NÉRON-SEVERI GROUPS UNDER SPECIALIZATION

DAVESH MAULIK AND BJORN POONEN

Abstract. André used Hodge-theoretic methods to show that in a smooth proper family $\mathcal{X} \to B$ of varieties over an algebraically closed field k of characteristic 0, there exists a closed fiber having the same Picard number as the geometric generic fiber, even if k is countable. We give a completely different approach to André's theorem, which also proves the following refinement: in a family of varieties with good reduction at p, the locus on the base where the Picard number jumps is p-adically nowhere dense. Our proof uses the "p-adic Lefschetz $(1,1)$ theorem" of Berthelot and Ogus, combined with an analysis of p-adic power series. We prove analogous statements for cycles of higher codimension, assuming a p-adic analogue of the variational Hodge conjecture, and prove that this analogue implies the usual variational Hodge conjecture. Applications are given to abelian schemes and to proper families of projective varieties.

1. Introduction

1.1. **The jumping locus.** For a smooth proper variety X over an algebraically closed field, let $\operatorname{NS} X$ be its Néron-Severi group, and let $\rho(X)$ be the rank of $\operatorname{NS} X$. (A variety is a separated scheme of finite type over a field, possibly non-reduced or reducible. See Sections 2 and 3 for further definitions and basic facts.)

Now suppose that we have a smooth proper morphism $\mathcal{X} \to B$, where B is an irreducible variety over an algebraically closed field k of characteristic 0. If $b \in B(k)$, then choices lead to an injection of the Néron-Severi group $\operatorname{NS} \mathcal{X}_{\bar{\eta}}$ of the geometric generic fiber into the Néron-Severi group $\operatorname{NS} \mathcal{X}_b$ of the fiber above b, so $\rho(\mathcal{X}_b) \geq \rho(\mathcal{X}_{\bar{\eta}})$: see Proposition 3.6. The jumping locus

$$B(k)_{\text{jumping}} := \{b \in B(k) : \rho(\mathcal{X}_b) > \rho(\mathcal{X}_{\bar{\eta}})\}$$

is a countable union of lower-dimensional subvarieties of B. If k is uncountable, it follows that $B(k)_{\text{jumping}} \neq B(k)$.

This article concerns the general case, in which k may be countable. Our goal is to present a p-adic proof of the following theorem, first proved by Y. André [And96, Théorème 5.2(3)]:

Theorem 1.1. *Let k be an algebraically closed field of characteristic 0. Let B be an irreducible variety over k. Let $\mathcal{X} \to B$ be a smooth proper morphism. Then there exists $b \in B(k)$ such that $\rho(\mathcal{X}_b) = \rho(\mathcal{X}_{\bar{\eta}})$.*

Remark 1.2. In fact, Y. André's result is more general, stated in terms of variation of the motivic Galois group in the context of his theory of "motivated cycles". On the other hand,

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our techniques, which are completely different, give new information about the jumping locus.

Special cases were proved earlier by T. Shioda [Shi81] and most notably T. Terasoma [Ter85]. The arguments of Terasoma and André involve, among other ingredients, an application of a version of Hilbert irreducibility for infinite algebraic extensions associated to ℓ-adic representations. We will say more about their methods and their relationship with ours in Section 8.

Remark 1.3. The condition $\rho(\mathcal{X}_b) = \rho(\mathcal{X}_{\bar{\eta}})$ is equivalent to the condition that the specialization map $\operatorname{NS}\mathcal{X}_{\bar{\eta}} \to \operatorname{NS}\mathcal{X}_b$ is an isomorphism: see Proposition 3.6.

Remark 1.4. Theorem 1.1 can be trivially extended to an arbitrary ground field k of characteristic 0, to assert the existence of a closed point $b \in B$ such that the *geometric* Picard number of $\mathcal{X}_b$ equals $\rho(\mathcal{X}_{\bar{\eta}})$. Similarly, one could relax the assumption on B and allow it to be any irreducible scheme of finite type over k.

