# REAL REPRESENTATIONS

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The goal of these notes is to explain the classification of real representations of a finite group. Throughout, $G$ is a finite group, $W$ is a $\mathbb{R}$-vector space or $\mathbb{R}G$-module, and $V$ is a $\mathbb{C}$-vector space or $\mathbb{C}G$-module (except in Section 2, where $V$ is over any field). Vector spaces and representations are assumed to be finite-dimensional.

## 1. Vector spaces over $\mathbb{R}$ and $\mathbb{C}$

1.1. **Constructions.** To get from $\mathbb{R}^n$ to $\mathbb{C}^n$, we can tensor with $\mathbb{C}$. In a more coordinate-free manner, if $W$ is an $\mathbb{R}$-vector space, then its complexification $W_{\mathbb{C}} := W \otimes_{\mathbb{R}} \mathbb{C}$ is a $\mathbb{C}$-vector space. We can view $W$ as an $\mathbb{R}$-subspace of $W_{\mathbb{C}}$ by identifying each $w \in W$ with $w \otimes 1 \in W_{\mathbb{C}}$. Then an $\mathbb{R}$-basis of $W$ is also a $\mathbb{C}$-basis of $W_{\mathbb{C}}$. In particular, $W_{\mathbb{C}}$ has the same dimension as $W$ (but is a vector space over a different field).

Conversely, we can view $\mathbb{C}^n$ as $\mathbb{R}^{2n}$ if we forget how to multiply by complex scalars that are not real. In a more coordinate-free manner, if $V$ is a $\mathbb{C}$-vector space, then its restriction of scalars is the $\mathbb{R}$-vector space ${}_{\mathbb{R}}V$ with the same underlying abelian group but with only scalar multiplication by real numbers. If $v_1, \ldots, v_n$ is a $\mathbb{C}$-basis of $V$, then $v_1, iv_1, \ldots, v_n, iv_n$ is an $\mathbb{R}$-basis of ${}_{\mathbb{R}}V$. In particular, $\dim({}_{\mathbb{R}}V) = 2 \dim V$.

Also, if $V$ is a $\mathbb{C}$-vector space, then the complex conjugate vector space $\overline{V}$ has the same underlying group but a new scalar multiplication $\cdot$ defined by $\lambda \cdot v := \bar{\lambda} v$, where $\bar{\lambda} v$ is defined using the original scalar multiplication.

Complexification and restriction of scalars are not inverse constructions. Instead:

**Proposition 1.1** (Complexification and restriction of scalars).

(a) *If $V$ is a $\mathbb{C}$-vector space, then the map*

$$
\begin{aligned}
({}_{\mathbb{R}}V)_{\mathbb{C}} &\longrightarrow V \oplus \overline{V} \\
v \otimes c &\longmapsto (cv, \bar{c}v)
\end{aligned}
$$

*is an isomorphism of $\mathbb{C}$-vector spaces.*

(b) *If $W$ is an $\mathbb{R}$-vector space, then*

$$ {}_{\mathbb{R}}(W_{\mathbb{C}}) \simeq W \oplus W. $$

*Proof.*

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(a) The map is $\mathbb{C}$-linear, by definition of the scalar multiplication on $\overline{V}$. It sends $x \otimes 1 + y \otimes i$ to $(x + iy; x - iy)$, and one can recover $x; y \in V$ uniquely from $(x + iy; x - iy)$, so the map is an isomorphism.

