HARMONIOUS PAIRS

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Dedicated to the memory of Felice Bateman, Paul Bateman, and Heini Halberstam

ABSTRACT. Let σ be the usual sum-of-divisors function. We say that a and b form a *harmonious pair* if $\frac{a}{\sigma(a)} + \frac{b}{\sigma(b)} = 1$; equivalently, the harmonic mean of $\frac{\sigma(a)}{a}$ and $\frac{\sigma(b)}{b}$ is 2. For example, 4 and 12 form a harmonious pair, since $\frac{4}{\sigma(4)} = \frac{4}{7}$ and $\frac{12}{\sigma(12)} = \frac{3}{7}$. Every amicable pair is harmonious, but there are many others. We show that the count of numbers that belong to a harmonious pair having both members in $[1, x]$ is at most $x/\exp((\log x)^{\frac{1}{12}+o(1)})$, as $x \to \infty$.

1. INTRODUCTION

Let $\sigma(n)$ denote the sum of the divisors of the natural number n. Recall that m and n are said to form an *amicable pair* if $\sigma(m) = \sigma(n) = m + n$. The study of amicable pairs dates back to antiquity, with the smallest such pair — 220 and 284 — known already to Pythagoras.

While amicable pairs have been of interest for 2500 years, many of the most natural questions remain unsolved. For example, although we know about 12 million amicable pairs, we have no proof that there are infinitely many. In the opposite direction, there has been some success in showing that amicable pairs are not so numerous. In 1955, Erdős showed that the set of n belonging to an amicable pair has asymptotic density zero [4]. This result has been subject to steady improvement over the past 60 years [11, 5, 8, 9, 10]. We now know that the count of numbers not exceeding x that belong to an amicable pair is smaller than

$$x/\exp((\log x)^{1/2}) \tag{1.1}$$

for all large x.

If m and n form an amicable pair, then $\sigma(m) = \sigma(n) = m + n$. From this, one sees immediately that $\frac{m}{\sigma(m)} + \frac{n}{\sigma(n)} = 1$. In this paper, we study solutions to this latter equation.

Definition. We say a and b form a *harmonious pair* if $\frac{a}{\sigma(a)} + \frac{b}{\sigma(b)} = 1$.

The terminology here stems from the following simple observation: a and b form a harmonious pair precisely when $\sigma(a)/a$ and $\sigma(b)/b$ have harmonic mean 2. While every amicable pair is harmonious, there are many examples not of this kind, for instance 2 and 120, or 3 and 45532800.

Our main theorem is an upper bound on the count of numbers belonging to a harmonious pair. While harmonious pairs certainly appear to be more thick on the ground than amicable pairs, we are able to get an upper estimate of the same general shape as (1.1).

Theorem 1. *Let $\epsilon > 0$. The number of integers belonging to a harmonious pair a, b with $\max\{a, b\} \le x$ is at most $x/\exp((\log x)^{\frac{1}{12}-\epsilon})$, for all $x > x_0(\epsilon)$.*

As a corollary of Theorem 1, the reciprocal sum of those integers that are the larger member of a harmonious pair is convergent. Note that Theorem 1 does *not* give a reasonable bound on the number of harmonious pairs lying in $[1, x]$.

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We are not aware of any previous work on harmonious pairs, as such. However, the following result can be read out of a paper of Borho [2]: If a, b form a harmonious pair and $\Omega(ab) = K$, then $ab \le K^{2^K}$. Borho states this for amicable pairs, but only the harmonious property of the pair is used in the proof.

We also discuss *discordant* numbers, being those numbers that are not a member of a harmonious pair. We show there are infinitely many discordant numbers, in fact, more than $x/(\log x)^\epsilon$ of them in $[1, x]$, when $\epsilon > 0$ is fixed and x is sufficiently large. Probably a positive proportion of numbers are discordant, but we have not been able to prove this. A weaker assertion that seems to escape us: it is not the case that the numbers that belong to some harmonious pair form a set of asymptotic density 1.

At the end of the paper we use harmonious pairs to disprove a conjecture in [7].

Notation. Throughout this paper, we use the Bachmann–Landau symbols O and o as well as the Vinogradov symbols $\ll$ and $\gg$ with their regular meanings. Recall that $A = O(B)$ and $A \ll B$ are both equivalent to the fact that the inequality $|A| < cB$ holds with some constant $c > 0$. Further, $A \gg B$ is equivalent to $B \ll A$, while $A = o(B)$ means that $A/B \to 0$. We write $\log_k x$ for the iteration of the natural log function, with the undertanding that x will be big enough to have $\log_k x \ge 1$. We let $P^+(n)$ denote the largest prime factor of n, with the convention that $P^+(1) = 1$. We write $s(n)$ for the sum of the proper divisors of n, so that $s(n) = \sigma(n) - n$. If p is prime, we write $v_p(n)$ for the exponent of p appearing in the prime factorization of n. We let $\tau(n)$ denote the number of positive divisors of n and let $\omega(n)$ denote the number of these divisors which are prime.

2. Proof of Theorem 1

The following proposition, whose proof constitutes the main part of the argument, establishes 'half' of Theorem 1. This proof largely follows the plan in [9, 10], though here we have more cases.

Proposition 2. *The number of integers $b \le x$ that are members of a harmonious pair a, b with $\max\{a, b\} \le x$ and $P^+(b) \ge P^+(a)$ is*

$$\ll x/\exp((\log x)^{\frac{1}{12}})$$

for all $x \ge 3$.

Proof. We may assume that $x > x_0$ where x_0 is some large, absolute constant. For α in $(0, 1)$ and $x \ge 3$ we put $\mathcal{L}_\alpha = \exp((\log x)^\alpha)$. We aim to bound the count of b-values by $O(x/\mathcal{L}_\alpha)$ with some fixed $\alpha \in (0, 1)$, whose size we will detect from our arguments. We will pile various conditions on b and keep track of the counting function of those $b \le x$ failing the given conditions.

1. We eliminate numbers $b \le x$ having a square full divisor $d > \frac{1}{2}\mathcal{L}_\alpha^2$. The counting functions of those is bounded above by

$$\sum_{\substack{d \ge \mathcal{L}_\alpha^2/2 \\ d \text{ squarefull}}} \frac{x}{d} \ll \frac{x}{\mathcal{L}_\alpha},$$

where the above estimate follows from the Abel summation formula using the fact that the counting function of the number of square full numbers $m \le t$ is $O(t^{1/2})$.

2. We eliminate numbers $b \le x$ for which $P^+(b) \le \mathcal{L}_{1-\alpha}$. Putting $y = \mathcal{L}_{1-\alpha}$, we have $u := \log x/\log y = (\log x)^\alpha$. By known estimates from the theory of smooth numbers (e.g., [3]), we have that the number of integers $b \le x$ with $P^+(b) \le x^{1/u}$ is

$$\le \frac{x}{\exp((1 + o(1))u \log u)} \qquad \text{as} \qquad x \to \infty, \tag{2.1}$$

when $u > (\log x)^{1-\epsilon}$, so certainly the above count is $< x/\mathcal{L}_\alpha$ once x is sufficiently large.

