

BMO 2008–2009 Round 1

Problem 1—Generalisation

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1 Introduction

Problem 1 is

Consider a standard 8×8 chessboard consisting of 64 small squares coloured in the usual pattern so that squares with both x and y coordinates odd are black and the others are white. A *zig-zag* path across the board is a collection of eight white squares, one in each row, which meet at their corners. How many zig-zag paths are there?

This problem generalises to any rectangular board. If both the number of rows and the number of columns are odd, it matters whether the colour specified is the colour of all the corner squares or none of them; otherwise symmetry considerations show that the numbers for each colour are equal. Thus we consider the problem

Let n and r be positive integers and consider a rectangular chessboard with r rows of n squares each. A *zig-zag* path across the board is a collection of r squares, one in each row, which meet at their corners. Let $m_{n,r}$ be the number of zig-zag paths, $c_{n,r}$ be the number of zig-zag paths whose colour is that of the square in the top right corner of the board, and $n_{n,r}$ be the number of zig-zag paths whose colour is the opposite of that of the square in the top right corner of the board. Determine $m_{n,r}$, $c_{n,r}$ and $n_{n,r}$.

Clearly $m_{n,r} = c_{n,r} + n_{n,r}$ and as noted above $c_{n,r} = n_{n,r}$ if n or r is even. The standard method of solving the original problem is simply to count the number of zig-zag paths from the top row to each square in each row.

in turn writing down s in the top row and then making each value in each subsequent row the sum of values above it on either side leading to the following diagram, this also readily allows computing values for other rectangular boards

1		1		1		1	
	2		2		2		1
2		4		4		3	
	6		8		7		3
6		14		15		10	
	20		29		25		10
20		49		54		35	
	69		103		89		35

2 Eigenvalue methods

The map above from the numbers in one row to the numbers in the next is a linear transformation given by a matrix $M = (M_{ij})$ where $M_{ij} = 1$ if $|i - j| = 1$ and $M_{ij} = 0$ otherwise

As a real symmetric matrix this has real eigenvalues λ_k and an orthonormal basis of eigenvectors \mathbf{v}_k . The sum of the elements in the s th row will be

$$\mathbf{u}^T M^s \mathbf{v} = \sum_k \mathbf{u} \cdot \mathbf{v}_k \lambda_k^s \mathbf{v} \cdot \mathbf{v}_k,$$

where \mathbf{u} is the all 1's vector and \mathbf{v} is the vector of values in the s th row

We claim that $\lambda_k = 2 \cos \frac{\pi k}{n}$ and that \mathbf{v}_k is the vector with s th coordinate $v_{k,s} = \sqrt{\frac{2}{n}} \sin \frac{\pi k s}{n}$. Certainly this is an eigenvector with the given eigenvalue since

$$\sin \left(\frac{\pi k s}{n} - \frac{\pi k}{n} \right) = \sin \left(\frac{\pi k s}{n} \right) \cos \frac{\pi k}{n} - \cos \left(\frac{\pi k s}{n} \right) \sin \frac{\pi k}{n},$$

so it remains to check the normalisation factor, that is to show that

$$\sum_s^n \sin \frac{\pi ks}{n} = \frac{n}{2}.$$

Now we have

$$\begin{aligned} \sum_s^n \sin \frac{\pi ks}{n} &= \sum_s^n \left(\sin \frac{\pi ks}{n} - \cos \frac{\pi k}{n} \sin \frac{\pi ks}{n} \right) \\ &= \frac{n}{2} - \sum_s^n \cos \frac{\pi k}{n} \sin \frac{\pi ks}{n} \end{aligned}$$

and since

$$\cos \frac{\pi ks}{n} \cos \frac{\pi k}{n} = \frac{1}{2} \left(\cos \left(\frac{\pi ks}{n} - \frac{\pi k}{n} \right) + \cos \left(\frac{\pi ks}{n} + \frac{\pi k}{n} \right) \right)$$

we have

$$\sum_s^n \cos \frac{\pi ks}{n} \cos \frac{\pi k}{n} = \cos \frac{\pi k}{n} \sum_s^n \cos \frac{\pi ks}{n}$$

from which the result follows as $\cos \frac{\pi k}{n} \neq 0$. The original sum may also be evaluated directly by expressing it in terms of geometric series in $\exp \left(\frac{\pi i k s}{n} \right)$.

