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On a class of Thue-Morse type sequences

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Abstract

We consider a class of binary sequences that generalize the Thue-Morse sequence. In particular, we investigate the occurrences of palindromes in such sequences. We also introduce the notion of the first difference of a binary sequence and characterize first differences of our class of Thue-Morse type sequences. Finally, we define the concept of a "change sequence" of a given binary sequence, a sequence which encodes the positions at which a binary sequence changes values. We characterize the change sequences corresponding to our class of Thue-Morse type sequences.

1 Introduction

The Thue-Morse sequence

$$\{t(n)\}_{n=0}^{\infty} = 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, \ldots$$

is defined by $t(n) = 0$ if n has an even sum of binary digits, and $t(n) = 1$ otherwise. This sequence has attracted much attention since its discovery by Axel Thue in the early 1900's, and is still the focus of much study. The Thue-Morse sequence can be constructed in a surprising variety of ways and has numerous applications in diverse fields such as differential geometry, algebra, number theory, and physics; see, for example, [3].

The Thue-Morse sequence can be generalized as follows: For $k \geq 2$, define $s_k(n)$ as the sum of digits in the base-k representation of a nonnegative integer n, and let $t_k(n) =$

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$s_k(n)$ mod 2. The sequence $\{t_k(n)\}_{n=0}^{\infty}$ is thus a binary sequence, and the special case $k = 2$ yields the Thue-Morse sequence $t_2(n) = t(n)$.

In a recent paper, Allouche and Shallit [4] investigated palindromes in the sequences $\{t_k(n)\}$. Here a palindrome is defined in the usual sense: a sequence digits is a palindrome if it reads the same forward and backward. For example, in the Thue-Morse sequence, the first four terms 0,1,1,0 form a palindrome, as do the first 16 terms. Allouche and Shallit proved the following result:

Theorem A (Allouche and Shallit [4]). *For all $k \geq 2$, the sequence $\{t_k(n)\}_{n=0}^{\infty}$ contains palindromes of arbitrary length.*

In our main result, Theorem 5.1, we generalize this result to a larger class of binary sequences (which contains the sequences $\{t_k(n)\}$). These sequences also can be defined as fixed points of a class of mappings on binary words.

In the course of proving this result, we introduce the concept of the first difference of a binary sequence. In Theorem 4.1 we characterize the first differences of our class of Thue-Morse type sequences. The characterization involves another class of maps, so-called Toeplitz maps, which we study in Section 2.

We relate the first difference to another concept, that of a change sequence of a binary sequence. This is a sequence which encodes the positions at which the sequence changes values. In a recent paper, Allouche et al. [1] determined the change sequence of the Thue-Morse sequence and proved the following result:

Theorem B (Allouche et al. [1]). *Let S be the set of integers n such that $t_2(n-1) \neq t_2(n)$, i.e.,*

$$S = \{1, 3, 4, 5, 7, 9, 11, 12, 13, 15, \ldots\}.$$

Then S is the set of integers n such that an even power of 2 exactly divides n.

In Theorem 3.1 we generalize this result to our class of Thue-Morse type sequences.

2 Notation and Preliminaries

We consider the set Σ^* of finite words over the alphabet $\Sigma = \{0, 1\}$. We also consider the set Σ^∞ of infinite and finite words over Σ. For a finite word w we let $|w|$ denote the length of w, and we set $|w| = \infty$ if w is an infinite word. We denote the ith letter of w by $w(i)$, i.e., $w(i) = t_i$ if $w = t_1 t_2 \ldots t_b$, where $t_k \in \Sigma$ for all k. Given two words w_1 and w_2, we let $w_1 w_2$ denote the word obtained by the concatenation of w_2 to the right of w_1. The concatenation of an arbitrary finite or infinite sequence of words is denoted analogously. We define the *complement* of w to be the word obtained by interchanging the letters 0 and 1 in w, or equivalently, by adding 1 modulo 2 to each letter of w. We denote the complement of w by $\overline{w}$.

We next define two morphisms, ϕ_w and ψ_w, on Σ^∞.