(b) We have ${}_{\mathbb{R}}(W \otimes_{\mathbb{R}} \mathbb{C}) = W \otimes_{\mathbb{R}} (\mathbb{R} \oplus i\mathbb{R}) = W \oplus iW \simeq W \oplus W$. □

1.2. **Linear maps between complexifications.** Tensoring $\mathrm{M}_{m;n}(\mathbb{R})$ with $\mathbb{C}$ yields $\mathrm{M}_{m;n}(\mathbb{C})$. The coordinate-free version of this is

**Proposition 1.2.** *If $W$ and $X$ are $\mathbb{R}$-vector spaces, then*

$$\mathrm{Hom}_{\mathbb{R}}(W; X) \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathrm{Hom}_{\mathbb{C}}(W_{\mathbb{C}}; X_{\mathbb{C}}):$$

**Corollary 1.3.** *If $W$ is an $\mathbb{R}$-vector space, then*

$$\mathrm{End}_{\mathbb{R}}(W) \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathrm{End}_{\mathbb{C}}(W_{\mathbb{C}}):$$

1.3. **Descent theory.** Let $V$ and $X$ be $\mathbb{C}$-vector spaces. A homomorphism $J : V \to X$ of abelian groups is called $\mathbb{C}$-antilinear if $J(\lambda v) = \bar{\lambda} J(v)$ for all $\lambda \in \mathbb{C}$ and $v \in V$; to give such a $J$ is equivalent to giving a $\mathbb{C}$-linear map $V \to \overline{X}$.

To recover $\mathbb{R}^n$ from its complexifcation $\mathbb{C}^n$ one takes the vectors fixed by coordinate-wise complex conjugation. More generally, given a $\mathbb{C}$-vector space $V$, finding a $\mathbb{R}$-vector space $W$ such that $W_{\mathbb{C}} \simeq V$ is equivalent to finding a "complex conjugation" on $V$; more precisely:

**Proposition 1.4.** *There is an equivalence of categories*

$$\{\mathbb{R}\text{-vector spaces}\} \leftrightarrow \{\mathbb{C}\text{-vector spaces equipped with } \mathbb{C}\text{-antilinear } J : V \to V \text{ such that } J \;\; = 1\}$$

$$W \mapsto (W_{\mathbb{C}}; 1_W \otimes (\text{complex conjugation}))$$

$$V^J := \{v \in V : Jv = v\} \leftarrow\!\shortmid (V; J):$$

*Sketch of proof.* The only tricky part is to show that given $(V; J)$, the map $V^J \otimes_{\mathbb{R}} \mathbb{C} \to V$ sending $v \otimes c$ to $cv$ is an isomorphism. For this, one can write down the inverse: map $v \in V$ to $\text{-}(v + Jv) \otimes 1 + \frac{}{i}(v - Jv) \otimes i \in V^J \otimes_{\mathbb{R}} \mathbb{C}$. □

*Remark* 1.5. More generally, given any Galois extension of fields $L{=}k$, an action of $\mathrm{Gal}(L{=}k)$ on an $L$-vector space $V$ is called semilinear if scalar multiplication is compatible with the actions of $\mathrm{Gal}(L{=}k)$ on $L$ and $V$, that is, if ${}^g(\grave{}\,v) = ({}^g\grave{}\,)({}^g v)$ for all $g \in \mathrm{Gal}(L{=}k)$, $\grave{} \in L$ and $v \in V$. Then the category of $k$-vector spaces is equivalent to the category of $L$-vector spaces equipped with a semilinear $\mathrm{Gal}(L{=}k)$-action. This is called descent, since it specifies what extra structure is needed on an $L$-vector space to make it "descend" to a $k$-vector space.

1.4. **Representations.** All the constructions and propositions above are natural. In particular, if $G$ acts on $W$, then it acts on any of the spaces constructed from $W$, and likewise for $V$. In particular,

- If $W$ is an $\mathbb{R}G$-module, then $W_{\mathbb{C}}$ is a $\mathbb{C}G$-module, and the matrix of $g \in G$ acting on $W$ with respect to a basis is the same as the matrix of $g$ acting on $W_{\mathbb{C}}$, so $\chi_{W_{\mathbb{C}}} = \chi_W$.
- If $V$ is a $\mathbb{C}G$-module, then $\overline{V}$ is another $\mathbb{C}G$-module, and $\chi_{\overline{V}} = \overline{\chi}_V$.
- If $V$ is a $\mathbb{C}G$-module, then ${}_{\mathbb{R}}V$ is an $\mathbb{R}G$-module. Taking the characters of both sides in Proposition 1.1 shows that $\chi_{{}_{\mathbb{R}}V} = \chi_V + \overline{\chi}_V$.