3. We assume that $\alpha < 1/2$. We eliminate numbers $b \le x$ having a divisor $d > \mathcal{L}_{2\alpha}$ with $P^+(d) \le \mathcal{L}_\alpha^2$. Put $y = \mathcal{L}_\alpha^2$. For each $t \ge \mathcal{L}_{2\alpha}$, we have $u = \log t/\log y \ge 0.5(\log x)^\alpha$. Thus, $u \log u \gg (\log x)^\alpha \log_2 x$, and in particular $u \log u > 3(\log x)^\alpha$ for $x > x_0$. So the number of such $d \le t$ is at most $t/\mathcal{L}_\alpha^2$, uniformly for $t \in [\mathcal{L}_{2\alpha}, x]$, assuming $x > x_0$. Fixing d, the number of $b \le x$ divisible by d is $\le x/d$. Using the Abel summation formula to sum the reciprocals of such d, we get that the number of such $b \le x$ is bounded by

$$\sum_{\substack{\mathcal{L}_{2\alpha} < d \le x \\ P^+(d) \le \mathcal{L}_\alpha^2}} \frac{x}{d} \ll \frac{x}{\mathcal{L}_\alpha^2} \int_{\mathcal{L}_{2\alpha}}^{x} \frac{dt}{t} \ll \frac{x \log x}{\mathcal{L}_\alpha^2} \ll \frac{x}{\mathcal{L}_\alpha}.$$

4. We eliminate numbers $b \le x$ having a prime factor $p > \mathcal{L}_\alpha^2$ such that $p \mid \gcd(b, \sigma(b))$. Let us take a closer look at such numbers. Suppose that $p > \mathcal{L}_\alpha^2$ and $p \mid \sigma(b)$. Then there is a prime power q^ℓ dividing b such that $p \mid \sigma(q^\ell)$. If $\ell \ge 2$, then $2q^\ell > \sigma(q^\ell) \ge p > \mathcal{L}_\alpha^2$, so $q^\ell > \mathcal{L}_\alpha^2/2$ and $q^\ell \mid b$ with $\ell \ge 2$, but such b's have been eliminated at **1**. So, $\ell = 1$, therefore $q \equiv -1 \pmod p$. Thus, b is divisible by pq for some prime $q \equiv -1 \pmod p$. The number of such numbers up to x is at most $\frac{x}{pq}$. Summing up the above bound over all primes $q \le x$ with $q \equiv -1 \pmod p$ while keeping p fixed, then over all primes $p \in (\mathcal{L}_\alpha^2, x]$ gives us a count of

$$\sum_{\mathcal{L}_\alpha^2 < p \le x} \sum_{\substack{q \equiv -1 \pmod p \\ q \le x}} \frac{x}{pq} \ll x(\log_2 x) \sum_{\mathcal{L}_\alpha^2 < p \le x} \frac{1}{p^2} \ll \frac{x}{\mathcal{L}_\alpha^2} \ll \frac{x}{\mathcal{L}_\alpha}.$$

5. We eliminate the numbers $b \le x/\mathcal{L}_\alpha$, since obviously there are only at most $x/\mathcal{L}_\alpha$ such values of b.

Let

$$d = \gcd(b, \sigma(b)).$$

Then $P^+(d) \le \mathcal{L}_\alpha^2$ by **4**, so by **3** we have $d \le \mathcal{L}_{2\alpha}$. Write $b = P_1 m_1$, where $P_1 = P^+(b)$. By **2**, we can assume that $P_1 > \mathcal{L}_{1-\alpha}$.

6. We eliminate $b \le x$ corresponding to some $a \le x/\mathcal{L}_{2\alpha}^2$. Indeed, let b have a corresponding a with the above property. With $c = \gcd(a, \sigma(a))$, we have an equality of reduced fractions

$$\frac{b/d}{\sigma(b)/d} = \frac{(\sigma(a) - a)/c}{\sigma(a)/c}.$$

Notice that c is determined uniquely in terms of a. Thus, $b/d = (\sigma(a) - a)/c$ is also determined by a. Since $d \le \mathcal{L}_{2\alpha}$, the number b is determined in at most $\mathcal{L}_{2\alpha}$ ways from a. So the number of b corresponding to some $a \le x/\mathcal{L}_{2\alpha}^2$ is at most $x/\mathcal{L}_{2\alpha} < x/\mathcal{L}_\alpha$.

7. Similar to **6**, we eliminate a bounded number of subsets of $b \le x$ which have some corresponding $a \le x$ with a counting function of size $O(x/\mathcal{L}_{2\alpha}^2)$.

In particular, by an argument similar to the one at **1**, we may assume that a has no divisor which is squarefull and larger than $\mathcal{L}_{2\alpha}^4/2$. In particular, if $p^2 \mid a$, then $p < \mathcal{L}_{2\alpha}^2$. We may further assume that $P^+(a) > \mathcal{L}_{1-2\alpha}$ by an argument similar to the one at **2**, and that if d_1 is the largest divisor of a such that $P^+(d_1) \le \mathcal{L}_{2\alpha}^4$, then $d_1 \le \mathcal{L}_{4\alpha}$, again by an argument similar to the one used at **3**. Assuming $\alpha \le \frac{1}{6}$, we then have

$$P^+(a) > \mathcal{L}_{1-2\alpha} \ge \mathcal{L}_{4\alpha} \ge d_1.$$

Further, $P^+(a) > \mathcal{L}_{1-2\alpha} > \mathcal{L}_{2\alpha}^2$. Hence, $P^+(a)^2 \nmid a$. Thus, $a = Q_1 n_1$, where $Q_1 = P^+(a)$ and $Q_1 \nmid n_1$.

8. Recall that $c = \gcd(a, \sigma(a))$. By an argument similar to **4**, we may eliminate numbers $b \le x$ with some corresponding a having the property that there exists a prime factor $p \mid c$ such that $p > \mathcal{L}_{2\alpha}^4$. Indeed, in this case $p \mid a$. Further, $p \mid \sigma(a)$ so there is a prime power q^ℓ dividing a such that $p \mid \sigma(q^\ell)$. If $\ell \ge 2$, then $2q^\ell > \sigma(q^\ell) \ge p > \mathcal{L}_{2\alpha}^4$, contradicting **7**. So, $\ell = 1$, $q \equiv -1 \pmod p$, and pq divides a, so the number of such $a \le x$ is at most x/pq. Summing up the above bound over all primes $q \equiv -1 \pmod p$ with $q \le x$, then over all primes $p \in (\mathcal{L}_{2\alpha}^4, x]$, we get a count on the number of such a of

$$\sum_{\mathcal{L}_{2\alpha}^4 < p \le x} \sum_{\substack{q \equiv -1 \pmod p \\ q \le x}} \frac{x}{pq} \ll x(\log_2 x) \sum_{\mathcal{L}_{2\alpha}^4 < p \le x} \frac{1}{p^2} \ll \frac{x}{\mathcal{L}_{2\alpha}^2},$$

and we are in a situation described at the beginning of **7**.

By **8**, we have that if $p \mid c$, then $p \le \mathcal{L}_{2\alpha}^4$. So from **7**, $c \le d_1 \le \mathcal{L}_{4\alpha}$.

9. We eliminate $b \le x$ for which $P^+(P_1 + 1) \le \mathcal{L}_{1-2\alpha}$. Assume that b satisfies this condition. Then $P_1 + 1 \le x/m_1 + 1 \le 2x/m_1$ is a number having $P^+(P_1+1) \le y = \mathcal{L}_{1-2\alpha}$, and $P_1 + 1 > P_1 > \mathcal{L}_{1-\alpha}$, by **2**. Thus, $u := \log(2x/m_1)/\log y \ge (\log x)^\alpha$, so that $u \log u > 3(\log x)^\alpha$ for $x > x_0$. Fixing m_1, the number of such P_1 (even ignoring the fact that they are prime) is, again by (2.1), at most

$$\frac{x}{L_\alpha^2 m_1}.$$

Summing over all $m_1 \le x$, we get at most $O(x(\log x)/\mathcal{L}_\alpha^2) = O(x/\mathcal{L}_\alpha)$ such b.

10. We may eliminate those $b \le x$ corresponding to an a with $P^+(Q_1 + 1) \le \mathcal{L}_{1-4\alpha}$. Indeed, assume that b satisfies the above property. Then $Q_1 + 1 \le x/n_1 + 1 \le 2x/n_1$. Further, $Q_1 + 1 > Q_1 > \mathcal{L}_{1-2\alpha}$ by **7** and $P^+(Q_1+1) \le \mathcal{L}_{1-4\alpha}$, so that with $y = \mathcal{L}_{1-4\alpha}$, we have $u := \log(2x/n_1)/\log y > (\log x)^{2\alpha}$. This shows that $u \log u > 4(\log x)^{2\alpha}$ for $x > x_0$. Thus, the number of possible numbers of the form $Q_1 + 1$ (even ignoring the fact that Q_1 is prime), is, again by (2.1), at most

$$\frac{x}{\mathcal{L}_{2\alpha}^3 n_1.}$$

Summing up the above bound for $n_1 \le x$, we see there are at most $O(x(\log x)/\mathcal{L}_{2\alpha}^3)$ possible a. So we are in the situation described at the beginning of **7**.