Let w be a word with $|w| \geq 2$. We let $\phi_w : \Sigma^\infty \to \Sigma^\infty$ be the morphism defined by

$$\phi_w(0) = w,$$
$$\phi_w(1) = \overline{w}.$$

It is not hard to see that if $|w| \geq 2$ and $w(1) = 0$, then there exists a unique fixed point beginning with 0 of the morphism ϕ_w; i.e., there is a unique infinite word $\mathbf{w}$ beginning with 0 such that

$$\mathbf{w} = \phi_w(\mathbf{w}).$$

Furthermore, $\mathbf{w}$ can be obtained by iterating ϕ_w:

$$\mathbf{w} = \phi_w^\infty(0) := \lim_{n\to\infty} \phi_w^n(0).$$

For example, if $w = 01$, then $\mathbf{w}$ is the Thue-Morse sequence mentioned in the introduction. The morphisms ϕ_w are special cases of so-called symmetric morphisms; see [7].

We also consider a second morphism $\psi_w : \Sigma^\infty \to \Sigma^\infty$ defined by

$$\psi_w(a) = wa,$$

where $a \in \Sigma$. Such a morphism is a special case of the *Toeplitz morphisms* (see [2, 5]). It is not hard to see that there is a unique fixed point of ψ_w beginning with $w(1)$; i.e., there is a unique infinite word w_* with $w_*(1) = w(1)$ such that

$$w_* = \psi_w(w_*),$$

and we can obtain w_* by iterating ψ_w:

$$w_* = \psi_w^\infty(w(1)) := \lim_{n\to\infty} \psi_w^n(w(1)).$$

The fixed point w_* is of the form

$$w_* = w\ w_*(1)\ w\ w_*(2)\ w\ w_*(3) \cdots .$$

This allows one to construct w_* by starting with a sequence of the form

$$w\ \_\ w\ \_\ w\ \_\ \cdots$$

and successively filling in the "holes" with the terms of this sequence.

Allouche et al. [1, pp. 456–458] showed that if $w = 101$, then w_* encodes the places at which the Thue-Morse sequence changes values, i.e., $w_*(n) = 1$ if and only if $t(n) \neq t(n-1)$.

In Sections 4 and 5 we shall need an operation $\oplus$ on the set Σ^* that is defined as follows: Let w and v be binary words of the same length b. Then

$$w_1 \oplus w_2 = (w_1(1) + w_2(1))(w_1(2) + w_2(2)) \cdots (w_1(b) + w_2(b)),$$

where addition is taken modulo 2.

To conclude this section, we define a product operation $\times$ on Σ^* as follows: For any words $w, v \in \Sigma^*$ with $|v| = c$, set

$$w \times 0 = w, \quad w \times 1 = \overline{w}, \tag{2.1}$$

and

$$w \times v = (w \times v(1))(w \times v(2)) \cdots (w \times v(c)). \tag{2.2}$$

This operation was introduced by Jacobs in [11] and was generalized by Hoit [9, 10] to words over alphabets $\{0, 1, \ldots, m\}$. This product is closely related to the morphisms ϕ_w defined above; indeed, it is clear that $w \times v = \phi_w(v)$.

3 Change Sequences

Let C be the set of indices n such that $t_2(n-1) \neq t_2(n)$, where

$$\{t_2(n)\}_{n=0}^{\infty} = \{0,1,1,0,1,0,0,1,1,0,0,1,0,1,1,0,\ldots\}$$

is the Thue-Morse sequence. As stated in the Introduction (see Theorem B), the set C is characterized by the property that $n \in C$ if and only if an even power of 2 exactly divides n. In this section we generalize this result to the class of all fixed points of the morphisms ϕ_w. To this end, we introduce the concept of a change sequence of a word as follows:

Definition 3.1. Let w be a finite or infinite word with $|w| \geq 2$. We define the change sequence C_w of w as the sequence of $n \in \{1,2,\ldots,|w|-1\}$ such that $w(n) \neq w(n+1)$.

