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1. **Introduction.** Ramanujan(1) defined a number to be highly composite if it has more divisors than any smaller number, and he used these numbers to determine the maximum order of $d(n)$. He investigated the factorization of the highly composite numbers, which is also of interest.

A number n is defined to be (1) *highly abundant* if $\sigma(n)>\sigma(m)$ for all $m<n$, (2) *superabundant* if $\sigma(m)/m<\sigma(n)/n$ for all $m<n$, and (3) *colossally abundant* if for some $\epsilon>0$, $\sigma(n)/n^{1+\epsilon}\geqq\sigma(m)/m^{1+\epsilon}$ for $m<n$ and $\sigma(n)/n^{1+\epsilon}>\sigma(m)/m^{1+\epsilon}$ for $m>n$. We do not make any attempt to estimate the maximum order of $\sigma(n)$, since this has already been done with great accuracy(2). But we shall give very precise results about the factorization of highly composite, superabundant, and colossally abundant numbers. In fact we prove that for superabundant numbers $n=2^{k_2}\cdots p^{k_p}$,

$$\log\left[(q^{k_q+1}-1)/(q^{k_q+1}-q)\right]>(\log q/p\log p)\left[1+O(\delta)\right],$$

$$\log\left[(q^{k_q+2}-1)/(q^{k_q+2}-q)\right]<(\log q/p\log p)\left[1+O(\delta)\right],$$

where $\delta=(\log\log p)^2/\log p\log q$ *if* $q^c<\log p$, *and* $\delta=\log p/q^{1-\theta}\log q$ *if* $q^{1-\theta}>\log p$. For highly composite numbers we prove that

$$\log(1+1/k_q)>\log q\log 2/\log p+O(\delta),$$

$$\log\left[1+1/(k_q+1)\right]<\log q\log 2/\log p+O(\delta),$$

where $\delta=(\log\log p)^3/(\log p)^3$ *if* $q^c<\log p$, *and* $\delta=1/q^{1-\theta}\log p$ *if* $q^{1-\theta}>\log p$. It is easy to see that these formulas determine k_q with an error of at most 1, and in most cases uniquely. This considerably sharpens Ramanujan's results for highly composite numbers(3).

The highly abundant numbers behave somewhat irregularly, but the irregularities are few. On the average they behave as superabundant numbers.

The principal tool used in this paper is the result of Ingham(4), which states that the number of primes between q and $q+cq^\theta$ is asymptotic to $cq^\theta/\log q$, for any $\theta>48/77$. Actually only the fact that θ is less than 1 is

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(1) *Collected works of S. Ramanujan*, p. 86.

(2) S. Wigert, *Sur quelques fonctions arithmetiques*, Acta Math. vol. 37 (1914) pp. 114–140.

(3) *Collected works*, pp. 99 ff.

(4) A. E. Ingham, *On the difference of two consecutive primes*, Quart. J. Math. Oxford Ser. vol. 8 (1937) p. 255.

used, so that all our results could be obtained from Hoheisel's original value, $\theta > 32999/33000$(5).

In comparing the magnitudes of $\sigma(n)$ and $\sigma(n')$ it is clear from the multiplicative property of $\sigma(n)$ that one need only consider the behavior of those primes which divide the two numbers to different powers. The same is true for $d(n)$ and $\sigma(n)/n$.

We prove that the quotient of two consecutive superabundant numbers tends to 1, and that the number of these numbers less than x is greater than $c \log x \cdot \log\log x/(\log\log\log x)^2$. On the other hand it would be easy to prove that the number of superabundant numbers is less than $(\log x)^{c \log\log x}$. The exponent could probably be reduced to $c \log\log\log x$.

For highly composite numbers we have the same upper limit, but we know that the number of highly composite numbers exceeds $(\log x)^{1+c}$. It would be interesting to know the exact order in both cases.

In the theory of colossally abundant numbers the most interesting question is whether the quotient of two consecutive colossally abundant numbers is a prime or not. This question leads to the following problem in Diophantine analysis. *If p and q are different primes, is it true that p^x and q^x are both rational only if x is an integer?*

For highly abundant numbers the results are less satisfactory. We do not know whether there are infinitely many highly abundant numbers which are not superabundant, nor do we know whether, if $n = 2^{k_2} \cdot 3^{k_3} \cdots q^{k_q} \cdots$ is highly abundant, then $k_2 \geqq k_3 \geqq \cdots$. Other open questions are the exact order of the largest prime factor of n, the exponent to which this prime divides n, and the number of highly abundant numbers less than n.

There is a section devoted to a desultory discussion of other multiplicative functions.

A table of highly abundant numbers less than 10^4 and a table of superabundant and colossally abundant numbers less than 10^{18} are appended. The highly abundant numbers were found by examining Glaisher's *Number-divisor tables*(6). The calculation of the superabundant numbers was materially aided by the result proved in §2, that the exponent to which 2 divides the superabundant n determines the exponents of all other primes with an error of 1 at most.

S. Pillai, in his paper *On $\sigma_{-1}(n)$ and $\phi(n)$*, Proceedings of the Indian Academy of Sciences vol. 17 (1943) p. 70, refers to certain results which appear to be connected with our work. We quote: "In the papers entitled 'Highly abundant numbers' and 'Totient numbers' which are unpublished and formed a part of my D. Sc. thesis, I proved the above results (concerning the maxi-

(5) G. Hoheisel, *Primzahlprobleme in der Analysis*, Berlin Math. Ges. Sitzungsber., 1930, pp. 550–558.

(6) J. L. Glaisher, *Number-divisor tables*, British Association for the Advancement of Science, Mathematical Tables, vol. 8.

mum order of $\sigma(n)$ and $\phi(n)$). But that proof depends on the properties of these numbers . . . ." Pillai's thesis is as yet inaccessible to us.

2. **Superabundant numbers.** A number n is said to be *superabundant* if $\sigma(m)/m<\sigma(n)/n$ for all $m<n$.

THEOREM 1. *If* $n=2^{k_2}\cdots p^{k_p}$, *then* $k_2\geqq k_3\geqq\cdots\geqq k_p$.

If the theorem is not true, there exist two primes q and r such that $q>r$ and $k_q>k_r$. We put $k_q=k$, $k_r=l$. Then since n is superabundant, and $n'=nr/q<n$, we must have $\sigma(n')/n'<\sigma(n)/n$. This inequality reduces to

$$(r^{l+2}-1)/(r^{l+2}-r)<(q^{k+1}-1)/(q^{k+1}-q).$$

As $(x^n-1)/(x^n-x)$ is a decreasing function of x and n for x, $n\geqq 2$, a simple calculation shows that our inequality is not satisfied.

THEOREM 2. *Let* $q<r$, *and set* $\beta=[k_q\log q/\log r]$. *Then* k_r *has one of the three values*: $\beta-1,\beta+1,\beta$.

Suppose first that $k_r=l\leqq\beta-2$. We let $k_q=k$, and define x by the inequality $q^{x-1}<r<q^x$. Then $k\geqq x$, for otherwise $q^k<r<r^{l+2}<q^k$, which is impossible. Now compare n with nr/q^x. Since $\sigma(n)$ is multiplicative we restrict our attention to the factors r and q. Then since n is superabundant, we must have

$$(q^x-1)r^{l+2}+r>(r-1)q^{k+1}+q^x.$$

But

$$(q^x-1)r^{l+2}+r\leqq(q^x-1)r^\beta+q^x\leqq(rq-r-1)r^\beta+q^x<(r-1)q^{k+1}+q^x,$$

which is a contradiction.

If l were greater than $\beta+1$, we could compare q^kr^l with $q^{k+x-1}r^{l-1}$, and we would obtain a contradiction by the same argument.

THEOREM 3. *If* p *is the largest prime factor of* n, *then* $k_p=1$, *except when* $n=4$, 36.

Let q be the second largest prime factor of n, and suppose that $k=k_p>1$. We put $k_q=l$, and let r be the prime just greater than p. On comparing n with nr/pq, we must have

$$1+1/r<[1+(p-1)/(p^{k+1}-p)][1+(q-1)/(q^{l+1}-q)].$$

But since $l\geqq k\geqq 2$, the right-hand side does not exceed

$$[1+1/(p^2+p)][1+1/(q^2+q)], \tag{1}$$

which is less than $(1+1/p^2)(1+1/q^2)$. By Tchebichef's theorem, this is less than $1+1/r$ if $p\geqq 11$. The numbers for which $3<p<11$ can be checked by using (1), and those for which $p=2$, 3 can be checked directly.

In order to prove sharper results several lemmas are needed. From here on p always denotes the largest prime factor of n.