A $\mathbb{C}$-representation $V$ of $G$ is said to be realizable over $\mathbb{R}$ if $V \simeq W_{\mathbb{C}}$ for some $\mathbb{R}$-representation $W$ of $G$. This implies that $\chi_V$ is real-valued, but we will see that the converse can fail.

## 2. Pairings

**2.1. Bilinear forms.** Let $V$ be a (finite-dimensional) vector space over any field $k$. A function $B \colon V \times V \to k$ is bi-additive if it is an additive homomorphism in each argument when the other is fixed; that is, $B(v_1 + v_2, w) = B(v_1, w) + B(v_2, w)$ for all $v_1, v_2, w \in V$, and $B(v, w_1 + w_2) = B(v, w_1) + B(v, w_2)$ for all $v, w_1, w_2 \in V$. The left kernel of $B$ is $\{v \in V : B(v, w) = 0 \text{ for all } w \in V\}$, and the right kernel is defined similarly.

A function $B \colon V \times V \to k$ is a bilinear form (or bilinear pairing) if it is $k$-linear in each argument; that is, $B$ is bi-additive and $B(\lambda v, w) = \lambda B(v, w)$ and $B(v, \lambda w) = \lambda B(v, w)$ for all $\lambda \in k$ and $v, w \in V$. We have

$$\{\text{bilinear forms on } V\} \simeq \operatorname{Hom}(V \otimes V, k) \simeq (V \otimes V)^* \simeq V^* \otimes V^* \simeq \operatorname{Hom}(V, V^*).$$

(here $\operatorname{Hom}$ is $\operatorname{Hom}_k$, and $\otimes$ is $\otimes_k$).

Let $B$ be a bilinear form.

- Call $B$ symmetric if $B(v, w) = B(w, v)$ for all $v, w \in V$.
- Call $B$ skew-symmetric if $B(v, w) = -B(w, v)$ for all $v, w \in V$.
- Call $B$ alternating if $B(v, v) = 0$ for all $v \in V$.

If $\operatorname{char} k \neq 2$, then alternating and skew-symmetric are equivalent. (If $\operatorname{char} k = 2$, then alternating is the stronger and better-behaved condition.) The map sending $(x, y) \mapsto B(x, y)$ to $(x, y) \mapsto B(y, x)$ is a linear automorphism of order $2$ of the space of bilinear forms, so if $\operatorname{char} k \neq 2$, it decomposes the space into $+1$ and $-1$ eigenspaces:

$$\{\text{bilinear forms}\} = \{\text{symmetric bilinear forms}\} \oplus \{\text{skew-symmetric bilinear forms}\},$$

which is the same as the decomposition

$$(V \otimes V)^* \simeq (\operatorname{Sym}^2 V)^* \oplus (\textstyle\bigwedge^2 V)^*.$$

**2.2. Sesquilinear and hermitian forms.** Now let $V$ be a $\mathbb{C}$-vector space.

- A sesquilinear form (or sesquilinear pairing) is a bi-additive pairing $(\ ,\ )$ that is $\mathbb{C}$-linear in the first variable and $\mathbb{C}$-antilinear in the second variable; that is $(\lambda v, w) = \lambda (v, w)$

and $(v, \lambda w) = \bar{\lambda}(v, w)$ for all $\lambda \in \mathbb{C}$ and $v, w \in V$. (The prefix "sesqui" means $1\frac{}{}$: the form is only $\mathbb{R}$-linear in the second argument.)