For example, if $w = 0010$ and $\mathbf{w}$ is the associated infinite word, i.e.,

$$\mathbf{w} = 0010001011010010\cdots,$$

then $C_w = \{2,3\}$ and $C_{\mathbf{w}} = \{2,3,6,7,8,10,11,12,\ldots\}$.

In the following theorem we give a general method for determining $C_{\mathbf{w}}$ for an arbitrary fixed points $\mathbf{w}$.

Theorem 3.1. *Let w be a word with $|w| = b \geq 2$ such that $w(1) = w(b) = 0$, let $\mathbf{w} = \phi_w^{\infty}(0)$, and let C_w be the change sequence of w. Then $C_{\mathbf{w}} = \{n \in \mathbb{N} : d_b(n) \in C_w\}$, where $d_b(n)$ is the last non-zero digit in the base-b representation of n.*

Proof. Let $w = t_1 t_2 \cdots t_b$. Since for $1 \leq r \leq b-1$ we have $d_b(n) = r$ if and only if $n = b^i(bk+r)$ for some $i,k \geq 0$, it suffices to show that

$$C_{\mathbf{w}} = \{b^i(bk+r) : i,k \geq 0, \quad r \in C_w\}. \tag{3.1}$$

To prove this, it suffices to show the following two equivalences:

$$r \in C_w \iff bk+r \in C_{\mathbf{w}} \qquad (k \geq 0,\ 1 \leq r \leq b-1). \tag{3.2}$$

$$m \in C_{\mathbf{w}} \iff bm \in C_{\mathbf{w}} \qquad (m \geq 1). \tag{3.3}$$

To prove (3.2), notice that $\mathbf{w} = w_0 w_1 w_2 w_3 \ldots$ with $w_k \in \{w, \overline{w}\}$ for all k. Clearly we have $C_w = C_{w_k}$ for all k. Since the words w_k all have length b, for $1 \leq r \leq b$ the $(bk+r)$th letter in $\mathbf{w}$ is equal to the rth letter in w_k. Hence, for $1 \leq r \leq b-1$ and all k, we have $\mathbf{w}(bk+r) \neq \mathbf{w}(bk+r+1)$ if and only if $w_k(r) \neq w_k(r+1)$; i.e., $bk+r \in C_{\mathbf{w}}$ if and only if $r \in C_{w_k} = C_w$. This proves (3.2).

To show that (3.3) holds, fix $m \geq 1$ and choose $j \geq 1$ such that $1 \leq m \leq b^j$. We set

$$v = \phi_w^j(t_1) = t_1 t_2 \cdots t_{b^j}$$

and note that $m \in C_{\mathbf{w}}$ if and only if $m \in C_v$. By definition, $m \in C_v$ if and only if $t_m \neq t_{m+1}$, which is equivalent to

$$\phi_w(t_m) = \overline{\phi_w(t_{m+1})}.$$

Now consider

$$\begin{aligned}\phi_w(v) &= \phi_w^{j+1}(t_1)\\ &= \phi_w(t_1)\phi_w(t_2)\cdots\phi_w(t_m)\phi_w(t_{m+1})\cdots\phi_w(t_{b^j})\\ &= \phi_w(t_1)\phi_w(t_2)\cdots\phi_w(t_m)\overline{\phi_w(t_m)}\cdots\phi_w(t_{b^j}).\end{aligned}$$

Since $|\phi_w(t_i)| = b$ for all i and $\phi_w(t_m)$ begins and ends with the same letter, it follows that $m \in C_{\mathbf{w}}$ is equivalent to $bm \in C_{\phi_w(v)} \subset C_{\mathbf{w}}$. □

Remark. Theorem 3.1 requires that w begins and ends with a 0. However, it is easy to check that for any word w with $w(1) = 0$, the word $v = \phi_w(w) = \phi_w(\phi_w(0))$ both begins and ends with a 0. Thus we can find the change sequence of $\mathbf{w}$ by applying Theorem 3.1 to the word v and noting that $\mathbf{v} = \lim_{n\to\infty} \phi_v^n(0) = \lim_{n\to\infty} \phi_w^n(0) = \mathbf{w}$.