LEMMA 1. $q^{k_q} < 2^{k_2+2}$.

This certainly holds if $k_q \leqq \beta = [k_2 \log 2/\log q]$. If $k_q = \beta+1$ and $2^{k_2+2} < q^{k_q}$ a contradiction will be reached by comparing $2^{k_2} q^{\beta+1}$ with $2^{k_2+x} q^{\beta}$, where $2^x < q < 2^{x+1}$.

LEMMA 2. *If* $r > (\log p)^c$, *then* $k_r = o(r^{1/c})$.

By Lemma 1, $k_r \log r < k_2$ if k_2 is large, and by Theorems 2 and 3, $k_2 < 5 \log p$. Hence $k_r < 5 \log p/\log r = o(r^{1/c})$.

LEMMA 3 (INGHAM)(7). *The number of primes in the interval* $(q, q+cq^{\theta})$ *is asymptotic to* $cq^{\theta}/\log q$, *where* $\theta \geqq 5/8$.

If Riemann's hypothesis is true any $\theta > 1/2$ can be used.

LEMMA 4. *If* q *is the greatest prime of exponent* k, *and if* $q^{1-\theta} > \log p$, *then all primes between* q *and* $q+q^{\theta}$ *have exponent* $k-1$.

This is obviously true if $k=1$. If $k>1$ and the lemma is assumed to be false for the prime r, a contradiction will be found on comparing $s^i q^k r^l$ with $s^{i-1} q^{k-1} r^{l+1}$, where r is the prime preceding q, $k_s = j$, and $k_r = l$. It is only necessary to note that j can be replaced by k and l by $k-2$.

THEOREM 4. *If* q *is either the greatest prime of exponent* k *or the least prime of exponent* $k-1$, *and if* $q^{1-\theta} > \log p$, *then*

$$q^k = (p \log p/\log q)[1 + O(\log p/q^{1-\theta} \log q)].$$

If q is the greatest prime of exponent k let $q_1 = q$, and let $q_2, \cdots, q_x$ be the immediate predecessors of q. Let $P_1, \cdots, P_y$ be the immediate successors of p. The integer y is chosen large and x is chosen so that $q_1 \cdots q_{x-1} < P_1 \cdots P_y < q_1 \cdots q_x$. Then $q_x^{x-1} < P_y^y$, and $q^x > p^y$. If $y = [q^{\theta}/\log p]$, then by Lemmas 3 and 4 it follows that

$$y/x = \log q/\log p + O(\log q/x \log p) + O(1/q^{1-\theta} \log p).$$

We now compare $n_1 = (q_1 \cdots q_x)^k$ with $n_2 = (q_1 \cdots q_x)^{k-1} \cdot P_1 \cdots P_y$. Since $n_2 < n_1$, it follows that $\sigma(n_2)/n_2 < \sigma(n_1)/n_1$. This reduces to

$$\prod_{i \leqq y} (1 + 1/P_i) < \prod_{j \leqq x} [1 + (q_j - 1)/(q_j^{k+1} - q_j)].$$

From the choice of y all the P_i can be replaced by $p+cp^{\theta}$, and the q_j by $q-cq^{\theta}$, so that

(7) *Ibid.*

$$1/P_i = 1/p + O(1/p^{2-\theta}), \qquad (q_i - 1)/(q_i^{k+1} - q_i) = 1/q^k + O(k/q^{k+1-\theta}).$$

Hence on taking logarithms we find that

$$y/p + O(y/p^{2-\theta}) < x/q^k + O(xk/q^{k+1-\theta}).$$

On combining this and the formulas for y and y/x, we find that

$$q^k \log q < p \log p[1 + O(\log q/q^\theta) + O(k/q^{1-\theta})].$$

Similarly by raising the exponents of the first u successors of q by unity and by dropping an appropriate number of the largest prime factors, the opposite inequality is found to hold. It is then possible to replace the k in the error term by $\log p/\log q$, and to drop the first of the two error terms.

If q is the least prime of exponent $k-1$ the formula holds for the predecessor r of q. By Lemma 3, $r=q-O(q^\theta)$, so that

$$\begin{aligned} r^k = q^k[1 + O(k/q^{1-\theta})] &= (p \log p/\log q)[1 + O(\log p/q^{1-\theta} \log q)] \\ &= (p \log p/\log r)[1 + O(\log p/q^{1-\theta} \log r)]. \end{aligned}$$

This expansion is valid, since $k=o(q^{1-\theta})$ by Lemma 3.

THEOREM 5. *If $k_q=k$ and $q<(\log p)^\alpha$, where α is a constant, then*

(i) $$\log \frac{q^{k+1} - 1}{q^{k+1} - q} > \frac{\log q}{p \log p}[1 + O((\log \log p)^2/\log p \log q)],$$

(ii) $$\log \frac{q^{k+2} - 1}{q^{k+2} - q} < \frac{\log q}{p \log p}[1 + O((\log \log p)^2/\log p \log q)].$$

Only the proof of the first part will be given in full, since the proof of the second is almost the same. We choose the prime s to be the least of exponent $\beta-1$, and r the greatest of exponent γ. Both will be in the range of Theorem 4, and once r is chosen, s is to be as large as possible consistent with $s<rq$. Then since $ns/rq<n$, we must have

$$(q^{k+1} - 1)(r^{\gamma+1} - 1)/(q^{k+1} - q)(r^{\gamma+1} - r) > (s^{\beta+1} - 1)/(s^{\beta+1} - s).$$

But $\log [(x^{n+1}-1)/(x^{n+1}-x)]=x^{-n}+O(x^{-n-1})$, so that by Theorem 4 we have

(2) $$\log [(q^{k+1} - 1)/(q^{k+1} - q)] > (\log s - \log r)/p \log p + O(1/pr^{1-\theta}).$$

If t is the least prime of exponent $\beta-2$, then $rq<t$. Hence

(3) $$0 < \log (qr/s) < \log (t/s).$$

By Theorem 4,

$$\log (t/s) = \log (p \log p)/\beta(\beta - 1) + \log \log s/\beta - \log \log t/(\beta - 1) + O(\log p/s^{1-\theta} \log s).$$

But again by Theorem 4, $\beta \sim \log p/\log s$, $\beta - 1 \sim \log p/\log t$. Hence

$$\log (t/s) \sim \log t \log s/\log p + O(\log p/s^{1-\theta} \log s).$$

Put $r = (\log p)^x$; since $r < s < rq$ and $t < s^2$, we have

$$\log (t/s) = O((\log \log p)^2/\log p) + O(1/\log s(\log p)^{x-x\theta-1}).$$

If we take $x > 2/(1-\theta)$ we find that

$$\log (t/s) = O((\log \log p)^2/\log p). \tag{4}$$

By combining (2), (3) and (4) we get the first inequality of the theorem.

To prove the second part, n is compared with nqr/s, where now s is the greatest prime of its exponent and r least. The argument is the same from this point on.

It seems possible to decrease the error term to $O(1/(\log p)^c)$ where c is any positive integer, by changing the exponents of more primes. As will be seen in §3 the error term in Theorem 4 cannot be substantially improved, but it is possible that the error term in both Theorems 4 and 5 can be improved to $O(1/p^c)$ for some $c > 0$. The proof of this would require a great deal more than is known about the Diophantine properties of the logarithms of primes.

By combining Theorems 4 and 5, the exponent of a prime can be uniquely determined except in a few cases.

Let δ denote the error term:

$$\delta = (\log \log p)^2/\log p \log q \quad \text{if} \quad q^{1-\theta} < \log p,$$
$$\delta = \log p/q^{1-\theta} \log q \quad \text{if} \quad q^{1-\theta} > \log p.$$

Then

$$\log p \log [(q^{k+1} - 1)/(q^{k+1} - q)] > \log q \log (1 + 1/p)[1 + O(\delta)],$$
$$\log p \log [(q^{k+2} - 1)/(q^{k+2} - q)] < \log q \log (1 + 1/p)[1 + O(\delta)].$$

By Theorem 5, this holds if $q^{1-\theta} < \log p$. If $q^{1-\theta} > \log p$, then by Theorem 4 the first inequality holds if $q = q_0$, where q_0 is the largest prime of exponent k. But if $q < q_0$, the inequality must hold since $(x^n - 1)/(x^n - x)$ is a decreasing function of x and n, for x, $n \geqq 2$. The second inequality is proved in the same way.

Let $K_q + 1$ be the integral part and θ_q the fractional part of $\log [(q^{1+\epsilon} - 1)/(q^\epsilon - 1)]/\log q$, where $\epsilon = \log (1 + 1/p)/\log p$. Then we must have $K_q + \theta_q - O(\delta) - 1 < k < K_q + \theta_q + O(\delta)$.