Remark 1.5. For explicit nontrivial examples of families $\mathcal{X} \to B$ over $\overline{\mathbb{Q}}$ with $b \in B(\overline{\mathbb{Q}})$ such that $\rho(\mathcal{X}_b) = \rho(\mathcal{X}_{\bar{\eta}})$, see [Shi81] and [vL07].

1.2. **The p-adic approach.** For our proof of Theorem 1.1, we embed a suitable finitely generated subfield of k in a p-adic field (see Section 6) and apply Theorem 1.7 below, which states that for a "family of varieties with good reduction" (in the strong sense of having a smooth proper family as in Setup 1.6, and not only the existence of models fiber-by-fiber), the jumping locus is *p-adically nowhere dense.*

Setup 1.6. Let K be a field that is complete with respect to a nontrivial discrete valuation, and let k be the residue field. For any valued field L, let $\mathcal{O}_L$ denote its valuation ring. Assume that $\operatorname{char} K = 0$ and $\operatorname{char} \mathsf{k} = p > 0$, and that k is perfect. Let C be the completion of an algebraic closure of K; then C also is algebraically closed (see [Kür13, §46] or [Rib99, p. 142]). Let B be an irreducible separated finite-type $\mathcal{O}_K$-scheme, and let $f \colon \mathcal{X} \to B$ be a smooth proper morphism.

Theorem 1.7. *Assume Setup 1.6. For $b \in B(\mathcal{O}_C) \subseteq B(C)$, let $\mathcal{X}_b$ be the C-variety above b. Then the set*

$$B(\mathcal{O}_C)_{\text{jumping}} := \{b \in B(\mathcal{O}_C) : \rho(\mathcal{X}_b) > \rho(\mathcal{X}_{\bar{\eta}})\}$$

is nowhere dense in $B(\mathcal{O}_C)$ for the analytic topology.

To prove Theorem 1.7, we apply a "p-adic Lefschetz $(1,1)$ theorem" of P. Berthelot and A. Ogus [BO83, Theorem 3.8] to obtain a down-to-earth local analytic description (Lemma 4.2) of the jumping locus in $B(\mathcal{O}_C)$. This eventually reduces the problem to a peculiar statement (Proposition 5.1) about linear independence of values of linearly independent p-adic power series.

Remark 1.8. It is well known (cf. [BLR90, p. 235]) that the archimedean analogue of Theorem 1.7 is false. For example, let B be an irreducible $\mathbb{C}$-variety, let $\mathcal{E} \to B$ be a family of elliptic curves such that the j-invariant map $j \colon B \to \mathbb{A}^1$ is dominant, and let $\mathcal{X} = \mathcal{E} \times_B \mathcal{E}$. For an elliptic curve E over an algebraically closed field, $\rho(E \times E) = 2 + \operatorname{rk} \operatorname{End} E$ (cf. the Rosati involution comment in the proof of Proposition 1.13). So $B(\mathbb{C})_{\text{jumping}}$ is the set of CM points in $B(\mathbb{C})$, i.e., the points for which the corresponding elliptic curve has complex multiplication. In the j-line, the set of CM points is the image of $\{z \in \mathbb{C} : \operatorname{im}(z) > 0 \text{ and } [\mathbb{Q}(z) : \mathbb{Q}] = 2\}$

under the usual analytic uniformization by the upper half plane. This image is dense in $\mathbb{A}^1(\mathbb{C})$, so its preimage under j is dense in $B(\mathbb{C})$.

Remark 1.9. Remark 1.8 is a particular case of a general topological density theorem of Mark Green [Voi03, Proposition 5.20], as we now explain. In the setting of Theorem 1.1 over $k = \mathbb{C}$, the Lefschetz $(1,1)$ theorem and [Voi03, Lemma 5.13] let us define countably many closed complex analytic subspaces Z_i of the analytification B^{an} such that the union of the associated subsets of $B(\mathbb{C})$ equals $B(\mathbb{C})_{\mathrm{jumping}}$. If some Z_i is reduced and of the "expected codimension" $h^{2,0}$, then [Voi03, Proposition 5.20] implies that $B(\mathbb{C})_{\mathrm{jumping}}$ is dense in $B(\mathbb{C})$ with respect to the complex topology. The argument ultimately relies on the topological density of $\mathbb{Q}$ inside $\mathbb{R}$.