By the result of Allouche et al. quoted in the Introduction (Theorem B), the change sequence of $\mathbf{01}$ is given by

$$C_{\mathbf{01}} = \{n = 2^{2i}m : i \geq 0, \quad m \text{ odd}\}. \tag{3.4}$$

We now verify this with Theorem 3.1. Since $w = 01$ ends in a 1, we must apply the theorem to $v = \phi_{01}(01) = 0110$. We have $|v| = 4$ and $C_v = \{1, 3\}$, and so by Theorem 3.1 it follows that

$$\begin{aligned}C_{\mathbf{01}} &= \{n = 4^i(4k + r) : i, k \geq 0, \quad r \in \{1, 3\}\}\\ &= \{n = 4^i m : i \geq 0, \quad m \text{ odd}\}\\ &= \{n = 2^{2i}m : i \geq 0, \quad m \text{ odd}\},\end{aligned}$$

which proves (3.4).

4 First Differences

In this section we define the concept of the first difference of a word, and we determine the first differences of words $\phi_w^\infty(0)$ for $w(1) = 0$. First Differences of words over a general alphabet have been used implicitly in a recent paper of Frid [7, pp. 359–360]. For first differences of infinite integer sequences in a different context, see Chalice [6].

Definition 4.1. We define the first difference of a finite or infinite word w to be the word

$$\Delta w = (w_1(2) - w_1(1))(w_1(3) - w_1(2))\cdots,$$

where subtraction is interpreted modulo 2.

Note that $|\Delta w| = |w| - 1$ for w of finite length. The following lemma relates the concept of a first difference to that of the change sequence introduced in Section 3. The proof is trivial.

Lemma 4.1. *Let w be a word. Then $\Delta w(i) = 1$ if and only if $i \in C_w$.*

Recall the sum operator $\oplus$, defined in Section 2 as the term-wise addition of two words modulo 2. The next lemma states the relationship between the first difference of a sum and the sum of the first differences:

Lemma 4.2. *Let w_1 and w_2 be finite words. Then $\Delta(w_1 \oplus w_2) = \Delta w_1 \oplus \Delta w_2$.*

We now turn to the first differences of fixed points $\mathbf{w}$. Notice, for example, that if $w = 011010$, then $\Delta w = 10111$. Now consider

$$\phi_w(w) = 011010\ 100101\ 100101\ 011010\ 100101\ 011010.$$

Computing the first difference of $\phi_w(w)$, we obtain

$$\begin{aligned}\Delta(\phi_w(w)) &= 10111\ \underline{1}\ 10111\ \underline{0}\ 10111\ \underline{1}\ 10111\ \underline{1}\ 10111\ \underline{1}\ 10111\\ &= \Delta w\ \underline{\Delta w(1)}\ \Delta w\ \underline{\Delta w(2)}\ \Delta w\ \underline{\Delta w(3)}\ \Delta w\ \underline{\Delta w(4)}\ \Delta w\ \underline{\Delta w(5)}\ \Delta w\\ &= \psi_{\Delta w}(\Delta w)\Delta w,\end{aligned}$$

where $\psi_{\Delta w}$ is the map introduced in Section 2. It is not hard to see that the latter word $\psi_{\Delta w}(\Delta w)\Delta w$ agrees with the word $\psi_{\Delta w}(\psi_{\Delta w}(\Delta w(1)))$ in all but the rightmost position. Since $\phi_w^\infty(0) = \mathbf{w}$ and $\psi_{\Delta w}^\infty(\Delta w(1)) = (\Delta w)_*$, this suggests the following connection between $\Delta\mathbf{w}$ and $(\Delta w)_*$:

Theorem 4.1. *Let w be a word with $|w| = b \geq 2$ and $w(1) = w(b) = 0$. Then the first difference of the fixed point of the morphism ϕ_w is exactly the fixed point of $\psi_{\Delta w}$, i.e., $\Delta\mathbf{w} = (\Delta w)_*$.*

Proof. It is sufficient to show that $\Delta\mathbf{w}(i) = 1$ if and only if $(\Delta w)_*(i) = 1$.