By choosing p large, the error term, denoted by ϵ_q, can be made less than $1/2$. It is then easy to see that the following theorem is true.

THEOREM 6. (i) *If* $\epsilon_q < \theta_q \leqq 1 - \epsilon_q$, *then* $k_q = K_q$.
(ii) *If* $\theta_q \leqq \epsilon_q$, *then* $k_q = K_q$ *or* $k_q = K_q - 1$.
(iii) *If* $1 - \epsilon_q < \theta_q$, *then* $k_q = K_q$ *or* $K_q + 1$.

The significance of this theorem will be seen in connection with the colossally abundant numbers.

THEOREM 7. $p \sim \log n$.

By Theorem 1 and the prime number theorem

$$\log n > \sum_{q \leqq p} \log q \sim p.$$

On the other hand by Theorems 4 and 5 we have

$$\log n = \sum_{q \leqq p} \log q^{k_q} < \sum_{q \leqq p} \log (2p \log p) = \pi(p) \log (2p \log p) \sim p.$$

It is easy to see by Theorem 2 that $\lim_n p = \infty$, which proves the theorem.

THEOREM 8. *The quotient of two consecutive superabundant numbers tends to* 1.

Take $q^{1-\theta} > \log p$, and q the greatest prime of exponent k. Choose r to be the least prime of exponent $k - a - 1$, where a is a constant. Then by using Theorem 4 it is easy to see that

$$\sigma(rn/q)/(rn/q) > \sigma(n)/n$$

if a is large enough. Hence a superabundant number must lie between these two numbers. But by Theorem 4

$$\log (r/q) \sim a \log p / k(k - a) \sim a \log q \log r / \log p,$$

so that if q and r are of the order of $(\log p)^c$, it will follow that r/q tends to 1.

THEOREM 9. *The number of superabundant numbers less than* x *exceeds*

$$c \log x \log \log x / (\log \log \log x)^2.$$

In the proof of Theorem 8 it was shown that the ratio of two consecutive superabundants n and n' is less than

$$1 + c(\log \log n)^2 / \log n.$$

The result stated follows immediately from this.

3. **Colossally abundant numbers.** A number n is said to be *colossally abundant* if, for some $\epsilon > 0$,

$$\sigma(n)/n^{1+\epsilon} \geqq \sigma(m)/m^{1+\epsilon} \quad \text{for} \quad m > n,$$
$$\sigma(n)/n^{1+\epsilon} > \sigma(m)/m^{1+\epsilon} \quad \text{for} \quad m < n.$$

It is obvious that these numbers are superabundant.

If P does not divide n it is easily seen that

$$\epsilon \geqq \log (1 + 1/P)/\log p > \log [(P^{x+1} - 1)/(P^{x+1} - P)]/\log p, \text{ for } x \geqq 0.$$

As the right-hand side is a decreasing function of P this inequality will hold for all P if it holds for the prime P just greater than the largest prime factor p of n.

If $q<p$, $k_q=k$, and $x\geqq 0$, then

$$\log [(q^{k+2+x} - 1)/(q^{k+2+x} - q)] \leqq \epsilon \log q < \log [(q^{k+1-x} - 1)/(q^{k+1-x} - q)].$$

It is easily seen that for a given positive ϵ, these inequalities uniquely determine a prime p and for each prime q an exponent k_q. Let $n=\prod_q q^{k_q}$. This number is colossally abundant. For the inequalities show that if y divides n and z is prime to y, $\sigma(nz/y)/(nz/y)^{1+\epsilon}$ is less than $\sigma(n)/n^{1+\epsilon}$ if $z<y$, and it is not greater than $\sigma(n)/n^{1+\epsilon}$ if $z>y$.

THEOREM 10. *If n_ϵ is the colossally abundant number associated with ϵ, and if $k_q(\epsilon)$ is the exponent of the prime q, then*

$$k_q(\epsilon) = [\log \{(q^{1+\epsilon} - 1)/(q^\epsilon - 1)\}/\log q] - 1.$$

This shows that the error term in Theorem 4 is nearly the best possible. Here $[x]$ denotes the greatest integer less than x.

The numbers n_ϵ and $k_q(\epsilon)$ do not decrease as ϵ decreases. Since $\log \{(q^{1+\epsilon}-1)/(q^\epsilon-1)\}/\log q$ is a continuous function of ϵ, $k_q(\epsilon)$ will increase by steps of at most 1, and this will occur when $\log \{(q^{1+\epsilon}-1)/(q^\epsilon-1)\}/\log q$ is an integer. But this makes q^ϵ rational. It is very likely that q^x and p^x can not be rational at the same time except if x is an integer. This would show that the quotient of two consecutive colossally abundant numbers is a prime. At present we can not show this. Professor Siegel has communicated to us the result that q^x, r^x and s^x can not be simultaneously rational except if x is an integer. Hence *the quotient of two consecutive colossally abundant numbers is either a prime or the product of two distinct primes.*

The following remark is of some interest: It follows from Theorems 4 and 5 that *for large superabundant n, $q^{k_q}<2^{k_2}$ for $q>11$.* For smaller q Theorem 10 shows this is not true.

4. **Highly composite numbers.** According to Ramanujan n is *highly composite* if $d(n)>d(m)$ for all $m<n$. As before p will denote the largest prime factor of n. Ramanujan proved the following results (among others)(8).

(8) *Collected works*, pp. 86 ff.

(1) *Except for the numbers* 4 *and* 36 *the exponent of* p *is unity.*
(2) *If* $n=2^{k_2}\cdots p^{k_p}$, then $k_2\geqq k_3\geqq\cdots\geqq k_p$.
(3) $k_q=O(\log p)$.

As in the case of superabundant numbers a lemma will be needed.

LEMMA 5. *If* q *is the greatest prime of exponent* k *and if* $q^{1-\theta}>\log p$, *then all primes between* q *and* $q+q^{\theta}$ *have exponent* $k-1$.

By Lemma 3, the number of primes between q and $q+cq^{\theta}$ is asymptotically $cq^{\theta}/\log q$, and by the third result quoted from Ramanujan, this will exceed k if $q^{1-\theta}>\log p$.

We now compare n with $nr_1\cdots r_k/q_1\cdots q_{k+1}$, where the r_i are the primes succeeding r_1, and q_j the predecessors of $q_1=q$. We assume that $r_1<q+q^{\theta}$, so that $r_k<q+3q^{\theta}$. Also we have $q_{k+1}>q-q^{\theta}$. Hence $q_1\cdots q_{k+1}>q^{k+1}(1-q^{\theta-1})^k>q^k(1+3q^{\theta-1})^k>r_1\cdots r_k$; for k is small compared with $q^{1-\theta}$, and if p is large, $q>(1+3q^{\theta-1})^k/(1-q^{\theta-1})^k$.

It follows that $n_1<n$, so that $d(n_1)<d(n)$. If the exponent of r_j is denoted by l_j and the exponent of q_i by k_i, this condition reduces to

$$\left[\prod_{i\leqq k+1}k_i\right]\left[\prod_{j\leqq k}(l_j+2)\right]<\left[\prod_{i\leqq k+1}(k_i+1)\right]\left[\prod_{j\leqq k}(l_j+1)\right].$$

But if $l_1<k-1$, we find that

$$\prod_i(1+1/k_i)\leqq(1+1/k)^{k+1}<(1+1/(k-1))^k<\prod_j(1+1/(l_j+1)),$$

which is a contradiction.

THEOREM 11. *If* q *is the greatest prime of exponent* k, *and if* $q^{1-\theta}>\log p$, *then*

$$\log(1+1/k)=\log q\log 2/\log p+O(1/q^{1-\theta}\log p).$$

Let $q_1=q$, and let $q_2,\cdots,q_x$ be the primes immediately preceding q and all of exponent k. Let $P_1,\cdots,P_y$ be the primes succeeding p. Then $n_0=nP_1\cdots P_y/q_1\cdots q_x$ will be less than n, so that $d(n_0)<d(n)$. Hence $(1+1/k)^x>2^y$ and $\log(1+1/k)>y\log 2/x$.

We choose x so that

$$q_1\cdots q_{x-1}<P_1\cdots P_y<q_1\cdots q_x,$$

and y so that $q_x>q-q^{\theta}$, $P_y<p+p^{\theta}$. This will be so if y is of the order of $q^{\theta}/\log p$. As in Theorem 4 we find that

$$y/x=\log q/\log p+O(\log q/x\log p)+O(1/q^{1-\theta}\log p).$$

Combining these two estimates we find that

$$\log(1+1/k)>\log 2\log q/\log p+O(\log q/x\log p)+O(1/q^{1-\theta}\log p).$$

From the choice of y, the second term can be included in the third.