11. Reducing the left and right-hand sides of the equation $\frac{a}{\sigma(a)} = \frac{\sigma(b)-b}{\sigma(b)}$ gives that $a/c = (\sigma(b) - b)/d$. Hence,

$$Q_1 n_1 = a = (c/d)(\sigma(b) - b) = (c/d)(P_1 s(m_1) + \sigma(m_1)), \tag{2.2}$$

and so

$$Q_1 n_1 d = c(P_1 s(m_1) + \sigma(m_1)).$$

Since $c \le \mathcal{L}_{4\alpha}$ and $Q_1 > \mathcal{L}_{1-2\alpha}$, it follows that $Q_1 \nmid c$. (Recall our assumption that $\alpha \le \frac{1}{6}$.) Hence, $\gcd(Q_1, c) = 1$, and $c \mid n_1 d$. Thus,

$$P_1 s(m_1) + \sigma(m_1) = Q_1 \lambda_1,$$

where $\lambda_1 = n_1 d/c$. Further, since $d \le \mathcal{L}_{2\alpha} < Q_1$ and $Q_1 \nmid n_1$, it follows that $Q_1 \nmid \lambda_1$.

We now break symmetry and make crucial use of our assumption that $P_1 \ge Q_1$.

We claim that $P_1 \nmid a$. Assume for a contradiction that $P_1 \mid a$. Recalling (2.2), and using the fact that $P_1 > \mathcal{L}_{1-\alpha} > \max\{c, d\}$, we get that $P_1 \mid \sigma(b) - b$, therefore $P_1 \mid \sigma(b)$, so $P_1 \mid d$, which is false.

Let $R_1 = P^+(P_1 + 1)$. We note that $R_1 \nmid a$. Indeed, the argument is similar to the argument that $P_1 \nmid a$. To see the details, assume that $R_1 \mid a$. Since $R_1 > \mathcal{L}_{1-2\alpha} \ge \max\{c, d\}$, it follows from (2.2) that $R_1 \mid \sigma(b) - b$. But $R_1 \mid P_1 + 1 \mid \sigma(b)$, therefore $R_1 \mid b$. Thus, $R_1 \mid d$, which is impossible since $R_1 > d$.

Now $R_1 \mid \sigma(b)/d = \sigma(a)/c$. Thus, there is some prime power Q_2^ℓ dividing a such that $R_1 \mid \sigma(Q_2^\ell)$. Hence, $\mathcal{L}_{1-2\alpha} < R_1 \le \sigma(Q_2^\ell) < 2Q_2^\ell$. If $\ell \ge 2$, we then get $\mathcal{L}_{1-2\alpha} < 2Q_2^\ell \le \mathcal{L}_{2\alpha}^4$ by **7**, which is false for $x > x_0$. Thus, $\ell = 1$, and we have that $R_1 \mid Q_2 + 1$. In particular, $Q_2 > R_1 > \mathcal{L}_{1-2\alpha}$. Since $Q_2 \le Q_1$ (the case $Q_2 = Q_1$ is possible), it follows that $Q_2 \le P_1$. We write $a = Q_2 n_2$ and going back to relation (2.2), we get

$$\sigma(b) - b = Q_2 \frac{n_2 d}{c}.$$

Note that $Q_2 > \mathcal{L}_{1-2\alpha} \ge \max\{c, d\}$, so indeed $c \mid n_2 d$. Write $\lambda_2 = n_2 d/c$. We then have

$$P_1 s(m_1) + \sigma(m_1) = \sigma(b) - b = Q_2 \lambda_2. \tag{2.3}$$

Note that $Q_2 \nmid s(m_1)$, for if not, then we would also get that $Q_2 \mid \sigma(m_1)$. Thus, $Q_2 \mid m_1 \mid b$ and $Q_2 \mid \sigma(m_1) \mid \sigma(b)$, therefore $Q_2 \mid d$, which is false since $Q_2 > d$.

Reduce now equation (2.3) with respect to R_1, using $P_1 \equiv Q_2 \equiv -1 \pmod{R_1}$, to get that

$$m_1 + \lambda_2 \equiv 0 \pmod{R_1}.$$

This shows that

$$\text{either} \qquad m_1 \ge R_1/2 \qquad \text{or} \qquad \lambda_2 \ge R_1/2. \tag{2.4}$$

So the situation splits into two cases. We treat an instance a bit stronger then the first case above throughout steps **12–15**, and the second situation in the subsequent steps **16–20**.

We first assume that

$$m_1 > \mathcal{L}_{1-6\alpha}^{1/4}. \tag{2.5}$$

Note that the left inequality (2.4) implies (2.5) for $x > x_0$. (The negation of the weak inequality (2.5) will be useful in **17**.)

12. We eliminate the numbers $b \le x$ for which $P^+(m_1) \le \mathcal{L}_{1-7\alpha}$. Fix P_1 and count the number of corresponding $m_1 \in (\mathcal{L}_{1-6\alpha}^{1/4}, x/P_1]$. If there are any such m_1, then with $y = \mathcal{L}_{1-7\alpha}$, we have $u := \log(x/P_1)/\log y \ge 0.25(\log x)^\alpha$. Hence, $u \log u > 3(\log x)^\alpha$ for $x > x_0$. By (2.1), the number of these m_1 is at most $x/(\mathcal{L}_\alpha^2 P_1)$ for $x > x_0$. Summing over all primes $P_1 \le x$, we get an upper bound of $O(x(\log_2 x)/\mathcal{L}_\alpha^2) = O(x/\mathcal{L}_\alpha)$ on the number of such b.

13. Let $P_2 = P^+(m_1)$ and put $m_1 = P_2 m_2$. Note that $P_2 \le x/(P_1 m_2)$ and $P_2 > \mathcal{L}_{1-7\alpha}$. Clearly, if $\alpha \le \frac{1}{11}$, then P_2 does not divide cd for large x because $P_2 > \mathcal{L}_{1-7\alpha} \ge \max\{c, d\}$. Also, since $\mathcal{L}_{1-7\alpha} > \mathcal{L}_\alpha^2/2$ for $x > x_0$, it follows that $P_2 \parallel b$. Thus, $P_2 + 1 \mid \sigma(b)$.

14. We eliminate $b \le x$ such that $P^+(P_2 + 1) \le \mathcal{L}_{1-8\alpha}$. Since $P_2 + 1 > \mathcal{L}_{1-7\alpha}$, for fixed P_1, m_2, by arguments similar to the preceding ones, we get that the number of such P_2 is at most $x/(\mathcal{L}_\alpha^2 P_1 m_2)$. Summing up the above inequality over all the primes $P_1 \le x$ and all positive integers $m_2 \le x$, we get a bound of $O(x(\log x)(\log_2 x)/\mathcal{L}_\alpha^2) = O(x/\mathcal{L}_\alpha)$ on the number of such b.