It remains to evaluate the dot products above for particular \mathbf{u} and \mathbf{v} .

Again this could be done directly with geometric series though here we demonstrate the results without this. For even k we have $\mathbf{u} \cdot \mathbf{v}_k = 0$ as positive and negative terms cancel. So suppose k is odd. First we claim that

$$\mathbf{v} \cdot \mathbf{v}_k = \sqrt{\frac{2}{n}} \frac{\sin \frac{\pi k}{n}}{1 - \cos \frac{\pi k}{n}}.$$

For put

$$S = \sum_s^n \sin \frac{\pi ks}{n}.$$

We then have

$$\begin{aligned} -\cos \frac{\pi k}{n} S &= \sum_s^n \left(\sin \frac{\pi ks}{n} - \cos \frac{\pi k}{n} \sin \frac{\pi ks}{n} \right) \\ &= \sum_s^n \left(\sin \frac{\pi ks}{n} - \sin \frac{\pi k(s-1)}{n} - \sin \frac{\pi k(s+1)}{n} \right) \\ &= \left(\sin \frac{\pi k}{n} - \sin \frac{\pi kn}{n} - \sin \frac{\pi kn}{n} + \sin \pi k \right) \\ &= \sin \frac{\pi k}{n}. \end{aligned}$$

Now

$$\mathbf{v} \cdot \mathbf{v}_k = \frac{\sin \frac{\pi k}{n}}{n - \cos \frac{\pi k}{n}} - \frac{\cos \frac{\pi k}{n}}{n - \cos \frac{\pi k}{n}}$$

so we conclude that

$$m_{n,r} = \frac{1}{n} \sum_{\substack{k=1 \\ k \text{ odd}}}^n \left(\cos \frac{\pi k}{n} \right)^{r-1} \left(\frac{\cos \frac{\pi k}{n}}{-\cos \frac{\pi k}{n}} \right)$$

with

$$c_{n,r} = n_{n,r} = \frac{1}{n} \sum_{\substack{k=1 \\ k \text{ odd}}}^n \left(\cos \frac{\pi k}{n} \right)^{r-1} \left(\frac{\cos \frac{\pi k}{n}}{-\cos \frac{\pi k}{n}} \right)$$

if either n or r is even

Similarly if we consider the vectors \mathbf{v} arising from restricting to a particular colour of squares when both n and r are odd we find

$$c_{n,r} = \frac{1}{n} \sum_{\substack{k=1 \\ k \text{ odd}}}^n \left(\cos \frac{\pi k}{n} \right)^{r-1} \left(\frac{\cos \frac{\pi k}{n}}{-\cos \frac{\pi k}{n}} \right)$$

and

$$n_{n,r} = \frac{1}{n} \sum_{\substack{k=1 \\ k \text{ odd}}}^n \left(\cos \frac{\pi k}{n} \right)^{r-1} \left(\frac{\cos \frac{\pi k}{n}}{-\cos \frac{\pi k}{n}} \right)$$

in this case

3 Combinatorial methods for the square case

The above results by eigenvalue methods give the asymptotics of the functions $m_{n,r}$, $c_{n,r}$ and $n_{n,r}$ as $r \rightarrow \infty$ for fixed n . The original problem however was for a square board $r = n$. In this case we may reason purely combinatorially

Consider zig zag paths on an infinitely wide extension of the given board that start within the columns of the original board but may go outside it. Then there is a one to one correspondence between

- zig zag paths that start and end within the columns of the original board but go outside it in intermediate rows and
- zig zag paths that start within the columns of the original board and end outside it but not in the columns immediately adjacent to it on either side

This correspondence is given simply by taking the first square at which the path going from top to bottom say goes outside the original board and reflecting the rest of the path in a vertical axis going through that square