The inequality in the other direction is proved in a similar fashion, raising the exponents of primes succeeding q and dropping an appropriate number of the largest prime factors.

THEOREM 12. *If the exponent of q is k, and if $q<(\log p)^c$, where c is constant, then*

(i) $$\log (1 + 1/k) > \log q \log 2/\log p + O((\log\log p)^3/(\log p)^3),$$

(ii) $$\log (1 + 1/(k + 1)) < \log q \log 2/\log p + O((\log\log p)^3/(\log p)^3).$$

As in the case of the superabundant numbers, we first let r and s be two large primes: r the greatest of exponent α and s the least of its exponent $\beta-1$: β is chosen so that the least prime t_1 of exponent $\beta-a-1$ is greater than rq, and if t_2 is the least prime of exponent $\beta-a$, then $t_2<rq$. Consequently we have

$$\log (t_2/s) < \log (qr/s) < \log (t_1/s).$$

From Theorem 11 we find that

$$\log (t_1/s) = a \log p/\beta^2 \log 2 + O(\log p/\beta^3),$$
$$\log (t_2/s) = (a - 1) \log p/\beta^2 \log 2 + O(\log p/\beta^3).$$

Hence if s is not too large, and $\epsilon>0$,

$$\log (qr/s) = (c - \theta_0) \log p/\beta^2 \log 2, \quad \text{where} \quad -\epsilon < \theta_0 < 1 + \epsilon.$$

We now take u to be the greatest exponent γ and v the least exponent $\gamma-b-1$. Then for b fixed, $\log (v/u)$ is a function of the integer γ. By Theorem 11 it is easily seen that $\beta\sim\log p/\log 2 \log r$. Take r equal $(\log p)^x$. Then if γ is as large as possible in the range of Theorem 11 we have $\log (v/u) <\log (qr/s)$, while if γ is as small as possible we have $\log (v/u) >\log (qr/s)$. Hence there is a least value of γ such that $\log (v/u) <\log (qr/s)$. If u_1 and v_1 are the primes corresponding to the exponent $\gamma-1$ then $\log (qr/s) <\log (u_1/v_1)$, so that

$$0 < \log (qru/sv) < \log (v_1/u_1) - \log (v/u).$$

But by Theorem 11

$$\log v = \log p \log [(\gamma - b + 1)/(\gamma - b)]/\log 2 + O(1/v^{1-\theta}),$$

and

$$\log u = \log p \log [(\gamma + 1)/\gamma]/\log 2 + O(1/u^{1-\theta}),$$

so that

$$\log (v/u) = \log p \log [\gamma(\gamma - b + 1)/(\gamma - b)(\gamma + 1)]/\log 2 + O(1/u^{1-\theta}),$$
$$\log (v_1/u_1) = \log p \log [(\gamma - 1)(\gamma - b)/(\gamma - b - 1)\gamma]/\log 2 + O(1/u_1^{1-\theta}).$$

It follows that

$$\log (v_1/u_1) - \log v/u = O(b \log p/\gamma^3) + O(1/u^{1-\theta}).$$

If we take $u^{1-\theta} > (\log p)^2$, then the first of these two terms will be the greater. In this case

$$\log v/u \sim b \log p/\gamma^2 \log 2,$$

and as $\log v/u$ must be asymptotic to $\log (qr/s)$, we have

$$\gamma \sim \log pb^{1/2}/x \log 2 \log \log p(c - \theta_0)^{1/2}.$$

From this we deduce that

$$\log u \sim x \log \log p(c - \theta_0)^{1/2}/b^{1/2},$$

so that we must make $x(c-\theta_0)^{1/2}/b^{1/2} > 2/(1-\theta)$. This can be done by taking $c=3$, $b=2$ and x large enough.

We now have the following estimate:

$$0 < \log (qru/sv) < M(\log \log p)^3/(\log p)^2.$$

The number nsv/qru is smaller than n and as n is highly composite we have

$$(k+1)(\alpha+1)\beta(\gamma+1)(\gamma-b) > k\alpha(\beta+1)\gamma(\gamma-b+1).$$

But by Theorem 11 and the estimate of qru/sv, we find that

$$\log (1 + 1/k) > \log 2 \log q/\log p + O((\log \log p)^3/(\log p)^3) + O(1/u^{1-\theta} \log p) + O(1/r^{1-\theta} \log p).$$

By our choice of r and u the last two terms can be included in the first error term. This proves the first inequality, and a similar argument proves the second.

Let ϵ_q be defined as follows:

$$\epsilon_q = O((\log \log p)^3/\log p(\log q)^2), \quad \text{if} \quad q^{1-\theta} \leqq \log p,$$

and

$$\epsilon_q = O(1/q^{1-\theta} \log p) \quad \text{if} \quad q^{1-\theta} > \log p.$$

THEOREM 13. *Let Λ_q and θ_q be the integral and fractional parts of $\{2^{\log q/\log p} - 1\}^{-1}$. Then:*

(1) *if $\epsilon_q < \theta_q \leqq 1 - \epsilon_q$, $k_q = \Lambda_q$;*

(2) *if $\theta_q \leqq \epsilon_q$, k_q is either Λ_q or $\Lambda_q - 1$;*

(3) *if $1 - \epsilon_q < \theta_q$, k_q is either Λ_q or $\Lambda_q + 1$.*

This formula is given by Ramanujan for large q[9]. It follows readily from Theorems 11 and 12.

(9) *Collected works*, pp. 99 ff.

This almost exact determination of the exponents k_q is remarkable, since no analogue of Theorem 2 can be proved for the highly composite numbers. In fact let us consider integers of the form $p^a \cdot q^b$. We can define highly composite numbers in the obvious way, and it can be shown that the exponent of p does not determine the exponent of q with a bounded error. In fact the quotient of two consecutive highly composite numbers of this sequence tends to 1.

As in the case of the superabundants it would no doubt be possible to improve the error term in Theorem 12. Probably the error term could be made $O(1/\log p)^c)$ for every c. However the study of the superior highly composite numbers shows that the error term in Theorem 11 can not be $O(1/q^{1+\epsilon})$. On the other hand it is possible that the error terms of Theorems 11 and 12 are both $O(p^{-c})$ for some $c>0$. To prove this would require again a great deal more than is known about the Diophantine properties of the logarithms of primes.

5. **Highly abundant numbers.** A number n is said to be *highly abundant* if $\sigma(n)>\sigma(m)$ for all $m<n$. Obviously all superabundant numbers are highly abundant, but the converse is not true. It is very likely that there are infinitely many highly abundant numbers which are not superabundant but this we cannot prove.

We define $f(x)$ to be the maximum of $\sigma(n)/n$ for all $n \leqq x$.

THEOREM 14. *If k is any positive constant there exist two absolute constants c_1 and c_2 such that for all sufficiently large x*

$$c_1 k \log\log x/\log x < f(x) - f(x/(\log x)^k) < c_2 k \log\log x/\log x.$$

If x is large and $f(x)=\sigma(n)/n$, then $n\sim x$, by Theorem 8. If $p_1, \cdots, p_r$ are the last r prime factors of n and $k+2>r\geqq k+1$, then by Theorem 7 $n_1=n/p_1\cdots p_r\sim x/(\log x)^r$, so that

$$f(x) - f(x/(\log x)^k) < \sigma(n)/n - \sigma(n_1)/n_1$$
$$= [\sigma(n)/n]\cdot\left[1 - \prod_i \{p_i/(p_i+1)\}\right].$$

It is known that $\sigma(n)/n<c \log\log n$([10]), and if k is fixed then

$$1 - \prod_i \{p_i/(p_i+1)\} \sim 1 - [\log x/(1+\log x)]^r \sim k/\log x.$$

This proves one-half of the inequality.

By adding new prime factors to n the same argument gives the other half.

([10]) S. Wigert, v.s.

THEOREM 15. *If n is highly abundant, then*

$$f(n) - \sigma(n)/n < c_3 \log\log n/\log n.$$

Let x be the greatest superabundant number such that $x<n/\log n$. Choose r so that $rx<n\leqq(r+1)x$. Then

$$\sigma(n)/n > \sigma(rx)/n = [\sigma(rx)/rx](rx/n) > [\sigma(x)/x][(n-x)/n]$$
$$> f(n/\log n)[1 - 1/\log n].$$

But, by Theorem 14, this exceeds $f(n)-c_3 \log\log n/\log n$.