15. Now we put $R_2 = P^+(P_2 + 1)$. Then $R_2 > d$ if $\alpha < \frac{1}{10}$, because $R_2 > \mathcal{L}_{1-8\alpha}$. Thus, $R_2 \mid \sigma(b)/d = \sigma(a)/c$, therefore there exists Q_3^ℓ dividing a such that $R_2 \mid \sigma(Q_3^\ell)$. Thus, $2Q_3^\ell > \sigma(Q_3^\ell) \ge R_2$. Since $\alpha < \frac{1}{10}$, we have $R_2 > \mathcal{L}_{1-8\alpha} > \mathcal{L}_{2\alpha}^4$ for $x > x_0$, so, by **7**, we get that $\ell = 1$. Thus, $Q_3 \parallel a$ and $Q_3 \equiv -1 \pmod{R_2}$ (the case $Q_3 = Q_1$ is possible). Now assume $\alpha \le \frac{1}{12}$. Then $Q_3 > R_2 > \mathcal{L}_{1-8\alpha} \ge c$. Since $a = (\sigma(b) - b)c/d$, it follows that $Q_3 \mid \sigma(b) - b$. Hence,

$$P_1 s(m_1) + \sigma(m_1) \equiv 0 \pmod{Q_3}.$$

Since $Q_3 > \mathcal{L}_{1-8\alpha}$, arguments similar to previous ones show that $Q_3 \nmid s(m_1)$. This puts $P_1 \le x/m_1$ in an arithmetic progression modulo Q_3. Since $Q_3 \le Q_1 \le P_1$, it follows that $x/m_1 \ge Q_3$, so that the number of such integers P_1 (even ignoring the fact that P_1 is prime) is $O(x/(m_1 Q_3))$. But $m_1 = P_2 m_2$, where $R_2 \mid \gcd(P_2+1, Q_3+1)$. Fixing R_2, P_2, Q_3 and summing up over $m_2 \le x$, we get a count of $O(x(\log x)/(P_2 Q_3))$. Now we sum up over primes P_2 and Q_3 both at most x and both congruent to $-1 \pmod{R_2}$ getting a count of $O(x(\log x)(\log_2 x)^2/R_2^2)$. We finally sum over primes $R_2 \in (\mathcal{L}_{1-8\alpha}, x]$ getting a bound of $O(x(\log x)(\log_2 x)^2/\mathcal{L}_{1-8\alpha}) = O(x/\mathcal{L}_\alpha)$ on the number of such b.

The steps **12 – 15** apply when m_1 satisfies (2.5). Now assume that m_1 fails (2.5). In this case, $m_1 < R_1/2$ (assuming $x > x_0$). So by (2.4), we must have $\lambda_2 \ge R_1/2 > 0.5\mathcal{L}_{1-2\alpha}$. Note that $\lambda_2 = n_2(d/c)$; therefore $n_2 = \lambda_2(c/d) > 0.5\mathcal{L}_{1-2\alpha}/d > \mathcal{L}_{1-2\alpha}^{0.5}$ for $x > x_0$, since $d \le \mathcal{L}_{2\alpha}$.

16. We eliminate $b \le x$ such that their corresponding a has the property that $P^+(n_2) \le \mathcal{L}_{1-4\alpha}$. Put $y = \mathcal{L}_{1-4\alpha}$. We fix Q_2 and count $n_2 \le x/Q_2$, with $n_2 > \mathcal{L}_{1-2\alpha}^{0.5}$, such that $P^+(n_2) \le y$. Since $u := \log(x/Q_2)/\log y \ge 0.5(\log x)^{2\alpha}$, it follows that $u \log u > 4(\log x)^{2\alpha}$ for $x > x_0$. Thus, for large x the number of corresponding $a \le x$ is at most $x/(\mathcal{L}_{2\alpha}^3 Q_2)$. Summing up the above bound over primes $Q_2 \le x$, we get a count of order $x(\log_2 x)/\mathcal{L}_{2\alpha}^3$. This is smaller than $x/\mathcal{L}_{2\alpha}^2$ for $x > x_0$, and so we are fine by **7**.

Now we put $n_2 = Q_3 n_3$, where $Q_3 = P^+(n_2) > \mathcal{L}_{1-4\alpha}$.

17. We eliminate $b \le x$ corresponding to a such that $P^+(Q_3+1) \le \mathcal{L}_{1-6\alpha}$. Fix Q_2 and n_3. Then $Q_3 \le x/(Q_2 n_3)$ and $Q_3 > \mathcal{L}_{1-4\alpha}$. Assuming that $P^+(Q_3+1) \le \mathcal{L}_{1-6\alpha}$, we get by previous arguments involving (2.1) that the count of such Q_3 is smaller than $x/(\mathcal{L}_{2\alpha}^3 Q_2 n_3)$, once $x > x_0$. Summing up the above bound over primes $Q_2 \le x$ and all positive integers $n_3 \le x$, we get an upper bound of order $x(\log x)(\log_2 x)/\mathcal{L}_{2\alpha}^3$ on the number of these a. Again, we are fine by **7**.

Write $R_2 = P^+(Q_3+1)$. Then $R_2 > \mathcal{L}_{1-6\alpha}$. Assume now that $\alpha < \frac{1}{10}$. Then $R_2 > \max\{c, d\}$ and $R_2 > \mathcal{L}_{2\alpha}^4$ for $x > x_0$. Since $Q_3 > \mathcal{L}_{1-4\alpha} > \mathcal{L}_{2\alpha}^2$ (for $x > x_0$), **7** gives that $Q_3 \parallel a$. Thus, $R_2 \mid Q_3+1 \mid \sigma(a)$. Since R_2 does not divide c, we get that R_2 divides $\sigma(a)/c = \sigma(b)/d$. Hence, $R_2 \mid \sigma(b)$. Since b has no squarefull divisors exceeding $\mathcal{L}_\alpha^2/2$, there is a prime $P_2 \parallel b$ such that $R_2 \mid P_2+1$. In fact, we can take $P_2 = P_1$, in other words, $R_2 \mid P_1+1$. Suppose otherwise. Then $R_2 \mid \frac{\sigma(b)}{P_1+1} = \sigma(m_1)$. However, since m_1 fails (2.5), $\sigma(m_1) \le m_1^2 \le (\mathcal{L}_{1-6\alpha})^{1/2} < R_2$.

Hence, $R_1 \mid P_1+1$ and $R_2 \mid P_1+1$.

18. Consider the case when $R_1 = R_2$. Then $a = Q_2 Q_3 n_3$ and both Q_2 and Q_3 are congruent to -1 modulo R_1. Fixing Q_2, Q_3, the number of such a is at most $x/Q_2 Q_3$. Summing this bound over all pairs of distinct primes Q_2, Q_3 up to x and congruent to -1 modulo R_1, we get a bound of $O(x(\log_2 x)^2/R_1^2)$. Now summing over all primes $R_1 \in (\mathcal{L}_{1-2\alpha}, x]$, we get a count that is $< x/\mathcal{L}_{1-2\alpha} < x/\mathcal{L}_{2\alpha}^2$ for $x > x_0$, and we are fine by **7**.

From now on, we assume that $R_1 \ne R_2$, so that $P_1 \equiv -1 \pmod{R_1 R_2}$.

19. We eliminate numbers $b \le x$ such that either $m_1 Q_2 R_1 \le x$ or $m_1 Q_3 R_2 \le x$. Suppose we are in the first case. Then $P_1 \equiv -1 \pmod{R_1}$, and

$$P_1 s(m_1) + \sigma(m_1) \equiv 0 \pmod{Q_2}.$$

Since $Q_2 \nmid s(m_1)$, this puts P_1 into an arithmetic progression modulo Q_2. By the Chinese remainder theorem, $P_1 \le x/m_1$ is in an arithmetic progression modulo $Q_2 R_1$, and the number of such numbers (ignoring the condition that P_1 is prime) is at most $1 + x/(m_1 Q_2 R_1) \le 2x/(m_1 Q_2 R_1)$. Here is where we use the condition that $m_1 Q_2 R_1 \le x$. We keep R_1 fixed and sum over all $m_1 \le x$, and primes $Q_2 \equiv -1 \pmod{R_1}$, getting a count of $O(x(\log x)(\log_2 x)/R_1^2)$. Then we sum over all primes $R_1 \in (\mathcal{L}_{1-2\alpha}, x]$, getting a count of $O(x(\log x)(\log_2 x)/\mathcal{L}_{1-2\alpha})$. This count of b values is $< x/\mathcal{L}_\alpha$ once $x > x_0$.

The same applies when $m_1 Q_3 R_2 \le x$. There, $P_1 \equiv -1 \pmod{R_2}$ and the congruence $P_1 s(m_1) + \sigma(m_1) \equiv 0 \pmod{Q_3}$ together with the fact that Q_3 does not divide $s(m_1)$ puts $P_1 \le x/m_1$ in an arithmetic progression modulo Q_3. By the Chinese remainder theorem, $P_1 \leq x/m_1$ is in an arithmetic progression modulo Q_3R_2, and the number of such possibilities (ignoring the fact that P_1 is prime) is at most $1 + x/(m_1Q_3R_2) \leq 2x/(m_1Q_3R_2)$. Here we used that $m_1Q_3R_2 \leq x$. Summing up the above bound over all $m_1 \leq x$ and primes $Q_3 \leq x$ in the arithmetic progression $-1 \pmod{R_2}$, we get a count of $O(x(\log x)(\log_2 x)/R_2^2)$. Summing up the above bound over all $R_2 > \mathcal{L}_{1-6\alpha}$, we get a count of $O(x(\log x)(\log_2 x)/\mathcal{L}_{1-6\alpha})$. So the number of these b is smaller than $x/\mathcal{L}_\alpha$ for $x > x_0$.