Remark 1.10. We can give a heuristic explanation of the difference between $\mathbb{C}$ and a field like $C = \mathbb{C}_p$. Namely, $[\mathbb{C} : \mathbb{R}] = 2$, but the analogous p-adic quantity $[\mathbb{C}_p : \mathbb{Q}_p]$ is infinite (in fact, equal to $2^{\aleph_0}$ [Lam86]). So a subvariety in $B(\mathbb{C}_p)$ of positive codimension can be thought of as having infinite $\mathbb{Q}_p$-codimension. This makes it less surprising that a countable union of such subvarieties could be p-adically nowhere dense.

Remark 1.11. G. Yamashita, in response to an earlier version of this article, has generalized the p-adic Lefschetz $(1,1)$ theorem from the smooth case to the semistable case, and has used our method to extend Theorem 1.7 to the case where $\mathcal{X} \to B$ is semistable [Yam11].

Remark 1.12. If $\mathcal{X} \to B$ is as in Remark 1.8, but over an algebraic closure k of a finite field $\mathbb{F}_p$, then again we have $\rho(\mathcal{X}_{\bar{\eta}}) = 3$, but now $\rho(\mathcal{X}_b) \geq 4$ for all $b \in B(k)$ since every elliptic curve E over k has endomorphism ring larger than $\mathbb{Z}$. Thus the characteristic p analogue of Theorem 1.1 fails. On the other hand, it seems likely that it holds for any algebraically closed field k that is not algebraic over a finite field.

1.3. **Applications to abelian varieties.** J.-P. Serre [Ser00, pp. 1–17] and R. Noot [Noo95, Corollary 1.5] used something like Terasoma's method, combined with G. Faltings' proof of the Tate conjecture for homomorphisms between abelian varieties, to prove that in a family of abelian varieties over a finitely generated field of characteristic 0, there exists a geometric closed fiber whose endomorphism ring equals that of the geometric generic fiber. Independently at around the same time, D. Masser [Mas96] used methods of transcendence theory to give a different proof, one that gives quantitative estimates of the number of fibers where the endomorphism ring jumps.

Theorem 1.1 reproves the existence result without Faltings' work or transcendence theory, and Theorem 1.7 strengthens this by showing that in the p-adic setting, the corresponding jumping locus is p-adically nowhere dense in the good reduction locus:

Proposition 1.13. *Assume Setup 1.6, and assume moreover that $\mathcal{X} \to B$ is an abelian scheme. Then*

$$\{b \in B(\mathcal{O}_C) : \operatorname{End} \mathcal{X}_{\bar{\eta}} \hookrightarrow \operatorname{End} \mathcal{X}_b \textit{ is not an isomorphism}\}$$

is nowhere dense in $B(\mathcal{O}_C)$ for the analytic topology.

Proof. Choose a polarization on $\mathcal{X}_\eta$, and replace B by a dense open subvariety to assume that it extends to a polarization of $\mathcal{X} \to B$. For a polarized abelian variety A over an algebraically closed field, let ι be the Rosati involution on $(\operatorname{End} A)_{\mathbb{Q}}$ (where the subscript denotes $\otimes\mathbb{Q}$). Then $(\operatorname{NS} A)_{\mathbb{Q}}$ is isomorphic to the fixed subspace $(\operatorname{End} A)^{\iota}_{\mathbb{Q}}$: see [Mum70, p. 190], for instance. This implies

$$\rho(A) = \dim(\operatorname{End} A)^{\iota}_{\mathbb{Q}}$$
$$\rho(A \times A) = 2\rho(A) + \dim(\operatorname{End} A)_{\mathbb{Q}}.$$