THEOREM 16. *If q^k is the highest power of q dividing the highly abundant number n, then there exists an absolute constant c_4 such that $q^{k+1}>c_4 \log n$.*

COROLLARY. *Every prime $q<c_4 \log n$ divides n.*

To prove this theorem it is only necessary to consider the primes less than $\log n$. By Theorems 14 and 15 we find that

$$f(qn) - \sigma(n)/n < c \log\log n/\log n.$$

But

$$\sigma(qn)/nq^{k+1} = \sigma(qn)/qn - \sigma(n)/n < f(qn) - \sigma(n)/n,$$

and $\sigma(n)/n>c \log\log n$, which completes the proof.

From Theorems 4, 5, and 7 it is easy to see that if n is superabundant, $\epsilon>0$, $n>n(\epsilon)$, and $q>Q(\epsilon)$, then

(5) $(1-\epsilon)\log n \log\log n/q \log q < q^{k_q} < (1+\epsilon)\log n \log\log n/\log q.$

Naturally this need not hold for the highly abundant numbers, but the exceptions will be proved to be few.

If q^{k_q} is the highest power of q dividing the highly abundant number n and if q^{k_q} falls outside the limits of (5), let x_q be the least non-negative integer which must be added to or subtracted from k_q in order to bring the new power of q between the two limits of (5). The powers q^{x_q} will be referred to as the defects of q. We shall prove that the product of all the defects of all primes q is less than $(\log n)^c$.

LEMMA 6. *The number of primes greater than $(1+\epsilon)\log n$ dividing n is bounded by a number depending only on ϵ.*

If the lemma is false, there are arbitrarily large n and r such that there are r primes $p_1, \cdots, p_r$ all greater than $(1+4\epsilon)\log n$ and all dividing n. As is well known, the product of all primes less than $(1+2\epsilon)\log n$ is greater than $n^{1+\epsilon}$ for large n. We can therefore find $r-2$ primes $q_1, \cdots, q_{r-2}$ less than $(1+2\epsilon)\log n$ which do not divide n, and an integer x such that

$$xQ = xq_1 \cdots q_{r-2} < P = p_1 \cdots p_r \leqq (x+1)Q.$$

We compare n with $n_1 = nxQ/P$. A simple calculation shows that

$$\sigma(n_1)/\sigma(n) > xQ \prod_i (1 + 1/q_i) \cdot \prod_j (1 - 1/P_j)/P.$$

But xQ/P can be replaced by $1-1/(\log n)^2$, q_i by $(1+2\epsilon)\log n$, and p_i by $(1+4\epsilon)\log n$. It is then easy to see that if r is large enough, $\sigma(n_1) > \sigma(n)$. This is a contradiction since $n_1 < n$.

LEMMA 7. *If p divides n then $p < (\log n)^3$.*

Suppose that the lemma is false. By Lemma 6 there is a prime q between $(1+\epsilon)\log n$ and $(1+2\epsilon)\log n$ which does not divide n. If x is defined by $xq < p \leqq (x+1)q$, then $n_1 = nxq/p < n$, and we can compare n and n_1. It is easy to see that

$$\sigma(n_1)/\sigma(n) > xq(1 + 1/q)(1 - 1/p)/p > 1$$

if n is large. But this is a contradiction.

THEOREM 17. *The products of the defects of all the primes is less than $(\log n)^c$, where $c = c_5(\epsilon)$.*

We first consider the primes q for which $q^{k_q} \log q > (1+\epsilon)\log n \log\log n$. By Lemma 7, if the product of the defects of these primes is not bounded by $(\log n)^c$, there are arbitrarily large numbers a and highly abundant numbers n such that, for a suitable choice of $y_q \leqq x_q$,

$$Q = \sum_q q^{y_q} = (\log n)^c, \qquad a \leqq c < a + 3.$$

Let $b = [c-2]$. Then if n is large enough, by Lemma 6 we can find primes $p_1, \cdots, p_b$ between $(1+\epsilon/4)\log n$ and $(1+\epsilon/2)\log n$ no one of which divides n. There is an integer x such that

$$xP = xp_1 \cdots p_b < Q \leqq (x+1)P,$$

and it is easy to see that $1 > xP/Q > 1 - 1/(\log n)^2$. We now compare n with $n' = nxP/Q$. It is easy to see that

$$\sigma(n')/\sigma(n) > xP \cdot \prod_i (1 + 1/p_i) \cdot \prod_q (1 - 1/q^{k_q+1-y_q})/Q,$$

where of course q runs through the primes described at the beginning of the proof. We have the following estimates,

$$\prod_i (1 + 1/p_i) > 1 + b/(1 + \epsilon/2)\log n,$$

and

$$\prod_q (1 - 1/q^{k_q+1-y_q}) > 1 - \sum_q \log q/(1+\epsilon)\log n \log\log n$$
$$\geqq 1 - c/(1+\epsilon)\log n.$$

Hence a contradiction will be reached if c is too large.

We turn now to the primes for which $q^{k_q+1} \log q < (1-\epsilon) \log n \log\log n$. Since $k_q \geqq 0$, we have $q \log q < (1-\epsilon) \log n \log\log n$, so that if n is large enough then $q < (1-\epsilon/2) \log n$. For these primes we shall prove the following stronger result:

THEOREM 17′. *If q runs through all primes such that $q^{k_q+1} < (1-\epsilon) \log n \cdot \log\log n/\log q$, then $\sum_q q^{x_q-1} \log q < c(\epsilon) \log\log n$.*

This is obviously much stronger than Theorem 17, since $q^{x_q-1} \geqq x_q$.

By Theorem 16 and the definition of x_q, we know that

$$q^{x_q-1} \log q < (1-\epsilon) \log n \log\log n/q^{k_q+1} < (1-\epsilon) \log\log n/c_4,$$

and therefore we have $\sum_{q<A} q^{x_q-1} \log q < A(1-\epsilon) \log\log n/c_4$. We can consequently restrict attention to the primes between A and $(1-\epsilon/2) \log n$. If the theorem is not true, then there are arbitrarily large numbers a and highly abundant n such that for appropriate $y_q \leqq x_q$, we have

$$\sum_q q^{y_q-1} \log q = c \log\log n, \qquad a \leqq c < a + (1-\epsilon)/c_4.$$

In order to show that this leads to a contradiction, we shall compare n with $nxQ/p_1 \cdots p_b$, where $Q = \prod_q q^{y_q}$, and the p_i are factors of n. It is therefore necessary to know that there are many prime factors of n near $\log n$.

LEMMA 8. *If b is any positive integer, then for sufficiently large n there are more than b prime factors of n between $(1-\epsilon) \log n$ and $(1+\epsilon) \log n$.*

Let $l_q = k_q - x_q$, $z_q = x_q$ if $q^{k_q} \log q > (1+\eta) \log n \log\log n$, and $l_q = k_q$, $z_q = 0$ otherwise; $\sum_q'$ extends over all the primes less than $(1-\epsilon) \log n$, and $\sum_q''$ extends over all larger prime factors of n. Then by the part of Theorem 17 that has just been proved, we have

$$\log n = \sum_q' l_q \log q + \sum_q'' l_q \log q + \sum_q z_q \log q$$
$$< \pi[(1-\epsilon) \log n] \log[(1+\eta) \log n \log\log n] + c_6 \log\log n + \sum_q'' l_q \log q.$$

Hence if $\eta = \epsilon/4$, and n is large, it follows from the prime number theorem that $\sum_q'' l_q \log q > (\epsilon/2) \log n$. In this sum $q > (1-\epsilon) \log n$, and $q^{l_q} < (1+\eta) \log n \log\log n$, so that if n is large enough we must have $l_q = 0$ or 1. Hence we have $\prod_q'' q > n^{\epsilon/2}$. There are only $c(\epsilon)$ primes above $(1+\epsilon) \log n$ which divide n and these are all less than $(\log n)^3$, so that if n is large enough there must be arbitrarily many prime factors of n between $(1-\epsilon) \log n$ and $(1+\epsilon) \log n$.

Now we return to the proof of our theorem. By Lemma 8 we can find

$b=[c+3]\geqq c+2$ prime factors $p_1, \cdots, p_b$ of n between $(1-3\epsilon/4)\log n$ and $(1+3\epsilon/4)\log n$, and an integer x such that $xQ<P=p_1\cdots p_b\leqq(x+1)Q$.

