

t. IIME has many of these — 140 083 177 093 009 024 070 040

u. America expert on artificial intellegence — 006 064 046 160 112 129 019

001r	002m	003g		004h	005b	006u	007d	008s	009t	010c	011k		013f	014n	015c	016l
017j		018s	019u		020g	021r	022m	023j	024t	025h	026p	027a	028q	029l	030b	
031g	032r		033b	034d	035a	036s	037q	038r		039h	040t		041q	042l	043b	
044p	045d	046u	047k	048f	049h	050o		051b	052g		053r	054a	055h	056m	057l	058c
	059s	060q		061l	062e		063m	064u	065l		066m	067g	068h	069c	070t	071l
	072r	073f	074a		075d	076l	077q	078b	079j	080h	081a		082l	083t	084o	
085c	086m	087p	088q	089r	090n		091l	092h		093t	094s	095d	096b	097j	098g	
099p	100b	101h	102g	103a		104r	105l	106f	107m	108o		109r	110l	111c	112u	113o
114b	115k	116l	117h	118o	119g		120l	121d	122r		123e	124h	125f	126m	127g	128k
	129u	130l		131c	132b	133s	134l	135m	136p	137n	138r	139h	140t	141l		142c
143l		144n	145k	146a	147r	148h	149s	150c	151p	152m		153l	154d	155h	156j	157a
	158r	159g	160u	161e	162s	163b	164m	165a	166p	167c		168h	169o	170n	171g	172j
173l	174g	175b	176q	177t	178r	179e	180l	181s	182h							

Last issue's mathacrostic was taken from (Regular and Semiregular) Polyhedra by H. S. M. Coxeter.

"...it always must be true: that if you have a polyhedron whose faces consist entirely of triangles, six coming together at some vertices and five at others, then the number of vertices where there are five triangles coming together is exactly twelve."

Contents

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