If in a family, $\dim(\operatorname{End} A)_{\mathbb{Q}}$ jumps, then so will $\rho(A\times A)$; conversely, if $\dim(\operatorname{End} A)_{\mathbb{Q}}$ does not jump, then neither does $\rho(A)$ (since the Rosati involution respects specialization), so $\rho(A\times A)$ also does not jump. Thus the $(\operatorname{End} A)_{\mathbb{Q}}$ jumping locus for $\mathcal{X} \to B$ equals the Picard number jumping locus for $\mathcal{X} \times_B \mathcal{X} \to B$. Apply Theorem 1.7 to $\mathcal{X} \times_B \mathcal{X}$. Finally, $(\operatorname{End} \mathcal{X}_{\bar\eta})_{\mathbb{Q}} \hookrightarrow (\operatorname{End} \mathcal{X}_b)_{\mathbb{Q}}$ is an isomorphism if and only if $\operatorname{End} \mathcal{X}_{\bar\eta} \hookrightarrow \operatorname{End} \mathcal{X}_b$ is an isomorphism, as one sees by considering the action on torsion points (this uses characteristic 0). ☐

Remark 1.14. Theorem 1.7 of [Noo95] states that for any algebraic group G arising as a Mumford-Tate group of a complex abelian variety, there exists an abelian variety A over a number field F such that the Mumford-Tate group of A equals G and such that moreover the Mumford-Tate conjecture holds; i.e., the action of $\operatorname{Gal}(\overline{F}/F)$ on a Tate module $T_\ell A$ gives an *open* subgroup in $G(\mathbb{Q}_\ell)$. A specialization result for the Mumford-Tate group follows easily from [And96, Théorème 5.2(3)] too.

It would be natural to conjecture a "nowhere dense" analogue, i.e., that the locus in a family of abelian varieties where the dimension of the Mumford-Tate group drops is p-adically nowhere dense in the good reduction locus. But we know how to prove this only if we assume Conjecture 9.2 from Section 9.

A proof similar to that of Proposition 1.13 yields another application of Theorem 1.1:

Proposition 1.15. *Let k be an algebraically closed field of characteristic* 0. *Let A be an abelian variety defined over k. Let B be an irreducible k-variety. Let $\mathcal{X} \to B$ be an abelian scheme such that $\mathcal{X}_b$ is isogenous to A for all $b \in B(k)$. Then $\mathcal{X}_{\bar\eta}$ is isogenous to $A_{\bar\eta} := A \times_k \bar\eta$.*

Sketch of proof. Let $A \sim \prod_{i=1}^r A_i^{n_i}$ be a decomposition of A up to isogeny into simple factors. Applying Theorem 1.1 to $\mathcal{X} \times A_i \to B$ shows that the multiplicity of $(A_i)_{\bar\eta}$ in the decomposition of $\mathcal{X}_{\bar\eta}$ equals n_i. Since the relative dimension of $\mathcal{X} \to B$ equals $\dim A$, this accounts for all simple factors of $\mathcal{X}_{\bar\eta}$. ☐

Remark 1.16. At least when B is integral and $\mathcal{X} \to B$ is projective (which is automatic if B is normal [FC90, 1.10(a)]), the conclusion of Proposition 1.15 implies that $\mathcal{X} \to B$ becomes constant after a finite étale base change $B' \to B$. This can be proved as follows. The kernel of an isogeny $A_{\bar\eta} \to \mathcal{X}_{\bar\eta}$ is the base extension of a finite group scheme G over k, since k is algebraically closed of characteristic 0. Replacing A by A/G, we may assume that $A_{\bar\eta} \simeq \mathcal{X}_{\bar\eta}$. Projectivity of $\mathcal{X} \to B$ yields a polarization on $\mathcal{X}$, and the corresponding polarization on $A_{\bar\eta}$ comes from a polarization defined over k (cf. Proposition 3.1). Choose $\ell \geq 3$, and replace B by a finite étale cover such that $\mathcal{X}[\ell] \simeq (\mathbb{Z}/\ell\mathbb{Z})^{2g}_B$. This lets us choose level-ℓ structures so that $A_{\bar\eta} \to \mathcal{X}_{\bar\eta}$ becomes an isomorphism of polarized abelian varieties with level-ℓ structure. Let $\mathcal{M}$ be the moduli scheme over k of polarized abelian varieties with level-ℓ structure. Then $\mathcal{X}$ gives rise to a k-morphism $B \to \mathcal{M}$ mapping $\bar\eta$ to a k-point. Since B is integral, $B \to \mathcal{M}$ is constant.