We compare n with $n_1=nxP/Q$, and it is easy to see that

$$\sigma(n_1)/\sigma(n) > xQ\prod_q[(q^{k_q+v_q+1}-1)/(q^{k_q+v_q+1}-q^{v_q})]$$
$$\cdot\prod_p[(p^{k_p+1}-p)/(p^{k_p+1}-1)]/P.$$

If $A=8/\epsilon$, we have the following estimates:

$$xQ/P > (1-c/\epsilon)/(\log n)^2,$$
$$(p^{k_p+1}-p)/(p^{k_p+1}-1) > 1-1/p > 1-1/(1-3\epsilon/4)\log n,$$
$$(q^{k_q+1+v_q}-1)/(q^{k_q+1+v_q}-q^{v_q}) > 1+(1-1/q)/q^{k_q+1}$$
$$> 1+(1-\epsilon/8)q^{v_q-1}\log q/(1-\epsilon)\log n\log\log n.$$

But if these are combined with the equations $b=[c+3]$ and $\sum_q q^{v_q-1}\log q=c\log\log n$, it is easy to see that for sufficiently large c a contradiction will result.

COROLLARY. *There are less than* $c(\epsilon)\log\log n/\log\log\log n$ *primes* q *which do not satisfy the inequality*:

$$(1-\epsilon)\log n\log\log n/q\log q < q^{k_q} < (1+\epsilon)\log n\log\log n/\log q.$$

The proof of this corollary is immediate, by using the prime number theorem. It shows that *only a finite number of highly abundant numbers can be highly composite.*

THEOREM 18. *For large* x *the number of highly abundant numbers less than* x *is greater than* $(1-\epsilon)(\log x)^2$, *for every* $\epsilon>0$.

Let n be highly abundant. The greatest prime factor p of n is greater than $(1-\epsilon)\log n$, so that the number $n'=n(p+1)/p$ exceeds n and $\sigma(n')>(p+1)\sigma(n/p)\geqq\sigma(n)$. Therefore there must be a highly abundant number between these two, and the ratio n_1/n of two consecutive highly abundant numbers is less than $1+1/(1-\epsilon)\log n$. The theorem follows immediately.

By using Theorem 17 it is possible to prove that the number of highly abundant numbers less than x is less than $(\log x)^{c\log\log x}$. This result can no doubt be improved.

An immediate consequence of Theorems 16 and 17 is the following result, which complements Lemma 6.

The number of primes less than $(1-\epsilon)\log n$ *which do not divide* n *is bounded by a number depending only on* ϵ.

THEOREM 19. *The largest prime factor of the highly abundant number n is less than $c \log n(\log\log n)^3$.*

Suppose the theorem to be false. By the corollary to Theorem 15 there is a constant c such that the least prime q satisfying (5) with $\epsilon=1/2$ is less than $c \log\log n$. Let $xq<p\leqq(x+1)q$, $p>d \log n(\log\log n)^3$, and compare n with $n_1=nxq/p$. Then we have

$$\begin{aligned}\sigma(n_1)/\sigma(n) &> xq(q^{k_q+2}-1)(p^{k_p+1}-p)/p(q^{k_q+2}-q)(p^{k_p+1}-1)\\ &> (1-q/p)(1-1/p)(1+1/2q^{k_q+1})\\ &> (1-2c/d \log n(\log\log n)^2)(1+\log 2/3c \log n(\log\log n)^2).\end{aligned}$$

But this is greater than 1 if d is too large.

It is very likely that this result is far from the best possible. It would be interesting to know whether the largest prime factor is asymptotic to $\log n$, or even less than $c \log n$.

THEOREM 20. *If p is the largest prime factor of the highly abundant number $n>216$, then the exponent of p is less than 3.*

Suppose that p^k divides n and that $p\geqq5$, $k\geqq3$. Replace p^2 by $24x=p^2-1$. Then a simple calculation shows that

$$\begin{aligned}\sigma(n_1)/\sigma(n) &> (2^{k_2+4}-1)(3^{k_3+2}-1)\\ &\quad\cdot 24x(p^{k-1}-1)/(2^{k_2+4}-8)(3^{k_3+2}-3)(p^{k+1}-1)\\ &> (1+7/16\cdot 2^{k_2})(1+2/9\cdot 3^{k_3})(1-2/p^2).\end{aligned}$$

This will be greater than 1 if $p^2\geqq46\cdot 2^{k_2}/7$, or if $p^2\geqq11\cdot 3^{k_3}$. Hence the theorem is true unless perhaps $p^2<46\cdot 2^{k_2}/7$ and $p^2<11\cdot 3^{k_3}$.

Since $p\geqq5$, then $k_2\geqq2$, and $k_3\geqq1$. Therefore we can replace $12p^2$ by xp_1, where p_1 is just greater than p and $xp_1<12p^2<(x+1)p_1$. Therefore the following inequality holds:

$$\begin{aligned}\sigma(n_1)/\sigma(n) &> xp_1(2^{k_2+1}-4)(3^{k_3+1}-3)(p_1+1)\\ &\quad\cdot(p^{k+1}-p^2)/12p^2p_1(2^{k_2+1}-1)(3^{k_3+1}-1)(p^{k+1}-1)\\ &> (1-p_1/12p^2)(1-12/7\cdot 2^{k_2})(1-3/4\cdot 3^{k_3})(1+1/p_1)(1-2/p^2).\end{aligned}$$

By Tchebichef's theorem and the estimates for x and p^2, the right-hand side exceeds

$$(1-1/6p)(1-552/49p^2)(1-33/4p^2)(1+1/2p)(1-2/p^2).$$

This is greater than 1 if $p\geqq67$, which is a contradiction. Thus the theorem is proved for all $p\geqq67$, and direct calculations and our tables show that 216 is the largest exceptional n.

We can prove without difficulty that 7200 is the largest highly abundant

number such that all its prime factors occur to powers greater than 1. Also 16 is the only highly abundant number where the largest prime factor occurs with exponent 4, and it never occurs with exponent 5 or more.

At present we cannot prove that, except for a finite number of cases, the exponent of the largest prime factor is unity. However, it is possible to do this if the following strong assumption about the distribution of primes is true: *in the arithmetic progression $ax+b$, if n is large, there is a prime with x between n and $n+n^{1/2-\epsilon}$.*

Nor can we prove that if $n=2^{k_2}\cdots p^{k_p}$, then $k_2\geqq\cdots\geqq k_p$. In fact, it seems quite likely that if n is superabundant, then np_1/p is highly abundant, where p_1 is the prime just greater than p.

6. **Some general remarks.** The functions ϕ and σ lead to a few other problems, but these seem to be of a much more trivial nature.

(1) *If n is such that for all $m<n$, $\sigma(m)/m>\sigma(n)/n$, then n is a prime.*

If n is not a prime it has a factor less than or equal to $n^{1/2}$, and then $\sigma(n)/n\geqq 1+1/n^{1/2}$. There is always a prime p between $n^{1/2}$ and n, and clearly $\sigma(p)/p=1+1/p\leqq 1+1/n^{1/2}$.

(2) *If $\phi(n)/n>\phi(m)/m$ for all $m<n$, then m is a prime.*

The proof is the same as for (1).

(3) *Let n be such that $\phi(n)\geqq\phi(m)$ for all $m<n$ and n greater than 4. Then if we assume that there is always a prime between x and $x-x^{1/2}+1$, n must be a prime.*

If n is not a prime it has a prime factor less than or equal to $n^{1/2}$, and if $n-n^{1/2}+1<p<n$, then

$$\phi(p) > n - n^{1/2} \geqq \phi(n).$$

(4) *If n is such that $\sigma(n)<\sigma(m)$ for $m<n$ and if there is a prime p between n and $n+n^{1/2}-2$, then $n=p$.*

The proof is the same as in (4).

(5) *If n is such that $\phi(n)/n<\phi(m)/m$ for all $m<n$, then $n=2\cdot 3\cdots p$.* (The proof is obvious.)

(6) Let n be such that $\phi(n)<\phi(m)$ for all $m>n$. These numbers are not trivial, but they seem to have a simpler structure than the highly abundant numbers. It is easy to see that the quotient of two consecutive numbers of this type tends to 1.

(7) Put $\sigma_r(n)=\sum_{d/n}d^r$. If $r>0$, the numbers n for which $\sigma_r(m)<\sigma_r(n)$ for $m<n$ are analogues of the highly abundant numbers, while if $r<0$ we get the analogues of the superabundant numbers. These numbers satisfy conditions similar to the highly abundant and superabundant numbers. If $r=0$ we get the highly composite numbers.

The highly composite numbers can be characterized as follows: *n is highly composite if and only if for some $\epsilon>0$, all $-\epsilon<r<0$, and for all $m<n$, $\sigma_r(m)<\sigma_r(n)$.* The proof is simple.