- A hermitian form (or hermitian pairing) is a bi-additive pairing $( \, , \, )$ such that $(\lambda v, w) = \lambda(v, w)$ and $(w, v) = \overline{(v, w)}$ for all $\lambda \in \mathbb{C}$ and $v, w \in V$.

A hermitian pairing is sesquilinear. We have

$$\{\text{sesquilinear forms on } V\} \simeq \operatorname{Hom}(V \otimes \overline{V}, \mathbb{C}) \simeq (V \otimes \overline{V}) \simeq V \otimes \overline{V} \simeq \operatorname{Hom}(\overline{V}, V).$$

2.3. **Nondegenerate and positive definite forms.** A bilinear form (or sesquilinear form) is called nondegenerate if its left kernel is $0$, or equivalently its right kernel is $0$, or equivalently the associated homomorphism $V \to V$ (respectively, $\overline{V} \to V$) is an isomorphism.

Suppose that $( \, , \, )$ is either a bilinear form on an $\mathbb{R}$-vector space or a hermitian form on a $\mathbb{C}$-vector space. Then $(v, v) \in \mathbb{R}$ for all $v$. Call $( \, , \, )$ positive definite if $(v, v) > 0$ for all nonzero $v \in V$. Positive definite forms are automatically nondegenerate.

## 3. Characters of symmetric and alternating squares

Let $V$ be an $n$-dimensional $\mathbb{C}$-representation of $G$. If $g \in G$ acts on $V$ with eigenvalues $\lambda, \ldots, \lambda_n$ (listed with multiplicity), then the eigenvalues of $g$ acting on associated vector spaces are as follows:

| Representation | Dimension | Eigenvalues |
|---|---|---|
| $V$ | $n$ | $\lambda, \ldots, \lambda_n$ |
| $\overline{V}$ | $n$ | $\bar{\lambda}, \ldots, \bar{\lambda}_n$ |
| $V$ | $n$ | $\bar{\lambda}, \ldots, \bar{\lambda}_n$ |
| $V \otimes V$ | $n$ | $\lambda_i \lambda_j$ for all $(i, j)$ |
| $\operatorname{Sym} V$ | $n(n+1)/2$ | $\lambda_i \lambda_j$ for $i \le j$ |
| $\bigwedge V$ | $n(n-1)/2$ | $\lambda_i \lambda_j$ for $i < j$ |

These are obvious if $V$ has a basis of eigenvectors (i.e., $\rho(g)$ is diagonalizable). In general, we have the Jordan decomposition $\rho(g) = d + n$, where $d$ is diagonalizable and $n$ is nilpotent, and $dn = nd$; then $d$ and $n$ induce commuting diagonalizable endomorphisms and nilpotent endomorphisms of each of the other representations, so the eigenvalues of $g$ are the same as the eigenvalues of $d$ on each of them.

## 4. Classification of division algebras over $\mathbb{R}$

**Lemma 4.1.** *The only finite-dimensional field extensions of $\mathbb{R}$ are $\mathbb{R}$ and $\mathbb{C}$.*

*Proof.* The fundamental theorem of algebra states that $\mathbb{C}$ is algebraically closed, so every finite extension of $\mathbb{R}$ embeds in $\mathbb{C}$. Since $[\mathbb{C} : \mathbb{R}] = 2$, there is no room for other fields in between. □

**Theorem 4.2** (Frobenius 1877). *The only finite-dimensional (associative) division algebras over $\mathbb{R}$ are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$.*