Remark 1.17. Under the appropriate hypotheses on k and $\mathcal{X} \to B$, Theorem 1.7 proves the analogous strengthening of Proposition 1.15. Namely, assuming Setup 1.6, if $\mathcal{X} \to B$ is an abelian scheme and $\mathcal{X}_{\bar{\eta}}$ is not isogenous to $A_{\bar{\eta}}$, then the set of $b \in B(\mathcal{O}_C)$ such that $\mathcal{X}_b$ is isogenous to A is p-adically nowhere dense.

1.4. **Outline of the article.** After introducing some notation in Section 2, we review some standard facts about Néron-Severi groups and specialization maps in Section 3. The next three sections prove Theorem 1.7 and use it to prove Theorem 1.1: Section 4 discusses some basic properties of crystalline cohomology and convergent isocrystals, and applies them to give a local description of the jumping locus; Section 5 proves the key p-adic power series proposition to be applied to understand this local description. Section 6 completes the proofs of Theorems 1.7 and 1.1.

Section 7 gives an application of Theorem 1.1: if all closed fibers in a smooth proper family are projective, then there exists a dense open subvariety of the base over which the family is projective, assuming that the base is a variety in characteristic 0. Section 8, which uses only étale and Betti cohomology, and some Hodge theory, sketches André's approach to Theorem 1.1, and compares the information it provides on the jumping locus to what is obtained from the p-adic approach.

Finally, Section 9 explains conditional generalizations of our results to cycles of higher codimension. The generalization of Theorem 1.7 is proved assuming a p-adic version of the variational Hodge conjecture (Conjecture 9.2). We also prove that the p-adic variational Hodge conjecture implies the classical variational Hodge conjecture.

2. NOTATION

If A is a commutative domain, let $\operatorname{Frac}(A)$ denote its fraction field. If $A \to B$ is a ring homomorphism, and M is an A-module, let M_B denote the B-module $M \otimes_A B$. If k is a field, then $\overline{k}$ denotes an algebraic closure, chosen consistently whenever possible. Given a prime number p, let $\mathbb{Z}_p$ be the ring of p-adic integers, let $\mathbb{Q}_p = \operatorname{Frac}(\mathbb{Z}_p)$, choose algebraic closures $\overline{\mathbb{Q}} \subseteq \overline{\mathbb{Q}}_p$, and let $\mathbb{C}_p$ denote the completion of $\overline{\mathbb{Q}}_p$.

For any S-schemes X and T, let X_T be the T-scheme $X \times_S T$. For a commutative ring R, we may write R as an abbreviation for $\operatorname{Spec} R$. If B is an irreducible scheme, let η denote its generic point. If $b \in B$, let $\kappa(b)$ be its residue field and let $\bar{b} = \operatorname{Spec} \overline{\kappa(b)}$. For example, if $\mathcal{X} \to B$ is a morphism, then $\mathcal{X}_{\bar{\eta}}$ is called the geometric generic fiber. Also let $\kappa(B)$ be the function field $\kappa(\eta)$. If B is a variety over a field F, let $|B|$ be the set of closed points of B; also choose an algebraic closure $\overline{F}$ and for all $b \in |B|$ view $\overline{\kappa(b)}$ as a subfield of $\overline{F}$. If X is a variety over a field equipped with an embedding in $\mathbb{C}$, then X^{an} denotes the associated complex analytic space.

If $\mathscr{X}$ is a complex analytic space and i is a nonnegative integer, then we have the Betti cohomology $\mathrm{H}^i(\mathscr{X}, F)$ for any field F. If X is a variety over a field k, and i and j are integers with $i \geq 0$, and ℓ is a prime not divisible by the characteristic of k, then we have the étale cohomology $\mathrm{H}^i_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell(j))$, which is equipped with a $\operatorname{Gal}(\overline{k}/k)$-action (replace $\overline{k}$ by a separable closure if k is not perfect).