Case 1. $i \not\equiv 0 \pmod b$. Then we have $i = bk + r$, where $k \geq 0$ and $1 \leq r \leq b-1$. By the construction of $(\Delta w)_*$ we have $(\Delta w)_*(i) = 1$ if and only if $\Delta w(r) = 1$. By Lemma 4.1 this holds if and only if $r \in C_w$. As in the proof of Theorem 3.1, we see that $r \in C_w$ holds if and only if $i = bk + r \in C_{\mathbf{w}}$. Hence, $(\Delta w)_*(i) = 1$ holds if and only if $i \in C_{\mathbf{w}}$, which by Lemma 4.1 is equivalent to $\Delta\mathbf{w}(i) = 1$.

Case 2. $i \equiv 0 \pmod b$. Then $i = b^j m$ for some $j \geq 0$, with $m \not\equiv 0 \pmod b$. Now as in the proof of Theorem 3.1 we see that $b^j m \in C_{\mathbf{w}}$ (i.e., $\Delta\mathbf{w}(b^j m) = 1$) if and only if $m \in C_{\mathbf{w}}$. Since $m \not\equiv 0 \pmod b$, by Case 1 we have $m \in C_{\mathbf{w}}$ if and only if $(\Delta w)_*(m) = 1$. By the construction of $(\Delta w)_*$ we see that the latter is equivalent to $(\Delta w)_*(i) = 1$. □

5 Palindromes

We now turn our attention to palindromes. Recall that the complement $\overline{w}$ is the word obtained by interchanging 0 and 1 in w. If $|w|$ is finite, then we define the *reversal* w^R to be the word w written "backwards"; that is, for $|w| = b$,

$$w^R = w(b)w(b-1)\cdots w(2)w(1).$$

The complement and reversal operations have the following properties, whose proofs are immediate from the definitions.

Proposition 5.1.

(i) $(\overline{w})^R = \overline{w^R}$.

(ii) $(w^R)^R = \overline{\overline{w}} = w$.

(iii) $(\Delta w)^R = \Delta(w^R)$.

(iv) $(w_1 w_2 \cdots w_n)^R = w_n^R w_{n-1}^R \cdots w_1^R$.

Definition 5.1. A *palindrome* is a word w such that $w^R = w$. A *skew-palindrome* is a word v such that $v^R = \overline{v}$. If a word is either a palindrome or a skew-palindrome, then it is said to be a *quasi-palindrome*. We denote the sets of palindromes, skew-palindromes, and quasi-palindromes by $\mathcal{P}$, $\mathcal{S}$, and $\mathcal{Q}$, respectively.

For example, the words 0110110 and 011001 are both quasi-palindromes. Specifically, the word 0110110 is a palindrome and the word 011001 is a skew-palindrome.

Proposition 5.2. *Let w and v be words of lengths b and c, respectively. Then:*

(i) *$w \in \mathcal{P}$ if and only if $w(i) = w(b-i+1)$ for $1 \le i \le b$.*

(ii) *$v \in \mathcal{S}$ if and only if $v(j) \ne v(c-j+1)$ for $1 \le j \le c$.*

(iii) *$v \in \mathcal{S}$ implies that $|v|$ is even.*

(iv) *$w \in \mathcal{P}$ if and only if $w \oplus w^R = 00\cdots0$; $v \in \mathcal{S}$ if and only if $w \oplus w^R = 11\cdots1$.*

(v) *$\mathcal{P} \cap \mathcal{S} = \emptyset$.*

Proof. (i) and (ii) follow from Definition 5.1. To prove (iii), observe that if $|v| = c$ were odd, then $v(j) = v(c-j+1)$ for $j = \lceil c/2 \rceil$, which violates (ii). The last two properties follow from (i) and (ii). □

Recall that the product of two words w and v is defined by

$$w \times v = \phi_w(v).$$

Given two sets of words $\mathcal{U}$ and $\mathcal{V}$, we let $\mathcal{U} \times \mathcal{V}$ denote the set of all words $u \times v$, with $u \in \mathcal{U}$ and $v \in \mathcal{V}$. We show that quasi-palindromes are closed with respect to this product operation.