This does not work for rectangular boards taller than they are wide because then the reflections can take a square of the path outside the board on one side to a square outside the board on the other. The cylindrical model of the next section shows how this method generalises to other boards but the resulting binomial coefficient sums do not appear particularly useful in general

Now the number of paths starting and ending in given squares but possibly going outside the board is simply given by a binomial coefficient, consider how many times each such coefficient is counted positively or negatively for different starting squares then pairing $\binom{r-}{k}$ and $\binom{r-}{r-k}$ we have for n even

$$\begin{aligned}
 c_{n,n} &= \sum_{k=0}^{(n/2)-1} (-1)^k k - n \binom{n-}{k} \\
 &\quad - n / n - \sum_{k=0}^{(n/2)-1} k \binom{n-}{k} \\
 &= -n \sum_{k=0}^{(n/2)-1} (-1)^k \sum_{k=0}^{(n/2)-1} n - \binom{n-}{k-} \\
 &= -n \sum_{k=0}^{(n/2)-1} (-1)^k n - \left(\sum_{k=0}^{(n/2)-1} \binom{n-}{n-k-} \right) \\
 &= -n \sum_{k=0}^{(n/2)-1} (-1)^k n - \left(\sum_{k=0}^{(n/2)-1} \binom{n-}{n-k-} \right)
 \end{aligned}$$

and so

$$m_{n,n} = n \sum_{k=0}^{(n/2)-1} (-1)^k n - \binom{n-}{n-k-}.$$

Similarly for odd n we have

$$c_{n,n} = \left(\sum_k^{(n-3)/2} \binom{n-k}{k} - n \binom{n-1}{n/2} \right) \frac{n-1}{2} \binom{n-1}{n/2}$$

and

$$n_{n,n} = \left(\sum_k^{(n-3)/2} \binom{n-k}{k} - n \binom{n-1}{n/2} \right) \frac{n-1}{2} \binom{n-1}{n/2}$$

so

$$m_{n,n} = n \binom{n-1}{n/2}.$$

4 The cylindrical model

Some of the complexity in understanding the linear map from one row to another appears to arise from the edge effects where some values in the next row are copies of values from the row above and others are sums of two values

Suppose the row $a \ a^- \ \dots \ a_n$ of length n is replaced by a cylindrical row of period n and values $a \ a^- \ \dots \ a_n \ -a_n \ \dots \ -a^- \ -a$

Then each cylindrical row is mapped to the next with *every* value in the next row being the sum of the values above it on either side

This generalises the argument in the previous section by showing that the desired number is given by the number of paths that end up in the right columns mod n minus those that end up in columns that are wrong by more than $\text{mod } n$. It also explains the eigenvalues that arise. For the map from one row to the next is the sum of two maps: one which shifts to the right and one which shifts to the left. Each of these maps clearly has as eigenvalues the n^{th} roots of unity and an eigenvector with eigenvalue ζ for one map has eigenvalue ζ^- for the other map. Each pair of roots ζ / ζ^- gives rise to a two dimensional eigenspace for the sum of the two maps whose eigenvalue $\zeta + \zeta^-$ is one of the cosines previously found and which has a one dimensional subspace of the form required to have arisen from a row in the original problem by the transformation given above. The roots ζ and $-\zeta$

each have a one dimensional eigenspace which does not include any nonzero vectors arising from the original problem

5 Integer sequences

These problems give rise to numerous integer sequences many of which are to be found in the On Line Encyclopedia of Integer Sequences

First there are at least six natural sequences for the square case $m_{n,n}$
 $c_{n,n}$ $n_{n,n}$ $c_{n,n}^-$ $c_{n,n}^+$ $n_{n,n}^-$ $n_{n,n}^+$ These appear in OEIS as A n
A n A n A n A n A n A n respectively

For each fixed n there are again at least six natural sequences if n is odd and r varies and at least two if n is even Some sequences appear many times in OEIS with different offsets and initial terms some may not appear with the given initial terms at all with the OEIS entries listed below being for versions that are the same as the sequences described here from some point onwards but with no terms or different terms before that point

First there are some sequences given by simple formulae The OEIS entries listed are those that appear more or less canonical for the formulae with no attempt in most cases to list versions with different initial terms