*Proof.* Let $D$ be a finite-dimensional (associative) division algebras over $\mathbb{R}$ not equal to $\mathbb{R}$ or $\mathbb{C}$. For any $d \in D - \mathbb{R}$, the $\mathbb{R}$-subalgebra $\mathbb{R}[d] \subseteq D$ generated by $d$ is a commutative domain of finite dimension over a field, so it is a field extension of finite degree over $\mathbb{R}$, hence a copy of $\mathbb{C}$. Fix one such copy, and let $i$ be a $\sqrt{-1}$ in it. View $D$ as a left $\mathbb{C}$-vector space. Conjugation by $i$ on $D$ (the map $x \mapsto ixi^{-}$ ) is a $\mathbb{C}$-linear automorphism of $D$, and it is of order $2$ since conjugation by $i \ = -1$ is the identity, so it decomposes $D$ into $+1$ and $-1$ eigenspaces $D$ and $D^-$. Explicitly,

$$D \ = \{x : ixi^{-} \ = x\} = \{x \text{ that commute with } i\} \supseteq \mathbb{C}$$
$$D^- = \{x : ixi^{-} \ = -x\}.$$

If $x \in D$ , then $\mathbb{C}[x]$ is commutative, hence a finite field extension of $\mathbb{C}$, but $\mathbb{C}$ is algebraically closed, so $\mathbb{C}[x] = \mathbb{C}$, so $x \in \mathbb{C}$. Thus $D \ = \mathbb{C}$.

Since $D \neq \mathbb{C}$, we have $D^- \neq 0$. Choose $j \in D^-$ such that $j \neq 0$. Right multiplication by $j$ defines a $\mathbb{C}$-linear map $D \ \to D^-$ (if $d \in D$ , then $i(dj)i^{-} \ = (idi^{-} \ )(iji^{-} \ ) = d(-j) = -dj$, so $dj \in D^-$), and it is injective since $D$ is a division algebra. Thus $\dim_{\mathbb{C}} D^- \leq \dim_{\mathbb{C}} D \ = 1$. Hence $D^- = \mathbb{C}j$. Since $\mathbb{R}[j]$ is another copy of $\mathbb{C}$, we have $j \ \in \mathbb{R} + \mathbb{R}j$. On the other hand $j \ \in D \ = \mathbb{C}$. Thus $j \ \in (\mathbb{R} + \mathbb{R}j) \cap \mathbb{C}$, which is $\mathbb{R}$, since $\mathbb{R} + \mathbb{R}j$ and $\mathbb{C}$ are different $2$-dimensional subspaces in $D$. Also, $j \ \neq 0$.

If $j \ > 0$, then $j \ = r$ for some $r \in \mathbb{R}$, so $(j + r)(j - r) = 0$, so $j = \pm r \in \mathbb{R}$, a contradiction since $D^- \cap \mathbb{R} = 0$.

Thus $j \ < 0$. Scale $j$ to assume that $j \ = -1$. Then $D = \mathbb{C} + \mathbb{C}j = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}ij$ with $i \ = -1$, $j \ = -1$, and $ij = -ji$, so $D \simeq \mathbb{H}$. ☐

If $D$ is an $\mathbb{R}$-algebra, then $D \otimes_{\mathbb{R}} \mathbb{C}$ is a $\mathbb{C}$-algebra.

**Proposition 4.3.** *We have*

$$\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C}$$
$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{C} \times \mathbb{C}$$
$$\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathrm{M} \ (\mathbb{C}).$$

*Proof.* The first isomorphism is a special case of the general isomorphism $A \otimes_A B \simeq B$.

The map $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ sending $a \otimes b$ to $(ab; a\bar{b})$ is an isomorphism by Proposition 1.1, and it respects multiplication.

There is a $\mathbb{C}$-algebra homomorphism $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \to \mathrm{M} \ (\mathbb{C})$. sending $h \otimes 1$ for each $h \in \mathbb{H}$ to the linear endomorphism $x \mapsto hx$ of the $2$-dimensional right $\mathbb{C}$-vector space $\mathbb{H}$ with basis $1; j$.