3. BASIC FACTS ON NÉRON-SEVERI GROUPS

3.1. **Picard groups and Néron-Severi groups.** For a scheme or formal scheme X, let $\operatorname{Pic} X$ be its Picard group. If X is a smooth proper variety over an algebraically closed field, let $\operatorname{Pic}^0 X$ be the subgroup consisting of isomorphism classes of line bundles algebraically equivalent to 0 (i.e., to $\mathscr{O}_X$), and define the Néron-Severi group $\operatorname{NS} X := \operatorname{Pic} X / \operatorname{Pic}^0 X$. The abelian group $\operatorname{NS} X$ is finitely generated [Nér52, p. 145, Théorème 2] (see [SGA 6, XIII.5.1] for another proof), and its rank is called the Picard number $\rho(X)$.

Proposition 3.1. *If $k \subseteq k'$ are algebraically closed fields, and X is a smooth proper k-variety, then the natural homomorphism $\operatorname{NS} X \to \operatorname{NS} X_{k'}$ is an isomorphism.*

Proof. The Picard scheme $\mathbf{Pic}_{X/k}$ is a group scheme that is locally of finite type over k (this holds more generally for any proper scheme over a field: see [Mur64, II.15], which uses [Oor62]). Then $\operatorname{Pic}^0 X$ is the set of k-points of the identity component of $\mathbf{Pic}_{X/k}$ [Kle05, 9.5.10]. So $\operatorname{NS} X$ is the group of components of $\mathbf{Pic}_{X/k}$. Thus $\operatorname{NS} X$ is unchanged by algebraically closed base extension. ☐

Remark 3.2. The Nakai-Moishezon criterion [Deb01, Theorem 1.21] implies that ampleness of a Cartier divisor on a proper scheme X over any field K depends only on its class in $\operatorname{NS} X_L$ for any algebraically closed field L containing K.

3.2. **Specialization of Néron-Severi groups.**

Proposition 3.3 (cf. [SGA 6, X App 7]). *Let R be a discrete valuation ring with fraction field K and residue field k. Fix an algebraic closure $\overline{K}$ of K. Choose a nonzero prime ideal $\mathfrak{p}$ of the integral closure $\overline{R}$ of R in $\overline{K}$, so $\overline{k} := \overline{R}/\mathfrak{p}$ is an algebraic closure of k. Let X be a smooth proper R-scheme. Then there is a natural homomorphism*

$$\operatorname{sp}_{\overline{K},\overline{k}} \colon \operatorname{NS} X_{\overline{K}} \to \operatorname{NS} X_{\overline{k}}.$$

depending only on the choices above. Moreover, if $\operatorname{sp}_{\overline{K},\overline{k}}$ maps a class $[\mathscr{L}]$ to an ample class, then $\mathscr{L}$ is ample.

Proof. As in [SGA 6, X App 7.8] or the proof of [BLR90, §8.4, Theorem 3], we have

$$\operatorname{Pic} X_K \xleftarrow{\sim} \operatorname{Pic} X \to \operatorname{Pic} X_k. \tag{3.4}$$

If $\mathscr{L}$ is a line bundle on X_K whose image in $\operatorname{Pic} X_k$ is ample, then the corresponding line bundle on X is ample relative to $\operatorname{Spec} R$ by [EGA III$_1$, 4.7.1], so $\mathscr{L}$ is ample too.

For each finite extension L of K in $\overline{K}$, the integral closure R_L of R in L is a Dedekind ring by the Krull-Akizuki theorem [Bou98, VII.2.§5, Proposition 5], and localizing at $\mathfrak{p} \cap R_L$ gives a discrete valuation ring R'_L. Take the direct limit over L of the analogue of (3.4) for R'_L to get $\operatorname{Pic} X_{\overline{K}} \to \operatorname{Pic} X_{\overline{k}}$ (cf. [SGA 6, X App 7.13.3]).