In case $r>0$ is used, we get the numbers n such that for all $m<n$, $d(n)\geqq d(m)$. We do not know whether there are infinitely many such numbers which are not highly composite. A similar problem arises for the functions $\sigma_r(n)$.

(8) If $f(n)$ is an increasing function we can consider numbers n such that $f(n)>f(m)$ for all $m<n$. Roughly speaking, the number of "highly abundant" numbers becomes greater if the rate of growth of $f(n)$ increases. In connection with this, the following results are true[11]:

(a) *for* $2^n d(n)$, *the density of "highly abundants" is zero*;

(b) *for* $2^n\sigma(n)$, *the density exists, is positive, and is different from* 1;

(c) *for* $2^n\nu(n)$, *where* $\nu(n)$ *is the number of prime divisors of* n, *the density is* 1.

(9) "Highly abundant" numbers can be defined for a general additive function $f(n)$. It seems very difficult to get any general results. If $f(n)\neq c\log n$, then the "highly abundant" numbers have density 0.

The proof of this is difficult and will be discussed elsewhere.

If $f(p^\alpha)=p^\alpha$, then the "highly abundants" are the primes and their powers. In general it is not true that the quotient of two consecutive "highly abundants" tends to 1; if $f(p^\alpha)=1$, then the "highly abundants" are the powers of 2. Another example is $f(p^\alpha)=2^{-p^\alpha}$.

7. **Tables.** The table for highly abundant numbers was obtained from Glaisher's table of $\sigma(n)$ to 10,000[12]. The superabundant numbers where computed by first limiting the numbers to be considered by preliminary computations, and then by the calculation of $\sigma(n)/n$. We give a sample of this preliminary computation.

The least superabundant number divisible by 2^{11} must be divisible by $3^6, 5^4, 7^3, 11^2, \cdots, 19^2, 23, \cdots, 227$. If 3^5 or a lower power were to occur, we compare $2^{11}\cdot 3^5$ with $2^9\cdot 3^6$. Similarly, in checking the power of 5 we drop a factor of 6 and add the factor 5, and for the other primes: add 7 and drop 10, add 19 and drop 20, add 227 and drop 228. This number exceeds $3\cdot 10^{100}$.

The colossally abundant numbers were determined directly from the definition, by computing the critical values of ϵ. For example, the first colossally abundant number divisible by 2^{11} occurs for $\epsilon=\log(4095/4094)/\log 2 = 0.00035232$, and its factorization is $2^{11}\cdot 3^6\cdot 5^4\cdot 7^3\cdot 11^2\cdots 29^2\cdots 461$.

It can be shown that 210 is the greatest highly abundant number in which 2 occurs with exponent 1. As we remarked before 7200 is the greatest highly abundant number all prime factors of which have exponents greater than 1, so that 7200 is the greatest for which $\sigma(n)$ is odd. It would be easy to see

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[11] For the method of proof, see P. Erdös and M. Kac, *The Gaussian law of errors in the theory of additive number theoretic functions*, Amer. J. Math. vol. 52 (1940) pp. 738–742 and P. Erdös, *On a problem of Chowla*, Proc. Cambridge Philos. Soc. vol. 32 (1936) pp. 530–540.

[12] *Ibid.*

that neither $n!$ nor the least common multiple of the first n integers can be highly abundant infinitely often. Also if n is highly abundant then $\sigma(n)$ can not be highly abundant infinitely often. The reason for this is that $\sigma(n)$ is divisible by an excessively high power of 2.

As far as the table of superabundants extends there are numbers n and $2n$ with no superabundant between them. Theorem 8 shows that this occurs only a finite number of times.

An inspection of the tables suggests the following conjecture: *if n is superabundant then there are two primes p and q such that both np and n/q are superabundant.* The same seems to be true for highly abundant and highly composite numbers.

HIGHLY ABUNDANT NUMBERS LESS THAN 10^4

n	Factors	$\sigma(n)$	n	Factors	$\sigma(n)$	n	Factors	$\sigma(n)$
2	2	3	288	$2^5\cdot 3^2$	819	2520	$2^3\cdot 3^2\cdot 5\cdot 7$	9360
3	3	4	300	$2^2\cdot 3\cdot 5^2$	868	2880	$2^6\cdot 3^2\cdot 5$	9906
4	2^2	7	336	$2^4\cdot 3\cdot 7$	992	3024	$2^4\cdot 3^3\cdot 7$	9920
6	$2\cdot 3$	12	360	$2^3\cdot 3^2\cdot 5$	1170	3120	$2^4\cdot 3\cdot 5\cdot 13$	10416
8	2^3	15	420	$2^2\cdot 3\cdot 5\cdot 7$	1344	3240	$2^3\cdot 3^4\cdot 5$	10890
10	$2\cdot 5$	18	480	$2^5\cdot 3\cdot 5$	1512	3360	$2^5\cdot 3\cdot 5\cdot 7$	12096
12	$2^2\cdot 3$	28	504	$2^3\cdot 3^2\cdot 7$	1560	3600	$2^4\cdot 3^2\cdot 5^2$	12493
16	2^4	31	540	$2^2\cdot 3^3\cdot 5$	1680	3780	$2^2\cdot 3^3\cdot 5\cdot 7$	13440
18	$2\cdot 3^2$	39	600	$2^3\cdot 3\cdot 5^2$	1860	3960	$2^3\cdot 3^2\cdot 5\cdot 11$	14040
20	$2^2\cdot 5$	42	630	$2\cdot 3^2\cdot 5\cdot 7$	1872	4200	$2^3\cdot 3\cdot 5^2\cdot 7$	14880
24	$2^3\cdot 3$	60	660	$2^2\cdot 3\cdot 5\cdot 11$	2016	4320	$2^5\cdot 3^3\cdot 5$	15120
30	$2\cdot 3\cdot 5$	72	720	$2^4\cdot 3^2\cdot 5$	2418	4620	$2^2\cdot 3\cdot 5\cdot 7\cdot 11$	16120
36	$2^2\cdot 3^2$	91	840	$2^3\cdot 3\cdot 5\cdot 7$	2880	4680	$2^3\cdot 3^2\cdot 5\cdot 13$	16380
42	$2\cdot 3\cdot 7$	96	960	$2^6\cdot 3\cdot 5$	3048	5040	$2^4\cdot 3^2\cdot 5\cdot 7$	19344
48	$2^4\cdot 3$	124	1008	$2^4\cdot 3^2\cdot 7$	3224	5760	$2^7\cdot 3^2\cdot 5$	19890
60	$2^2\cdot 3\cdot 5$	168	1080	$2^3\cdot 3\cdot 5$	3600	5880	$2^3\cdot 3\cdot 5\cdot 7^2$	20520
72	$2^3\cdot 3^2$	195	1200	$2^4\cdot 3\cdot 5^2$	3844	6120	$2^3\cdot 3^2\cdot 5\cdot 17$	21060
84	$2^2\cdot 3\cdot 7$	224	1260	$2^2\cdot 3^2\cdot 5\cdot 7$	4368	6240	$2^5\cdot 3\cdot 5\cdot 13$	21168
90	$2\cdot 3^2\cdot 5$	234	1440	$2^5\cdot 3^2\cdot 5$	4914	6300	$2^2\cdot 3^2\cdot 5^2\cdot 7$	22568
96	$2^5\cdot 3$	252	1560	$2^3\cdot 3\cdot 5\cdot 13$	5040	6720	$2^6\cdot 3\cdot 5\cdot 7$	24384
108	$2^2\cdot 3^3$	280	1620	$2^2\cdot 3^4\cdot 5$	5082	7200	$2^5\cdot 3^2\cdot 5^2$	25389
120	$2^3\cdot 3\cdot 5$	360	1680	$2^4\cdot 3\cdot 5\cdot 7$	5952	7560	$2^3\cdot 3^3\cdot 5\cdot 7$	28800
144	$2^4\cdot 3^2$	403	1800	$2^3\cdot 3^2\cdot 5^2$	6045	7920	$2^4\cdot 3^2\cdot 5\cdot 11$	29016
168	$2^3\cdot 3\cdot 7$	480	1980	$2^2\cdot 3^2\cdot 5\cdot 11$	6552	8400	$2^4\cdot 3\cdot 5^2\cdot 7$	30752
180	$2^2\cdot 3^2\cdot 5$	546	2100	$2^2\cdot 3\cdot 5^2\cdot 7$	6944	8820	$2^2\cdot 3^2\cdot 5\cdot 7^2$	31122
210	$2\cdot 3\cdot 5\cdot 7$	576	2280	$2^3\cdot 3\cdot 5\cdot 19$	7200	9240	$2^3\cdot 3\cdot 5\cdot 7\cdot 11$	34560
216	$2^3\cdot 3^3$	600	2340	$2^2\cdot 3^2\cdot 5\cdot 13$	7644	10080	$2^5\cdot 3^2\cdot 5\cdot 7$	39312
240	$2^4\cdot 3\cdot 5$	744	2400	$2^5\cdot 3\cdot 5^2$	7812			