Proposition 5.3. *Let w and v be finite words. Then w,$v \in \mathcal{Q}$ if and only if $w \times v \in \mathcal{Q}$. Moreover, we have the following containment relations:*

$$\mathcal{P} \times \mathcal{P} \subset \mathcal{P}, \tag{5.1}$$
$$\mathcal{P} \times \mathcal{S} \subset \mathcal{S}, \tag{5.2}$$
$$\mathcal{S} \times \mathcal{P} \subset \mathcal{S}, \tag{5.3}$$
$$\mathcal{S} \times \mathcal{S} \subset \mathcal{P}. \tag{5.4}$$

Proof. Suppose first that $w,v \in \mathcal{P}$ with $|w| = b$ and $|v| = c$. Then

$$w \times v = \phi_w(v) = w_1 w_2 \cdots w_c,$$

with $w_i = \phi_w(v(i))$ for $1 \leq i \leq c$. Notice for each i that we have $w_i \in \{w, \overline{w}\} \subset \mathcal{P}$. Applying Proposition 5.1 (iv), we obtain

$$\begin{aligned}(\phi_w(v)) \oplus (\phi_w(v))^R &= w_1 w_2 \cdots w_c \oplus w_c^R w_{c-1}^R \cdots w_1^R \\ &= (w_1 \oplus w_c^R)(w_2 \oplus w_{c-1}^R) \cdots (w_c \oplus w_1^R).\end{aligned}$$

Since $v \in \mathcal{P}$, Proposition 5.2 (i) implies

$$\begin{aligned}w_i &= \phi_w(v(i)) \\ &= \phi_w(v(c-i+1)) \\ &= w_{c-i+1}.\end{aligned}$$

Since $w_i \in \mathcal{P}$, it follows from this and Proposition 5.2 (iv) that

$$\begin{aligned}(w_i \oplus w_{c-i+1}^R) &= (w_i \oplus w_i^R) \\ &= 00 \cdots 0.\end{aligned}$$

Hence

$$(\phi_w(v)) \oplus (\phi_w(v))^R = 00 \cdots 0,$$

which implies that $\phi_w(v) \in \mathcal{P}$. This proves (5.1). The relations (5.2)–(5.4) can be proved by similar arguments. It follows from (10.1)–(10.4) that $\phi_w(v) \in \mathcal{Q}$ whenever $w \in \mathcal{Q}$ and $v \in \mathcal{Q}$.

Conversely, suppose $\phi_w(v) \in \mathcal{Q}$. Then $\phi_w(v)$ is given by

$$\phi_w(v) = \phi_w(v(1))\phi_w(v(2)) \cdots \phi_w(v(c)).$$

By definition $\phi_w(v) \in \mathcal{Q}$ implies that either

$$\phi_w(v) = (\phi_w(v))^R$$

or

$$\overline{\phi_w(v)} = (\phi_w(v))^R.$$

In particular, we have

$$\phi_w(v(1)) = (\phi_w(v(c)))^R$$

or

$$\overline{\phi_w(v(1))} = (\phi_w(v(c)))^R.$$

It follows that either $w = w^R$ or $\overline{w} = w^R$, and in any case we have $w \in \mathcal{Q}$. It remains to be shown that $v \in \mathcal{Q}$, and to do this we again distinguish several cases. Suppose first that $w \in \mathcal{P}$ and $\phi_w(v) \in \mathcal{S}$. Now $\phi_w(v)$ is of the form

$$\phi_w(v) = \phi_w(v(1))\phi_w(v(2)) \cdots \phi_w(v(c)).$$

By our assumption that $\phi_w(v) \in \mathcal{S}$ it follows that

$$\overline{\phi_w(v(i))} = (\phi_w(v(c-i+1)))^R \tag{5.5}$$

for $1 \leq i \leq c$. If $v(i) = 0$, then (5.5) and our assumption $w \in \mathcal{P}$ implies that $v(c-i+1) = 1$. Likewise, $v(i) = 1$ implies that $v(c-i+1) = 0$. Hence we have $v(i) \neq v(c-i+1)$ for all i, so $v \in \mathcal{S}$ by Proposition 5.2 (ii). The proofs of the remaining cases are similar. □

We now characterize first differences of palindromes.