This induces $\operatorname{NS} X_{\overline{K}} \to \operatorname{NS} X_{\overline{k}}$ (cf. [SGA 6, X App 7.12.1]); a sketch of an alternative argument for this is as follows: First we can pass from R to its completion to reduce to the case that R is excellent. It suffices to show the following (after replacing R and K by finite extensions): Given a smooth proper geometrically connected K-curve C_K and a line bundle $\mathscr{L}_K$ on $X_K \times C_K$, any two fibers above points in $C_K(K)$ specialize to algebraically equivalent line bundles on X_k. By Lipman's resolution of singularities for 2-dimensional excellent schemes [Lip78], C_K extends to a regular proper flat R-scheme C, and C_k is geometrically connected by Stein factorization (cf. [EGA III$_1$, 4.3.12]). The two specialized line bundles are fibers above points of $C(k)$ of an extension of $\mathscr{L}_K$ to the regular scheme $X \times_R C$, so they are algebraically equivalent.

The ampleness claim follows from Remark 3.2 and the statement for Pic already discussed. ☐

Remark 3.5. In Proposition 3.3, if R is complete, or more generally henselian, then there is only one choice of $\mathfrak{p}$.

Proposition 3.6. *Let B be a noetherian scheme. Let $s, t \in B$ be such that s is a specialization of t (i.e., s is in the closure of $\{t\}$). Let $p = \operatorname{char} \kappa(s)$. Let $\mathcal{X} \to B$ be a smooth proper morphism. Then it is possible to choose a homomorphism*

$$\operatorname{sp}_{\bar{t},\bar{s}} \colon \operatorname{NS} \mathcal{X}_{\bar{t}} \to \operatorname{NS} \mathcal{X}_{\bar{s}}$$

with the following properties:

(a) *If $p = 0$, then $\operatorname{sp}_{\bar{t},\bar{s}}$ is injective and $\operatorname{coker}(\operatorname{sp}_{\bar{t},\bar{s}})$ is torsion-free.*
(b) *If $p > 0$, then* (a) *holds after tensoring with $\mathbb{Z}[1/p]$.*
(c) *If $\operatorname{sp}_{\bar{t},\bar{s}}$ maps a class $[\mathscr{L}]$ to an ample class, then $\mathscr{L}$ is ample.*
Moreover:
(i) *In all cases, $\rho(\mathcal{X}_{\bar{s}}) \geq \rho(\mathcal{X}_{\bar{t}})$.*

Proof. A construction of $\operatorname{sp}_{\bar{t},\bar{s}}$ is explained at the beginning of [SGA 6, X App 7.17]: the idea is to choose a discrete valuation ring R with a morphism $\operatorname{Spec} R = \{s', t'\} \to B$ mapping s' to s and t' to t, to obtain

$$\operatorname{NS} \mathcal{X}_{\bar{t}} \xrightarrow{\sim} \operatorname{NS} \mathcal{X}_{\bar{t}'} \xrightarrow{\operatorname{sp}_{\bar{t}',\bar{s}'}} \operatorname{NS} \mathcal{X}_{\bar{s}'} \xleftarrow{\sim} \operatorname{NS} \mathcal{X}_{\bar{s}},$$

with the outer isomorphisms coming from Proposition 3.1.

For any prime $\ell \neq p$, there is a commutative diagram

$$\begin{tikzcd} \operatorname{NS} \mathcal{X}_{\bar{t}} \otimes \mathbb{Z}_\ell \arrow[r, hook] \arrow[d, "\operatorname{sp}_{\bar{t},\bar{s}}"'] & \mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\bar{t}}, \mathbb{Z}_\ell(1)) \arrow[d, equal] \\ \operatorname{NS} \mathcal{X}_{\bar{s}} \otimes \mathbb{Z}_\ell \arrow[r, hook] & \mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\bar{s}}, \mathbb{Z}_\ell(1)) \end{tikzcd} \tag{3.7}$$

(cf. [SGA 6, 7.13.10]: there everything is tensored with $\mathbb{Q}$, but the explanation shows that in our setting we need only tensor with $\mathbb{Z}[1/(i-1)!]$ with $i = 1$). This proves the injectivity in (a) and (b). By (3.7), $\operatorname{coker}(\operatorname{sp}_{\bar{t},\bar{s}}) \otimes \mathbb{Z}_\ell$ is contained in $\operatorname{coker}\left(\operatorname{NS} \mathcal{X}_{\bar{t}} \otimes \mathbb{Z}_\ell \to \mathrm{H}^2_{\text{ét}}(\mathcal{X}_{\bar{t}}, \mathbb{Z}_\ell(1))\right)$. Using the Kummer sequence, one shows [Mil80, V.3.29(d)] that the latter is $T_\ell \operatorname{Br} \mathcal{X}_{\bar{t}} := \varprojlim_n (\operatorname{Br} \mathcal{X}_{\bar{t}})[\ell^n]$, which is automatically torsion-free; this proves the torsion-freeness in (a) and (b). Finally, (c) follows from the corresponding part of Proposition 3.3, and (i) follows from (a) and (b). ☐