20. We now look at the instance $m_1Q_2R_1 > x$ and $m_1Q_3R_2 > x$. We will show that this set is empty for $x > x_0$. Indeed, write $Q_2 = R_1\ell_1 - 1,\ Q_3 = R_2\ell_2 - 1$ for some even integers $\ell_1, \ell_2 > 0$. The inequalities

$$m_1Q_2R_1 > x \qquad \text{and} \qquad m_1Q_3R_2 > x$$

yield $m_1\sqrt{Q_2Q_3R_1R_2} > x$. Since $R_1R_2 = (Q_2+1)(Q_3+1)/(\ell_1\ell_2) \ll Q_2Q_3/\ell_1\ell_2$, we get

$$\frac{m_1Q_2Q_3}{\sqrt{\ell_1\ell_2}} \gg x. \tag{2.6}$$

Now

$$P_1s(m_1) + \sigma(m_1) = \sigma(b) - b = a(d/c) = Q_2Q_3(n_3d/c).$$

Since $\min\{Q_2, Q_3\} > \max\{d, c\}$, we have $c \mid n_3d$. Put $\ell_3 = n_3d/c$. Then $Q_2Q_3\ell_3 = \sigma(b) - b < \sigma(b) \ll x\log_2 x$. Thus, $Q_2Q_3 \ll x(\log_2 x)/\ell_3$. Hence, using (2.6),

$$\frac{xm_1\log_2 x}{\ell_3\sqrt{\ell_1\ell_2}} \gg \frac{m_1Q_1Q_2}{\sqrt{\ell_1\ell_2}} \gg x \qquad \text{giving} \qquad \ell_3\sqrt{\ell_1\ell_2} \ll m_1\log_2 x.$$

In particular,

$$\ell_1 \ll m_1^2(\log_2 x)^2, \qquad \ell_2 \ll m_1^2(\log_2 x)^2, \qquad \ell_3 \ll m_1\log_2 x. \tag{2.7}$$

Write $P_1 = R_1R_2\ell - 1$. We then have

$$(R_1R_2\ell - 1)s(m_1) + \sigma(m_1) = (R_1\ell_1 - 1)(R_2\ell_2 - 1)\ell_3,$$

which is equivalent to

$$(\ell s(m_1) - \ell_1\ell_2\ell_3)R_1R_2 + m_1 = -R_1\ell_1\ell_3 - R_2\ell_2\ell_3 + \ell_3. \tag{2.8}$$

Moving m_1 to the other side, dividing by R_1R_2 and using (2.7), we get

$$|\ell s(m_1) - \ell_1\ell_2\ell_3| = O\left(\frac{m_1}{R_1R_2} + \frac{\ell_1\ell_3}{R_2} + \frac{\ell_2\ell_3}{R_1}\right) = O\left(\frac{m_1^3(\log_2 x)^3}{\mathcal{L}_{1-6\alpha}}\right) = o(1) \qquad (x \to \infty),$$

where the last estimate above comes from the fact that m_1 fails (2.5). Since the left–hand side above is an integer, it must be 0 for $x > x_0$. Returning to (2.8), we get

$$m_1 = -R_1\ell_1\ell_3 - R_2\ell_2\ell_3 + \ell_3 < 0,$$

a contradiction. Hence, this case cannot occur once $x > x_0$.

Denouement. Glancing back through the argument, we find that every step can be carried out with $\alpha = \frac{1}{12}$. Hence, the total count of b-values is $O(x/\exp((\log x)^{1/12}))$. $\square$

To count values of a paired with b having $P^+(a) \leq P^+(b)$, we use a method introduced by Wirsing [16]. Wirsing showed that the number of solutions $n \leq x$ to an equation of the form $\sigma(n)/n = \beta$ is at most $\exp(O(\log x/\log\log x))$, uniformly in β. The next lemma provides a sharper bound if the number of primes dividing n is not too large.

Lemma 3. *Let k be a positive integer and let $x \geq 10^5$. Let $\beta \geq 1$ be a rational number. The number of integers $n \leq x$ with $\omega(n) \leq k$ and $\frac{\sigma(n)}{n} = \beta$ is at most $(2\log x)^{3k}$.*

Proof. Write $\beta = \lambda/\mu$, where the right-hand fraction is in lowest terms. If $\sigma(n)/n = \lambda/\mu$, then $\mu \mid n$. So we may assume that $\omega(\mu) \le k$. Given an n with $\sigma(n)/n = \beta$, put $\mathscr{P} = \{p \le 2k\} \cup \{p \mid \mu\}$, and write

$$n = AB, \qquad \text{where} \qquad A = \prod_{p \notin \mathscr{P}} p^{v_p(n)}, \quad B = \prod_{p \in \mathscr{P}} p^{v_p(n)}.$$

Note that $\mu \mid B$. The main idea of the proof is to show that B nearly determines its cofactor A. Specifically, we will show that for any given B, the number of corresponding A is at most $(\log x)^k$.

Since $\gcd(A, B) = 1$, we have

$$\sigma(A)\sigma(B) = \frac{\lambda}{\mu} AB. \tag{2.9}$$

Moreover,

$$1 \ge A/\sigma(A) > \prod_{p \mid A} (1 - 1/p) \ge 1 - \sum_{p \mid A} \frac{1}{p} \ge 1 - \frac{k}{2k+1} > \frac{1}{2}.$$

As a consequence,

$$\frac{1}{2}\left(\frac{\lambda}{\mu} B\right) < \sigma(B) = \frac{A}{\sigma(A)} \left(\frac{\lambda}{\mu} B\right) \le \frac{\lambda}{\mu} B. \tag{2.10}$$

Thus, $\sigma(B) \nmid \frac{\lambda}{\mu} B$ unless the final inequality is an equality, which occurs only if $A = 1$ and $\sigma(B) = \frac{\lambda}{\mu} B$.

Suppose that $\sigma(B) \nmid \frac{\lambda}{\mu} B$. Then there is a prime dividing $\sigma(B)$ to a higher power than $\frac{\lambda}{\mu} B$. Let p_1 be the least such prime and observe that p_1 is entirely determined by β and B. By (2.9), $p_1 \mid A$. Suppose that $p_1^{e_1} \parallel A$. Set $A_1 = A/p_1^{e_1}$ and $B_1 = Bp_1^{e_1}$. Then (2.9)–(2.10) hold with A and B replaced by A_1 and B_1, respectively. From the analogue of (2.10), we find that if $\sigma(B_1) \mid \frac{\lambda}{\mu} B_1$, then $A_1 = 1$, so that $A = p_1^{e_1}$.

Suppose that $\sigma(B_1) \nmid \frac{\lambda}{\mu} B_1$. There is a prime dividing $\sigma(B_1)$ to a higher power than it divides $\frac{\lambda}{\mu} B_1$. Let p_2 be the smallest such prime. Then p_2 is entirely determined by β, B, and e_1, and $p_2 \mid A_1$. If $p_2^{e_2} \parallel A_1$, we set $A_2 = A_1/p_2^{e_2}$ and $B_2 = B_1 p_1^{e_2}$. If $\sigma(B_2) \mid \frac{\lambda}{\mu} B_2$, then $A = p_1^{e_2} p_2^{e_2}$. If not, there is a prime dividing B_2 to a higher power than $\frac{\lambda}{\mu} B_2$, which allows us to continue the argument.