Explicitly, we have

$$1 \otimes 1 \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$i \otimes 1 \longmapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$j \otimes 1 \longmapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$ij \otimes 1 \longmapsto \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}:$$

For example, to get the image of $i \otimes 1$, observe that

$$i1 = 1 \cdot i + j \cdot 0$$
$$ij = 1 \cdot 0 + j \cdot (-i):$$

The four matrices on the right are linearly independent over $\mathbb{C}$, so $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \to \mathrm{M}\ (\mathbb{C})$ is an isomorphism of $4$-dimensional $\mathbb{C}$-algebras. □

## 5. Real and complex representations

Let $G$ be a finite group. Let $W$ be an irreducible $\mathbb{R}$-representation of $G$. Let $V$ be one irreducible $\mathbb{C}$-subrepresentation of $W_{\mathbb{C}}$. The following table gives facts about this situation.

| $D$ | $\mathrm{End}_G(W_{\mathbb{C}})$ | $W_{\mathbb{C}}$ | ${}_{\mathbb{R}}V$ | $\dim_{\mathbb{R}} W$ | $\dim_{\mathbb{C}} V$ | $V$ realiz. over $\mathbb{R}$? | $V \simeq \overline{V}$? $\chi_V$ real-valued? | $V \simeq V$ ? $\exists$ $G$-inv. $B$? | $\mathrm{FS}(V)$ |
|---|---|---|---|---|---|---|---|---|---|
| $\mathbb{R}$ | $\mathbb{C}$ | $V$ | $W \oplus W$ | $n$ | $n$ | YES | YES | YES (symmetric) | $1$ |
| $\mathbb{C}$ | $\mathbb{C} \times \mathbb{C}$ | $V \oplus \overline{V}$ | $W$ | $2n$ | $n$ | NO | NO | NO | $0$ |
| $\mathbb{H}$ | $\mathrm{M}\ (\mathbb{C})$ | $V \oplus V$ | $W$ | $4n$ | $2n$ | NO | YES | YES (skew-sym.) | $-1$ |

The columns are as follows:

- First, $D := \mathrm{End}_G W$. By Schur's lemma, $D$ is a division algebra over $\mathbb{R}$, so $D$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. Accordingly, $V$ is said to be of real type, complex type, or quaternionic type. Let $n$ be the dimension of $W$ as a right $D$-vector space.
- We have $\mathrm{End}_G(W_{\mathbb{C}}) \simeq (\mathrm{End}_G W) \otimes_{\mathbb{R}} \mathbb{C} = D \otimes_{\mathbb{R}} \mathbb{C}$ by taking $G$-invariants in Corollary 1.3.
- The $W_{\mathbb{C}}$ column gives the decomposition of $W_{\mathbb{C}}$ into irreducible $\mathbb{C}$-representations.
- The ${}_{\mathbb{R}}V$ column gives the decomposition of ${}_{\mathbb{R}}V$ into irreducible $\mathbb{R}$-representations.
- The $\dim_{\mathbb{R}} W$ column gives $\dim_{\mathbb{R}} W = [D : \mathbb{R}] \dim_D W = [D : \mathbb{R}]n$.

- The $\dim_{\mathbb{C}} V$ column entries follow from the $W_{\mathbb{C}}$ column and the column giving $\dim_{\mathbb{R}} W = \dim_{\mathbb{C}}(W_{\mathbb{C}})$.
- Is $V$ realizable over $\mathbb{R}$? That is, is $V \simeq X_{\mathbb{C}}$ for some $\mathbb{R}$-representation $X$ of $G$?
- Is $V \simeq \overline{V}$ as a $\mathbb{C}$-representation of $G$? Equivalently, is $\chi_V = \overline{\chi}_V$? That is, is it true that $\chi_V(g$

answers. By Section 2.1, we have isomorphisms

$$\mathrm{Hom}(V; V\ ) \simeq \{\text{symmetric bilinear forms}\} \oplus \{\text{skew-symmetric bilinear forms}\}:$$

Taking $G$-invariants yields

$$\mathrm{Hom}_G(V; V\ ) \simeq \{G\text{-invariant symm. bilinear forms}\} \oplus \{G\text{-invariant skew-symm. bilinear forms}\}:$$