Lemma 5.1. *Given a finite word $w \in \Sigma^*$ with $|w| \geq 2$, we have $w \in \mathcal{Q}$ if and only if $\Delta w \in \mathcal{P}$.*

Proof. Applying Proposition 5.2 (iv), it follows that $\Delta w \in \mathcal{P}$ is equivalent to

$$\begin{aligned} 00\cdots 0 &= \Delta w \oplus (\Delta w)^R \\ &= \Delta w \oplus \Delta(w^R) && \text{(Prop. 5.1(iii))} \\ &= \Delta(w \oplus w^R) && \text{(Lemma 4.2).} \end{aligned}$$

Hence $w \oplus w^R$ is either $00\cdots 0$ or $11\cdots 1$, and by Proposition 5.2 (iv) this is equivalent to $w \in \mathcal{Q}$. □

We now consider palindromes in infinite words of the form

$$w_* = \psi_w^\infty(w(1))$$

introduced in Section 2.

Lemma 5.2. *Let w be a finite word. If w_* contains arbitrarily long palindromes, then w itself is a palindrome.*

Proof. Let v be a subword of w_* that is a palindrome. Then v is of the form

$$v = x[u_1]w[u_2]w\cdots w[u_n]y, \tag{5.6}$$

where $[u_i]$ for $1 \leq i \leq n$ are the letters filling the "holes" arising in the construction of w_*, and x and y are a suffix and prefix of w, respectively. (We allow the possibility that x or y or both are empty.)

Suppose first that $|x| = |y|$. If we subtract $|x|+1$ letters from each side of (5.6), then the remaining word

$$v_0 = w[u_2]w[u_3]\cdots[u_{n-1}]w \tag{5.7}$$

is still a palindrome. Notice that v_0 both begins and ends with w. Since $v_0 \in \mathcal{P}$, it follows that $w = w^R$, and hence $w \in \mathcal{P}$.

Now assume without loss of generality that $|x| > |y|$. Let $|w| = b$, and let $|x| = b - m$ for some m with $0 \leq m < b$. Then $|y| = b - m - r$ for some r with $1 \leq r \leq b - m$ (since $|x| > |y|$). Now let x_1 be the suffix of x of length $|x| - |y| - 1$, so that $|x_1| = r - 1$. If we remove $|y| + 1$ letters from each side of v, then the resulting word

$$v_1 = x_1[u_1]w[u_2]\cdots[u_{n-1}]w \tag{5.8}$$

must still be a palindrome. Since x_1 is a suffix of w, we can write w as

$$w = aw(b-r+1)x_1, \tag{5.9}$$

where a is the prefix of w of length $b-r$, and $w(b-r+1)$ is the $(b-r+1)$st letter of w. Substituting (5.9) into (5.8) we obtain

$$v_1 = x_1[u_1]aw(b-r+1)x_1[u_2]\cdots[u_{n-1}]aw(b-r+1)x_1. \tag{5.10}$$

Subtracting $r-1$ letters from each side of (5.10), we obtain

$$v_2 = [u_1]aw(b-r+1)x_1[u_2]\cdots[u_{n-1}]aw(b-r+1), \tag{5.11}$$

which again is a palindrome. Since $v_2 \in \mathcal{P}$, it follows that

$$\begin{aligned}[u_1]aw(b-r+1) &= ([u_{n-1}]aw(b-r+1))^R \\ &= w(b-r+1)a^R[u_{n-1}].\end{aligned}$$

Hence $[u_1] = w(b-r+1) = [u_{n-1}]$. Proceeding inductively (by removing at each step $b+1$ letters from each side) we see that $[u_1] = [u_2] = \cdots = [u_{n-1}]$. Since, by assumption, w_* contains arbitrarily long palindromes, we can take n in (5.6) arbitrarily large. In particular, the "holes" arising in the construction of w_* must include arbitrarily long strings of consecutive 0's or consecutive 1's. This is only possible if $w = 00\cdots 0$ or $w = 11\cdots 1$. Thus w is a palindrome. □

Remark. The converse of the lemma is also true (though we will not need this fact here): If w is a palindrome, then w_* contains arbitrarily long palindromes.