We carry out this process until $A_r = 1$, which happens in $r \le k$ steps. Then $A = p_1^{e_1} \cdots p_r^{e_r}$. Here each prime p_{i+1} is entirely determined by β, B, and $e_1, \dots, e_i$. Thus, A is entirely determined by β, B, and the exponent sequence $e_1, \dots, e_r$. Clearly,

$$3^{e_i} \le (2k+1)^{e_i} \le \prod_{i=1}^{r} p_i^{e_i} = A \le x,$$

and so each $e_i \in [1, \log x/\log 3]$. Extend $e_1, \dots, e_r$ to a sequence $e_1, \dots, e_k$ by putting $e_i = 0$ for $r < i \le k$. Since each $e_i \in [0, \frac{\log x}{\log 3}]$, the number of possibilities for $e_1, \dots, e_r$ is at most $(1 + \frac{\log x}{\log 3})^k \le (\log x)^k$, using in the last step that $x \ge 10^5$.

To bound the number of possibilities for $n = AB$, it now suffices to estimate the number of possibilities for B. We have $B = \prod_{p \in \mathscr{P}} p^{f_p}$, where each $f_p \in [0, \frac{\log x}{\log 2}]$. Thus, B belongs to a set of size at most

$$\left(1 + \frac{\log x}{\log 2}\right)^{\#\mathscr{P}} \le (2\log x)^{\#\mathscr{P}} \le (2\log x)^{\pi(2k)+\omega(\mu)} \le (2\log x)^{k+k} = (2\log x)^{2k}.$$

Putting everything together gives a final upper bound of $(\log x)^k \cdot (2\log x)^{2k} \le (2\log x)^{3k}$. $\square$

Proof of Theorem 1. In view of Proposition 2, it is enough to estimate the number of a involved in a pair a, b with $\max\{a, b\} \le x$ and $P^+(b) \ge P^+(a)$. With $K \ge 1$ to be specified shortly, we partition these a according to whether or not $\omega(a) \le K$. Since $\frac{\sigma(a)}{a}$ is determined by b, Lemma 3 and Proposition 2 show that the number of a with $\omega(a) \le K$ is

$$\ll (2\log x)^{3K} \cdot x/\exp((\log x)^{1/12}).$$

On the other hand, since $\tau(a) \geq 2^{\omega(a)}$, we have trivially that the number of a with $\omega(a) > K$ is

$$\ll \frac{x \log x}{2^K}.$$

Adding these two estimates and taking $K = (\log x)^{1/12}(\log\log x)^{-2}$ finishes the proof. □

Remark. We have shown that there are not many integers which are the member of some harmonious pair contained in $[1, x]$. It would be interesting to show that there are not too many such harmonious pairs. Note that the upper bound $x \exp(O(\log x/\log\log x))$ follows trivially from Wirsing's theorem. One cannot immediately derive a sharper estimate from Theorem 1, since a single integer may be shared among many pairs. However, Theorem 1 and Lemma 3 imply (arguing similarly to the proof just given) that the number of pairs with $\max\{a, b\} \leq x$ and $\min\{\omega(a), \omega(b)\} \leq (\log x)^{\frac{1}{12}-\delta}$ is at most $x/\exp((\log x)^{\frac{1}{12}+o(1)})$, for any fixed $\delta > 0$.

3. Discordant numbers

Given a number a, is there a number b for which the pair a, b is harmonious? If not, we say that a is *discordant*. Since a and b form a harmonious pair exactly when $\sigma(b)/b = \sigma(a)/s(a)$, deciding whether a is discordant amounts to solving a special case of the following problem:

Problem (Recognition problem for $\sigma(n)/n$). Decide whether a given rational number belongs to the image of the function $\sigma(n)/n$.

Rational numbers not in the range of $\sigma(n)/n$ have been termed *abundancy outlaws*. In the early 1970s, C.W. Anderson [1] conjectured that the set $\{\sigma(n)/n\}$ is *recursive*: In other words, an algorithm exists for deciding whether or not a given rational number is an outlaw. This conjecture is still open, but some partial results can be found in [8]. See also [15, 12, 6, 14].

Difficulties arise when trying to decide discordance even for small values of a. The smallest number whose status is unresolved seems to be $a = 11$; to answer this, we would need to know whether or not 12 is an abundancy outlaw. Anderson noted that $\frac{\sigma(b)}{b} = \frac{5}{3}$ if and only if $5b$ is an odd perfect number with $5 \nmid b$. Since $\frac{\sigma(24)}{s(24)} = \frac{5}{3}$, it follows that $a = 24$ is a member of a harmonious pair if and only if there is an odd perfect number precisely divisible by 5.

It is perhaps not immediately clear that there are infinitely many discordant numbers. Here we prove the following modest lower bound.

Proposition 4. *The number of discordant integers $n \leq x$ is at least $x/(\log x)^{(e^{-\gamma}+o(1))/\log_3 x}$ as $x \to \infty$.*

The following simple lemma can be found in [1] and [15].

Lemma 5. *Suppose v and u are coprime positive integers. If $v < \sigma(u)$, then v/u is an abundancy outlaw.*

Proof. Suppose $\frac{v}{u} = \frac{\sigma(n)}{n}$. Then $u \mid n$, so that $\frac{v}{u} = \frac{\sigma(n)}{n} \geq \frac{\sigma(u)}{u}$. Hence, $v \geq \sigma(u)$. □

Lemma 5 implies the following criterion for discordance.

Lemma 6. *If n, u, v are positive integers with $s(n)/\sigma(n) = u/v$, $\gcd(u, v) = 1$, and*

$$\frac{n}{\sigma(n)} + \frac{u}{\sigma(u)} < 1, \tag{3.1}$$

then n is discordant.

Proof. Our assumptions imply that $\frac{u}{\sigma(u)} < 1 - \frac{n}{\sigma(n)} = \frac{s(n)}{\sigma(n)} = \frac{u}{v}$, so that $v < \sigma(u)$. From Lemma 5, $\sigma(n)/s(n)$ is an outlaw; hence, n is discordant. □

We now are ready to prove Proposition 4.

Proof. Let $\epsilon > 0$ be arbitrary but fixed and let π be the largest prime number smaller than $e^{\gamma-\epsilon}\log_3 x$. Thus, for large enough x we have $\pi \geq 5$. Let $B = (\log_2 x)/(\log_3 x)^2$ and let A_0 denote the least common multiple of integers in $[1, B]$ coprime to π. Further, let A be the product of A_0 and all primes r with the property that $\pi \mid \sigma(r^{v_r(A_0)})$. That is, if r is a prime and $r^\alpha \parallel A_0$ with $\pi \mid \sigma(r^\alpha)$, we multiply by r. We have

$$\pi \nmid A\sigma(A), \quad A = \exp((1+o(1))B), \quad \sigma(A)/A = (e^\gamma + o(1))\log_3 x,$$

as $x \to \infty$.

Let k run over integers to $x^{1/4}$ such that $A \mid k$ and $\pi \nmid k\sigma(k)$. We would like a lower bound for $\sum 1/k$. For this, we restrict our attention to numbers of the form Aj, where $j \leq x^{1/5}$ is squarefree with no prime factors below B and no prime factors in the residue class $-1 \pmod{\pi}$. Let $i_0 = \lfloor 3\log\log(x^{1/5})\rfloor$ and let $\mathcal{S}$ denote the set of primes r in $(B, x^{1/i_0}]$ with $r \not\equiv -1 \pmod{\pi}$. For an integer $i \leq i_0$, the sum S_i of reciprocals of squarefree numbers $j \leq x^{1/5}$ composed solely of primes in $\mathcal{S}$ satisfies

$$S_i \geq \frac{1}{i!}\left(\sum_{r\in\mathcal{S}}\frac{1}{r}\right)^i - \frac{1}{(i-2)!}\sum_{r\in\mathcal{S}}\frac{1}{r^2}\left(\sum_{r\in\mathcal{S}}\frac{1}{r}\right)^{i-2} > \frac{1}{(i-2)!}\left(\sum_{r\in\mathcal{S}}\frac{1}{r}\right)^{i-2}\left(\frac{1}{i^2}\left(\sum_{r\in\mathcal{S}}\frac{1}{r}\right)^2 - \sum_{r\in\mathcal{S}}\frac{1}{r^2}\right).$$