Suppose that $V \simeq V$ . Then $\mathrm{Hom}_G(V; V\ ) \simeq \mathrm{End}_G V \simeq \mathbb{C}$ by Schur's lemma, so there exists a unique nondegenerate $G$-invariant bilinear form $B$ up to a scalar in $\mathbb{C}^\times$, and it is either symmetric or skew-symmetric. Since $B$ is nondegenerate, the $\mathbb{C}$-linear functional $(-; w)$ equals $B(-; Jw)$ for a unique $Jw \in V$. Then $J := V \to V$ is $\mathbb{C}$-antilinear, and it is an isomorphism since $( \ ; \ )$ too is nondegenerate. Now $J$ is a $\mathbb{C}$-*linear* automorphism of the representation $V$, so by Schur's lemma, $J$ is multiplication-by-$r$ for some $r \in \mathbb{C}^\times$. Also by Schur's lemma, every other $\mathbb{C}$-antilinear $G$-equivariant isomorphism is $cJ$ for some $c \in \mathbb{C}$, and replacing $J$ by $cJ$ changes $r$ to $c\bar{c}r$ (Proof: For $v \in V$, if $JJv = rv$, then $cJ(cJ(v)) = c\bar{c}J(J(v)) = c\bar{c}rv$).

- If $B$ is symmetric, then for any choice of nonzero $v \in V$,

  $$(Jv; Jv) = B(Jv; J\ v) = B(Jv; rv) = rB(Jv; v) = rB(v; Jv) = r(v; v)$$

  but $( \ ; \ )$ is positive definite, so $r$ is a positive real number.
- If $B$ is skew-symmetric, the same calculation shows that $r$ is a negative real number.

Finally, the following are equivalent:

- $V$ is realizable over $\mathbb{R}$
- We can choose $c \in \mathbb{C}^\times$ so that $(cJ) \ = 1$.
- We can choose $c \in \mathbb{C}^\times$ so that $c\bar{c}r = 1$.
- $r$ is positive.
- $B$ is symmetric.

*Frobenius–Schur indicator*: We have

$$\begin{aligned}
\overline{\mathrm{FS}(V)} &= \frac{1}{\#G}\sum_g \chi_{V^*}(g\ ) \\
&= \frac{1}{\#G}\sum_g \Big(\chi_{\ ^2V\ ^*}(g) - \chi_{\wedge\ V)}\ (g)\Big) \quad \text{(by the formulas in Section 3)} \\
&= (\mathbb{C}; (\mathrm{Sym}\ V)\ ) - (\mathbb{C}; (\textstyle\bigwedge\ V)\ ) \\
&= \dim\{G\text{-invariant symm. bilinear forms}\} - \dim\{G\text{-invariant skew-symm. bilinear forms}\} \\
&= \begin{cases} 1-0 \\ 0-0 \\ 0-1 \end{cases} = \begin{cases} 1; & \text{if } D = \mathbb{R}; \\ 0; & \text{if } D = \mathbb{C}; \\ -1; & \text{if } D = \mathbb{H}. \end{cases}
\end{aligned}$$

□

**Proposition 5.1.** *Every irreducible $\mathbb{C}$-representation $V$ of $G$ occurs in $W_{\mathbb{C}}$ for a unique irreducible $\mathbb{R}$-representation $W$ of $G$.*

*Proof.* By Proposition 1.1(a), $V$ occurs in $({}_{\mathbb{R}}V)_{\mathbb{C}}$, so $V$ occurs in $W_{\mathbb{C}}$ for some irreducible $\mathbb{R}$-subrepresentation $W$ of ${}_{\mathbb{R}}V$. If $W$ is any irreducible $\mathbb{R}$-representation such that $V$ occurs in $W_{\mathbb{C}}$, then the ${}_{\mathbb{R}}V$ column of the table shows that $W$ equals the unique irreducible $\mathbb{R}$-subrepresentation of ${}_{\mathbb{R}}V$, so $W$ is uniquely determined by $V$. □