- $m_{n,r} = c_{n,r} - c_{n,r}^-$ $r = 1, 2, \dots$ A n $r = 1, 2, \dots$
- $n_{n,r} = c_{n,r}^- - n_{n,r}^-$ A n $r = 1, 2, \dots$
- $m_{n,r}^-$ A n $r = 1, 2, \dots$
- $c_{n,r}^-$ A n $r = 1, 2, \dots$
- $m_{3,r} = c_{3,r}^- - c_{3,r}^-$ $r = 1, 2, \dots$ A n $r = 1, 2, \dots$ see also A n A
- $c_{3,r} = n_{3,r} - [r/2]$ A n $r = 1, 2, \dots$ see also A n A n
- $c_{3,r}^- = c_{3,r}^- - n_{3,r}^-$ $r = 1, 2, \dots$ A n $r = 1, 2, \dots$
- $m_{4,r} = F_r - F_r^-$ A n $r = 1, 2, \dots$ see also A n A n A n
- $c_{4,r} = F_r - F_r^-$ A n $r = 1, 2, \dots$

As n increases fewer familiar sequences appear although there are still some variants with different initial terms

- $m_{5,r} = A \quad r -$
- $c_{5,r} = A \quad r$
- $n_{5,r} = A \quad r$
- $c_{5,r} = A \quad r -$ see also $A \quad A \quad A$
- $c_{5,r} = A \quad r$ see also $A \quad A \quad A$
- $n_{5,r} = A \quad r$ see also $A \quad A \quad A$
- $m_{,r} = A \quad r -$
- $c_{,r} = A \quad r -$
- $m_{,r} = A \quad r$
- $c_{,r} = A \quad r$
- $n_{,r} = A \quad r$
- $c_{,r} = A \quad r$
- $c_{,r} = A \quad r$
- $n_{,r} = A \quad r$ see also $A \quad A \quad A$
- $m_{,r} = A \quad r$
- $c_{,r} = A \quad r -$
- $m_{,r} = A \quad r$
- $c_{,r} = A \quad r$
- $n_{,r} = A \quad r$
- $c_{,r} = A \quad r$
- $c_{,r} = A \quad r$
- $n_{,r} = A \quad r$
- $m_{,r} = A \quad r$
- $c_{,r} = A \quad r -$

- $m_{n,r} = A_{n,r}^{(1)} \quad r$
- $c_{n,r} = A_{n,r}^{(2)} \quad r$
- $n_{n,r} = A_{n,r}^{(3)} \quad r$
- $c_{n,r}^{\overline{}} = A_{n,r}^{(4)} \quad r$
- $c_{n,r-}^{\overline{}} = A_{n,r-}^{(5)} \quad r$
- $n_{n,r-}^{\overline{}} = A_{n,r-}^{(6)} \quad r$
- $m_{n,r}^{\overline{}} = A_{n,r}^{(7)} \quad r$
- $c_{n,r}^{\overline{}} = A_{n,r}^{(8)} \quad r$
- The sequences for $\leq n \leq$ and probably larger n not checked do not seem to appear in OEIS

Of particular note appear to be the relations to sequences concerning the numbers of differential operations on \mathbb{R}^n as enumerated in [1] and [2]. It appears that $m_{n,r}$ counts differential operations on \mathbb{R}^{n-r} and $c_{n,r}^{\overline{}}$ counts differential operations on \mathbb{R}^{n-} although I do not have a proof or a direct combinatorial correspondence between the two problems and the offsets listed in OEIS for the differentiation operation sequences do not seem consistent across all such sequences. Any particular individual case may readily be verified using the recurrences given for the sequences for differential operations.

Supposing those relations $A_{n,r}^{(1)}$ gives a table corresponding to a complicated arrangement of the present sequences while $A_{n,r}^{(2)}$ gives the sequence $c_{4,3}^{\overline{}}, m_{3,3}, c_{4,4}, m_{4,5}, c_{5,5}, m_{5,5}, c_{6,6}, m_{6,6}, c_{7,7}, m_{7,7},$

References

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