We are now ready to state and prove our main result, giving us a necessary and sufficient condition for a fixed point $\mathbf{w}$ to contain palindromes of arbitrary length. For similar subject matter, see Hof, Knill, and Simon [8].

Theorem 5.1. *Let w be a finite word with $|w| = b \geq 2$, and $w(1) = 0$. Then $\mathbf{w}$ contains arbitrarily long palindromes if and only if w is a quasi-palindrome.*

Proof. Suppose first that $w \in \mathcal{Q}$. Then by Proposition 5.3 we have $\phi_w(w) \in \mathcal{P}$, and hence $\phi_w^{2n+1}(w) \in \mathcal{P}$ for all n. Thus $\phi_w^{\infty}(w) = \mathbf{w}$ contains arbitrarily long palindromes.

Conversely, if $\mathbf{w}$ contains arbitrarily long palindromes, then by Lemma 5.1 its first difference $\Delta\mathbf{w}$ also contains arbitrarily long palindromes. If $w(b) = 0$, then we can apply Theorem 4.1 to conclude that $(\Delta w)_*$ contains arbitrarily long palindromes. By Lemma 5.2 it follows that $\Delta w \in \mathcal{P}$, which by Lemma 5.1 implies that $w \in \mathcal{Q}$. If $w(b) = 1$, then we cannot apply Theorem 4.1 directly to w. However, we may apply the theorem to the word $u = \phi_w(w)$ to obtain $\Delta\mathbf{w} = \Delta\mathbf{u} = (\Delta u)_*$. It follows that $(\Delta u)_*$ contains arbitrarily long palindromes. As before, this implies that $\phi_w(w) = u \in \mathcal{Q}$. By Proposition 5.3 it follows that $w \in \mathcal{Q}$. □

References

[1] Jean-Paul Allouche, André Arnold, Jean Berstel, Srečko Brlek, Simon Plouffe, and Bruce E. Sagan, A relative of the Thue-Morse sequence, *Discrete Math.* **139** (1995), 455–461.

[2] Jean-Paul Allouche and Roland Bacher, Toeplitz sequences, paperfolding, Towers of Hanoi and progression-free sequences of integers, *Enseign. Math.* **38** (1992), 315–327.

[3] Jean-Paul Allouche and Jeffrey Shallit, The ubiquitous Prouhet-Thue-Morse sequence, *Sequences and Their Applications: Proceedings of SETA '98*, Springer Ser. Discrete Math. Theor. Comput. Sci., Springer, London, 1999, pp. 1–16.

[4] Jean-Paul Allouche and Jeffrey Shallit, Sums of digits, overlaps, and palindromes, *Discrete Math. Theor. Comp. Sci.* **4** (2000), 1–10 (electronic).

[5] Julien Cassaigne and Juhani Karhumäki, Toeplitz words, generalized periodicity and periodically iterated morphisms, *European J. Combin.* **18** (1997), 497–510.

[6] D. Chalice, How to differentiate and integrate sequences, *Amer. Math. Monthly* **108** (2001), 911–921.

[7] Anna E. Frid, Overlap-free symmetric D0L words, *Discrete Math. Theor. Comput. Sci.* **4** (2001), 357–362.

[8] A. Hof, O. Knill, and B. Simon, Singular continuous spectrum for palindromic Schrödinger operators, *Comm. Math. Phys.* **174** (1995), 149–159.

[9] A. Hoit, The distribution of generalized sum-of-digits functions, Ph.D. thesis, Univ. of Illinois, Urbana-Champaign, 1999.

[10] A. Hoit, Block products, preprint, 2002.

[11] K. Jacobs, Maschinenerzeugte 0-1-folgen, *Selecta Math.* **1** (1969), 1–27.

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