**Theorem 5.2** (Frobenius–Schur). *We have*

$$\#\{g \in G : g^2 = 1\} = \sum_V (\dim V)\ \mathrm{FS}(V),$$

*where $V$ ranges over the irreducible $\mathbb{C}$-representations of $G$ up to isomorphism.*

*Proof.* The character of the regular representation $\mathbb{C}G$ is given by

$$\chi(g) = \begin{cases} \#G, & \text{if } g = 1; \\ 0, & \text{if } g \neq 1. \end{cases}$$

Thus

$$\begin{aligned} \#\{g \in G : g^2 = 1\} &= \frac{1}{\#G} \sum_g \chi(g^2) \\ &= \mathrm{FS}(\mathbb{C}G) \\ &= \sum_V (\dim V)\ \mathrm{FS}(V), \end{aligned}$$

since $\mathbb{C}G \simeq \bigoplus_V (\dim V) V$. □

*Remark* 5.3. Everything above for finite groups $G$ holds also for *compact* groups $G$. The only changes required are:

- All representations should be given by *continuous* homomorphisms.
- Averages over $G$ (such as in the definition of the Frobenius–Schur indicator) should be defined as *integrals* with respect to normalized Haar measure.
- Theorem 5.2 might fail or even fail to make sense.

*Remark* 5.4. Let $k$ be a field such that $\operatorname{char} k \nmid \#G$. Let $X_1, \ldots, X_r$ be the irreducible $k$-representations of $G$. Let $D_i = \operatorname{End}_G X_i$. Let $n_i$ be the dimension of $X_i$ as a right $D_i$-vector space. Then

$$\begin{aligned} kG &\simeq \prod_i^r \operatorname{End}_{D_i} X_i \\ &\simeq \prod_i^r \mathrm{M}_{n_i}(D_i). \end{aligned}$$

In particular,

$$\mathbb{R}G \simeq \prod \mathrm{M}_{d_i}(\mathbb{R}) \times \prod \mathrm{M}_{e_j}(\mathbb{C}) \times \prod \mathrm{M}_{f_k}(\mathbb{H})$$

for some positive integers $d_i; e_j; f_k$, and tensoring with $\mathbb{C}$ yields

$$\mathbb{C}G \simeq \prod \mathrm{M}_{d_i}(\mathbb{C}) \times \prod \left(\mathrm{M}_{e_j}(\mathbb{C}) \times \mathrm{M}_{e_j}(\mathbb{C})\right) \times \prod \mathrm{M}_{f_k}(\mathbb{C}):$$

## 6. Some conclusions to remember

- Every irreducible $\mathbb{C}$-representation $V$ of $G$ occurs in $W_{\mathbb{C}}$ for a unique irreducible $\mathbb{R}$-representation of $G$.
- The representation $V$ is said to be of real, complex, or quaternionic type according to whether $\mathrm{End}_G W$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.
- The type can be determined from the character $\chi_V$ by computing the Frobenius–Schur indicator.
- The representation $V$ is realizable over $\mathbb{R}$ if and only if $V$ is of real type, which happens if and only if there exists a nondegenerate $G$-invariant *symmetric* bilinear form $B: V \times V \to \mathbb{C}$.
- The representation $V$ is of complex type if and only if $V \not\simeq V$ ; in this case, there does not exist any nondegenerate $G$-invariant bilinear form $B: V \times V \to \mathbb{C}$.
- The representation $V$ is of quaternionic type if and only if there exists a nondegenerate $G$-invariant *skew-symmetric* bilinear form $B: V \times V \to \mathbb{C}$.
- If $V$ is realizable over $\mathbb{R}$, then $\chi_V$ is real-valued. *The converse is not true in general* (it fails exactly in the quaternionic case).

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