By the prime number theorem for residue classes (cf. [8, Theorem 1]),

$$\sum_{r\in\mathcal{S}}\frac{1}{r} = \left(1 - \frac{1}{\pi-1}\right)\log\log x - \log\log B + O(1). \tag{3.2}$$

Thus, since $\sum_{r\in\mathcal{S}} 1/r^2 \ll 1/B$, this sum is small compared with $(1/i^2)(\sum_{r\in\mathcal{S}} 1/r)^2$, so that

$$\sum_k \frac{1}{k} \geq \frac{1}{A}\sum_j \frac{1}{j} \geq \frac{1}{A}\sum_{i\leq i_0} S_i \gg \frac{1}{A}\sum_{i\leq i_0}\frac{1}{i!}\left(\sum_{r\in\mathcal{S}}\frac{1}{r}\right)^i = \frac{1}{A}e^{\sum_{r\in\mathcal{S}}\frac{1}{r}} - \frac{1}{A}\sum_{i>i_0}\frac{1}{i!}\left(\sum_{r\in\mathcal{S}}\frac{1}{r}\right)^i = T_1 - T_2,$$

say. By (3.2), $T_1 \gg (\log x)^{1-1/(\pi-1)}/(A\log B)$. Also note that by (3.2),

$$T_2 \leq \sum_{i>i_0}\frac{1}{i!}\left(\sum_{r\in\mathcal{S}}\frac{1}{r}\right)^i \ll \frac{1}{i_0!}\left(\sum_{r\in\mathcal{S}}\frac{1}{r}\right)^{i_0} \leq \left(\frac{e}{i_0}\sum_{r\in\mathcal{S}}\frac{1}{r}\right)^{i_0} = o(1)$$

as $x \to \infty$. Thus,

$$\sum_k \frac{1}{k} \geq \frac{1}{A\log B}(\log x)^{1-(1+o(1))/\pi} = (\log x)^{1-(1+o(1))/\pi}, \quad x \to \infty.$$

Next, for each k chosen, let q run over primes to $x^{1/2}/k$, with $q \nmid k$, $\pi \nmid q(q+1)$, and

$$\pi \nmid qs(k) + \sigma(k) = s(qk).$$

To arrange for this last condition, note that if $\pi \mid s(k)$, it is true automatically, and if $\pi \nmid s(k)$, then there are at least $\pi - 3$ allowable residue classes for q modulo π (we discard the classes $0, 1, -\sigma(k)/s(k)$). For $m = qk$ so chosen, we have $A \mid m$, $\pi \nmid ms(m)\sigma(m)$, and by the prime number theorem for residue classes,

$$\sum_m \frac{1}{m} = \sum_k \frac{1}{k}\sum_q \frac{1}{q} \gg \sum_k \frac{1}{k} \geq (\log x)^{1-(1+o(1))/\pi}, \quad x \to \infty.$$

Finally, for each m, we let p run over primes to x/m where $p \nmid m$ and

$$\pi \mid ps(m) + \sigma(m) = s(pm). \tag{3.3}$$

For (3.3), we take p in the residue class $-\sigma(m)/s(m)$ modulo π. Since $\pi \nmid \sigma(m)s(m)$, this is a nonzero residue class. Further, it is not the class -1 since $\pi \nmid m$ implies that $\sigma(m) \not\equiv s(m) \pmod{\pi}$. Thus, if we choose p satisfying (3.3), then $\pi \nmid p+1$. For $n = pm$ so chosen we have by the prime number theorem for residue classes

$$\sum_{n \le x} 1 = \sum_m \sum_p 1 \gg \sum_m \frac{x/m}{\pi \log(x/m)} \gg \frac{x}{\pi \log x} \sum_m \frac{1}{m} \ge \frac{x}{(\log x)^{(1+o(1))/\pi}} \tag{3.4}$$

as $x \to \infty$.

It remains to note that for each number n constructed we have $n \le x$, $A \mid n$, and if $s(n)/\sigma(n) = u/v$ with u, v coprime, then $\pi \mid u$. Thus, as $x \to \infty$,

$$\frac{n}{\sigma(n)} + \frac{u}{\sigma(u)} \le \frac{A}{\sigma(A)} + \frac{\pi}{\pi+1} \le \frac{1}{(e^\gamma + o(1)) \log_3 x} + 1 - \frac{1}{e^{\gamma-\epsilon} \log_3 x + 1},$$

and this expression is smaller than 1 for all large x. Thus, by (3.1), n is discordant. Since $\epsilon > 0$ is arbitrary, (3.4) implies the proposition. □

The criterion (3.1) is sufficient for discordance but not necessary; there are abundancy outlaws of the form $\sigma(a)/s(a)$ not captured by Lemma 5. In order to detect (some) of these, we combine Lemma 5 with a bootstrapping procedure described in the following result.

Lemma 7 (Recursive criterion for outlaws)**.** *Let v and u be coprime positive integers. Let P be the product of any finite set of primes p for which $\frac{p^{v_p(u)}}{\sigma(p^{v_p(u)})} \cdot \frac{v}{u}$ is known to be an outlaw. If $\frac{\sigma(uP)}{uP} > \frac{v}{u}$, then $\frac{v}{u}$ is an outlaw.*

Proof. If $\sigma(n)/n = v/u$, then $u \mid n$. Let p be a prime dividing P. If $p^{v_p(u)} \parallel n$, then

$$\frac{\sigma(n/p^{v_p(u)})}{n/p^{v_p(u)}} = \frac{p^{v_p(u)}}{\sigma(p^{v_p(u)})} \cdot \frac{v}{u},$$

contradicting that the right-hand side is an outlaw. Thus, $p^{v_p(u)+1} \mid n$ for all p dividing P, and so $uP \mid n$. Hence, $\frac{v}{u} = \frac{\sigma(n)}{n} \ge \frac{\sigma(uP)}{uP}$, contrary to assumption. □

Example. As an illustration, let us show that 888 is discordant. We have $\frac{\sigma(888)}{s(888)} = \frac{95}{58}$. Then $2 \parallel 58$, and $\frac{2}{\sigma(2)} \cdot \frac{95}{58} = \frac{95}{87}$. Since $\sigma(87) = 120 > 95$, the fractional $\frac{95}{87}$ is a known outlaw by Lemma 5. Moreover, $\frac{\sigma(2 \cdot 58)}{2 \cdot 58} = \frac{105}{58} > \frac{95}{58}$. So Lemma 7, with $P = p = 2$, implies that $\frac{95}{58}$ is an outlaw.

Table 1 displays the counts of numbers up to 2^k, for $k = 1, 2, \ldots, 26$, that belong to a harmonious pair with other member at most 10^j, where $j = 8, 9, \ldots, 18$. To collect this data, we modified a gp script posted by Michael Marcus to the Online Encyclopedia of Integer Sequences (see [13]). Given a rational number $\frac{v}{u}$ and a search limit L, the script (provably) finds all $b \le L$ with $\frac{\sigma(b)}{b} = \frac{v}{u}$. For each $1 < a \le 2^k$, we used this script to determine whether or not the equation $\frac{\sigma(b)}{b} = \frac{\sigma(a)}{s(a)}$ has any solutions $b \le 10^j$.

Table 2 summarizes three counts: Numbers known to be harmonious because they are members of a pair contained in $[1, 10^{18}]$, numbers proved to be discordant, and numbers which fall into neither camp. The count of discordant numbers was obtained by tallying those $a > 1$ for which $\sigma(a)/s(a)$ could be determined to be an abundancy outlaw by at most five iterations of Lemma 7. (Here the 0th iteration corresponds to outlaws detected by Lemma 5.) We then added 1 to the counts, since 1 is discordant but not detected in this fashion.

It would be interesting to prove (or disprove) that the set of discordant integers has positive lower density. It seems possible that we could detect further classes of discordant numbers by developing some of the ideas introduced in [14] for finding abundancy outlaws; this deserves further study. In the opposite direction, we do not know how to show that there are infinitely many non-discordant integers, i.e., that there are infinitely many harmonious pairs.