SUPERABUNDANT NUMBERS LESS THAN 10^{18}

Factorization of n	$\sigma(n)/n$	Factorization of n	$\sigma(n)/n$
2	1.500[13]	$2^5\cdot3^2\cdot5^2\cdot7\cdot11\cdot13$	4.855[13]
2^2	1.750	$2^4\cdot3^3\cdot5\cdot7\cdot11\cdot13\cdot17$	4.896
$2\cdot3$	2.000[13]	$2^4\cdot3^2\cdot5^2\cdot7\cdot11\cdot13\cdot17$	4.933
$2^2\cdot3$	2.333[13]	$2^5\cdot3^3\cdot5\cdot7\cdot11\cdot13\cdot17$	4.975
$2^3\cdot3$	2.500	$2^5\cdot3^2\cdot5^2\cdot7\cdot11\cdot13\cdot17$	5.013
$2^2\cdot3^2$	2.527	$2^6\cdot3^3\cdot5\cdot7\cdot11\cdot13\cdot17$	5.015
$2^4\cdot3$	2.583	$2^4\cdot3^3\cdot5^2\cdot7\cdot11\cdot13\cdot17$	5.059
$2^2\cdot3\cdot5$	2.800[13]	$2^5\cdot3^3\cdot5^2\cdot7\cdot11\cdot13\cdot17$	5.141[13]
$2^3\cdot3\cdot5$	3.000[13]	$2^4\cdot3^3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot19$	5.154
$2^2\cdot3^2\cdot5$	3.033	$2^6\cdot3^3\cdot5^2\cdot7\cdot11\cdot13\cdot17$	5.182
$2^4\cdot3\cdot5$	3.100	$2^5\cdot3^4\cdot5^2\cdot7\cdot11\cdot13\cdot17$	5.184
$2^3\cdot3^2\cdot5$	3.250[13]	$2^4\cdot3^2\cdot5^2\cdot7\cdot11\cdots19$	5.193[14]
$2^4\cdot3^2\cdot5$	3.358	$2^5\cdot3^3\cdot5\cdot7\cdot11\cdots19$	5.237
$2^3\cdot3\cdot5\cdot7$	3.428	$2^5\cdot3^2\cdot5^2\cdot7\cdot11\cdots19$	5.276
$2^2\cdot3^2\cdot5\cdot7$	3.466	$2^6\cdot3^3\cdot5\cdot7\cdot11\cdots19$	5.279
$2^4\cdot3\cdot5\cdot7$	3.542	$2^4\cdot3^3\cdot5^2\cdot7\cdot11\cdots19$	5.326
$2^3\cdot3^2\cdot5\cdot7$	3.714[13]	$2^5\cdot3^3\cdot5^2\cdot7\cdot11\cdots19$	5.412[13]
$2^4\cdot3^2\cdot5\cdot7$	3.838[13]	$2^6\cdot3^3\cdot5^2\cdot7\cdot11\cdots19$	5.455
$2^5\cdot3^2\cdot5\cdot7$	3.900	$2^5\cdot3^4\cdot5^2\cdot7\cdot11\cdots19$	5.457
$2^4\cdot3^3\cdot5\cdot7$	3.936	$2^7\cdot3^3\cdot5^2\cdot7\cdot11\cdots19$	5.476
$2^4\cdot3^2\cdot5^2\cdot7$	3.966	$2^6\cdot3^4\cdot5^2\cdot7\cdot11\cdots19$	5.500
$2^3\cdot3^2\cdot5\cdot7\cdot11$	4.051	$2^5\cdot3^3\cdot5^2\cdot7^2\cdot11\cdots19$	5.508
$2^4\cdot3^2\cdot5\cdot7\cdot11$	4.187[13]	$2^4\cdot3^3\cdot5^2\cdot7\cdot11\cdots23$	5.557
$2^5\cdot3^2\cdot5\cdot7\cdot11$	4.254	$2^5\cdot3^3\cdot5^2\cdot7\cdot11\cdots23$	5.647[13]
$2^4\cdot3^3\cdot5\cdot7\cdot11$	4.294	$2^6\cdot3^3\cdot5^2\cdot7\cdot11\cdots23$	5.692[13]
$2^4\cdot3^2\cdot5^2\cdot7\cdot11$	4.326	$2^5\cdot3^4\cdot5^2\cdot7\cdot11\cdots23$	5.694
$2^5\cdot3^3\cdot5\cdot7\cdot11$	4.368	$2^7\cdot3^3\cdot5^2\cdot7\cdot11\cdots23$	5.714
$2^5\cdot3^2\cdot5^2\cdot7\cdot11$	4.396	$2^6\cdot3^4\cdot5^2\cdot7\cdot11\cdots23$	5.739
$2^6\cdot3^3\cdot5\cdot7\cdot11$	4.398	$2^5\cdot3^3\cdot5^2\cdot7^2\cdot11\cdots23$	5.748
$2^4\cdot3^2\cdot5\cdot7\cdot11\cdot13$	4.509[13]	$2^7\cdot3^4\cdot5^2\cdot7\cdot11\cdots23$	5.762
$2^5\cdot3^2\cdot5\cdot7\cdot11\cdot13$	4.581[13]	$2^6\cdot3^3\cdot5^2\cdot7^2\cdot11\cdots23$	5.793
$2^4\cdot3^3\cdot5\cdot7\cdot11\cdot13$	4.624	$2^5\cdot3^4\cdot5^2\cdot7^2\cdot11\cdots23$	5.796
$2^4\cdot3^2\cdot5^2\cdot7\cdot11\cdot13$	4.659	$2^7\cdot3^3\cdot5^2\cdot7^2\cdot11\cdots23$	5.816
$2^5\cdot3^3\cdot5\cdot7\cdot11\cdot13$	4.699[13]	$2^5\cdot3^3\cdot5^2\cdot7\cdot11\cdots29$	5.84225
$2^5\cdot3^2\cdot5^2\cdot7\cdot11\cdot13$	4.734	$2^6\cdot3^4\cdot5^2\cdot7^2\cdot11\cdots23$	5.84226
$2^6\cdot3^3\cdot5\cdot7\cdot11\cdot13$	4.736	$2^6\cdot3^3\cdot5^2\cdot7\cdot11\cdots29$	5.888[13]
$2^4\cdot3^3\cdot5^2\cdot7\cdot11\cdot13$	4.778	$2^5\cdot3^4\cdot5^2\cdot7\cdot11\cdots29$	5.890

[13] Colossally abundant.

[14] This is the first superabundant which is not highly composite.

Factorization of n	$\sigma(n)/n$
$2^7 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdots 29$	5.911
$2^6 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdots 29$	5.937
$2^5 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 29$	5.946
$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdots 29$	5.961
$2^6 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 29$	5.993
$2^5 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 29$	5.996
$2^7 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 29$	6.017
$2^5 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdots 31$	6.030
$2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 29$	6.043
$2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdots 31$	6.078[13]
$2^5 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdots 31$	6.080
$2^7 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdots 31$	6.102
$2^6 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdots 31$	6.129
$2^5 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 31$	6.138
$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdots 31$	6.153
$2^6 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 31$	6.187[13]
$2^5 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 31$	6.189

Factorization of n	$\sigma(n)/n$
$2^7 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 31$	6.211
$2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 31$	6.238[13]
$2^6 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdots 37$	6.242
$2^7 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 31$	6.263
$2^7 \cdot 3^3 \cdot 5^2 \cdot 7 \cdot 11 \cdots 37$	6.267
$2^8 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 31$	6.275
$2^6 \cdot 3^4 \cdot 5^3 \cdot 7^2 \cdot 11 \cdots 31$	6.278
$2^6 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdots 37$	6.294
$2^5 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 37$	6.304
$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdots 37$	6.319
$2^6 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 37$	6.354
$2^5 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 37$	6.356
$2^7 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 37$	6.379
$2^6 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 37$	6.407[13]
$2^7 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 37$	6.432
$2^8 \cdot 3^4 \cdot 5^2 \cdot 7^2 \cdot 11 \cdots 37$	6.445

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