	2^4	2^5	2^6	2^7	2^8	2^9	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}	2^{15}	2^{16}	2^{17}	2^{18}
10^8	10	18	37	70	127	226	367	594	944	1456	2227	3310	4838	6823	9493
10^9	10	18	38	74	135	240	397	657	1057	1663	2601	3962	5972	8701	12539
10^{10}	10	18	38	74	135	240	403	682	1126	1823	2888	4497	6936	10429	15457
10^{11}	10	19	39	77	141	250	420	715	1207	1978	3176	5009	7831	12076	18307
10^{12}	10	19	39	77	143	254	427	732	1254	2075	3390	5397	8599	13516	20895
10^{13}	10	19	39	79	146	258	434	745	1295	2157	3567	5742	9269	14755	23139
10^{14}	10	19	39	79	146	260	438	755	1322	2221	3704	6028	9796	15758	25025
10^{15}	10	19	39	79	147	262	444	765	1348	2273	3805	6254	10280	16674	26715
10^{16}	10	19	39	80	148	264	449	774	1362	2305	3895	6463	10684	17483	28223
10^{17}	10	19	39	80	148	266	457	787	1381	2339	3964	6616	11019	18139	29580
10^{18}	10	19	39	80	149	268	461	796	1398	2368	4031	6757	11275	18663	30640

	2^{19}	2^{20}	2^{21}	2^{22}	2^{23}	2^{24}	2^{25}	2^{26}
10^8	13035	17600	23294	30445	39200	49779	62363	77374
10^9	17792	24835	33953	45853	60956	79901	103318	131954
10^{10}	22586	32500	45843	63695	87100	117548	156567	205675
10^{11}	27393	40371	58276	83122	116711	161754	221399	298577
10^{12}	31939	47994	70793	103288	148490	210543	294805	406975
10^{13}	36008	55037	82861	123112	180642	261391	373696	526878
10^{14}	39631	61539	94240	142449	212625	313250	455894	655103
10^{15}	42844	67456	104686	160569	243473	364106	538838	787186
10^{16}	45660	72740	114179	177347	272600	413431	620475	920261
10^{17}	48190	77413	122830	192819	299822	460478	700065	1051622
10^{18}	50291	81454	130287	206485	324537	504113	775476	1179215

TABLE 1. Number of positive integers up to 2^k belonging to a harmonious pair with other member at most 10^j.

	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}	2^{15}	2^{16}	2^{17}	2^{18}	2^{19}
Harmonious	461	796	1398	2368	4031	6757	11275	18663	30640	50291
Discordant	27	49	103	209	418	822	1598	3154	6114	11849
Not classified	536	1203	2595	5615	11935	25189	52663	109255	225390	462148

	2^{20}	2^{21}	2^{22}	2^{23}	2^{24}	2^{25}	2^{26}
Harmonious	81454	130287	206485	324537	504113	775476	1179215
Discordant	22985	44710	87056	169084	329189	641109	1250156
Not classified	944137	1922155	3900763	7894987	15943914	32137847	64679493

TABLE 2. Counts up to various heights of numbers belonging to a harmonious pair in $[1, 10^{18}]$, numbers known to be discordant, and numbers fitting neither classification.

4. CONCLUDING REMARKS

Harmonious pairs have a surprising connection with a different generalization of amicable pairs recently studied by two of us [7]. Say that m and n form a δ-*amicable* pair if $\sigma(m) = \sigma(n) = m + n + \delta$. When $\delta = 0$, this reduces to the usual notion of an amicable pair. It was shown in [7] that for each fixed $\delta \neq 0$, the set of numbers in $[1, x]$ belonging to a δ-amicable pair has size $O_\delta(x(\log_2 x)^4/(\log x)^{1/2})$. The same authors conjectured that for arbitrary B, this count is

$$\ll_{\delta,B} x/(\log x)^B. \tag{4.1}$$

	2^{10}	2^{11}	2^{12}	2^{13}	2^{14}	2^{15}	2^{16}	2^{17}	2^{18}	2^{19}
$H_{\text{single}}(x)$	93	170	251	379	584	897	1323	1965	2909	4377
$H_{\text{pair}}(x)$	56	99	146	222	336	515	764	1130	1666	2500
$\Delta(x)$	46	74	118	187	285	432	651	979	1449	2181

	2^{20}	2^{21}	2^{22}	2^{23}	2^{24}	2^{25}	2^{26}
$H_{\text{single}}(x)$	6630	9865	14689	21537	31961	47311	69798
$H_{\text{pair}}(x)$	3787	5631	8383	12310	18279	27067	39934
$\Delta(x)$	3320	4934	7378	10959	16215	24055	35605

TABLE 3. Values of $H_{\text{single}}(x) = \#$ of n involved in a harmonious pair $a \le b \le x$, $H_{\text{pair}}(x) = \#$ of pairs $a \le b \le x$, and $\Delta(x) = \#$ of values of $\delta = a + b \le x$.

The conjectured upper bound (4.1) turns out to be too optimistic. To explain why, we first describe how to associate to a harmonious pair a, b a family of δ-amicable numbers with $\delta = a+b$. Since $a/\sigma(a)+b/\sigma(b) = 1$, the fractions $a/\sigma(a)$ and $b/\sigma(b)$ have the same denominator in lowest terms, say d. Thus, we can write

$$a/\sigma(a) = u/d \quad \text{and} \quad b/\sigma(b) = v/d,$$

where both right-hand fractions are reduced and $u + v = d$. Write

$$a = ua_0, \quad b = vb_0, \quad \sigma(a) = da_0, \quad \sigma(b) = db_0.$$

Put $n = ap$ and $m = bq$, where $p \nmid a$, $q \nmid b$ are primes. Then the equation $\sigma(n) = \sigma(m)$ amounts to requiring $\sigma(a)(p + 1) = \sigma(b)(q + 1)$, or equivalently, $a_0(p + 1) = b_0(q + 1)$. This holds precisely when

$$p = \frac{b_0}{(a_0, b_0)}t - 1 \quad \text{and} \quad q = \frac{a_0}{(a_0, b_0)}t - 1 \tag{4.2}$$

for some positive integer t. In that case,

$$\begin{aligned} n + m + \delta = ap + bq + \delta &= ua_0\left(\frac{b_0}{(a_0, b_0)}t - 1\right) + vb_0\left(\frac{a_0}{(a_0, b_0)}t - 1\right) + (a + b) \\ &= \frac{a_0b_0t}{(a_0, b_0)}(u + v) = \frac{da_0b_0t}{(a_0, b_0)}. \end{aligned}$$

But this last fraction is equal to both $\sigma(m)$ and $\sigma(n)$, and thus m and n form a δ-amicable pair.

We have constructed a pair of δ-amicable numbers from each pair of primes p, q satisfying (4.2), as long as $p \nmid a$ and $q \nmid b$. One expects that there are always infinitely many such pairs. When $b_0 = a_0$, which corresponds to the case when a, b form an amicable pair, this follows immediately from the prime number theorem for arithmetic progressions. In that case, the above construction produces $\gg x/\log x$ members of a δ-amicable pair not exceeding x, which is much larger than allowed by (4.1). If $b_0 \ne a_0$, we cannot rigorously prove the existence of infinitely many prime pairs satisfying (4.2), but this follows from the prime k-tuples conjecture. Here we expect the construction to produce $\gg x/(\log x)^2$ numbers in $[1, x]$ that belong to a δ-amicable pair. Again, this contradicts the conjectured bound (4.1).

The following related questions seem attractive but difficult.

Question. *Does the bound* (4.1) *hold if δ cannot be written as $a + b$ for any harmonious pair a, b?*

Question. *Let $\Delta(x)$ be the number of $\delta \le x$ that can be written as a sum of two members of a harmonious pair. Can one show that $\Delta(x) = o(x)$, as $x \to \infty$? Of course this would follow if we could show that the count $H_{\text{pair}}(x)$ of harmonious pairs in $[1, x]$ is $o(x)$. Perhaps $\Delta(x) \sim H_{\text{pair}}(x) \sim \frac{1}{2}H_{\text{single}}(x)$, where $H_{\text{single}}(x)$ is the quantity bounded in Theorem 1. See Table 3.*

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