Proposition 3.8. *Let B be a noetherian scheme. For a smooth proper morphism $\mathcal{X} \to B$ and a nonnegative integer n, define*

$$B_{\geq n} := \{b \in B : \rho(\mathcal{X}_{\bar{b}}) \geq n\}.$$

(a) *The set $B_{\geq n}$ is a countable union of Zariski closed subsets of B.*
(b) *If we base change by a morphism $\iota \colon B' \to B$ of noetherian schemes, then $B'_{\geq n} = \iota^{-1}(B_{\geq n})$.*

Proof. Proposition 3.1 proves (b).

Now we prove (a). Proposition 3.6(i) says that $B_{\geq n}$ contains the closure of any point in $B_{\geq n}$. So if $B = \operatorname{Spec} A$ for some finitely generated $\mathbb{Z}$-algebra A, then $B_{\geq n}$ is the (countable) union over $b \in B_{\geq n}$ of the closure of $\{b\}$. Combining this with (b) proves (a) for any noetherian affine scheme. Finally, if B is any noetherian scheme, write $B = \bigcup_{i=1}^n B_i$ with B_i affine, let C_i be the union of the closures in B of the generic points of all the irreducible components of the closed subsets of B_i appearing in the countable union for $(B_i)_{\geq n}$, and let $C = \bigcup_{i=1}^n C_i$. Then $B_{\geq n} = \bigcup_{i=1}^n (B_i)_{\geq n} \subseteq C$ and the opposite inclusion follows using Proposition 3.6(i) again as above. ☐

Corollary 3.9. *Let $k \subseteq k'$ be algebraically closed fields. Let B be an irreducible k-variety. For a smooth proper morphism $\mathcal{X} \to B$, the jumping locus*

$$B(k')_{\text{jumping}} := \{b \in B(k') : \rho(\mathcal{X}_b) > \rho(\mathcal{X}_{\overline{\eta}})\}$$

is the union of $Z(k')$ where Z ranges over a countable collection of closed k-subvarieties of B.

Proof. Proposition 3.8(a) yields subvarieties Z for the case $k' = k$. The same subvarieties work for larger k' by Proposition 3.8(b). ☐

3.3. **Pathological behavior in positive characteristic.** The material in this section is not needed elsewhere in this article. Let R be a discrete valuation ring, and define $K, k, \overline{K}, \overline{k}$ as in Section 3.2. The two examples below show that $\operatorname{sp}_{\overline{K},\overline{k}}$ is not always injective.

Example 3.10. There exist R of equicharacteristic 2 and a smooth proper morphism $X \to \operatorname{Spec} R$ such that $X_{\overline{K}}$ and $X_{\overline{k}}$ are Enriques surfaces of type $\mathbb{Z}/2\mathbb{Z}$ and α_2, respectively [BM76, p. 222]. (The type refers to the isomorphism class of the scheme $\mathbf{Pic}^\tau$ parametrizing line bundles numerically equivalent to 0.) In this case $\operatorname{NS} X_{\overline{K}} \to \operatorname{NS} X_{\overline{k}}$ has a nontrivial kernel, generated by the canonical class of $X_{\overline{K}}$, an element of order 2.

Example 3.11. There exist R of mixed characteristic $(0, 2)$ and a smooth proper morphism $X \to \operatorname{Spec} R$ such that $X_{\overline{K}}$ and $X_{\overline{k}}$ are Enriques surfaces of type $\mathbb{Z}/2\mathbb{Z}$ and μ_2, respectively [Lan83, Theorem 1.3], so again we have a nontrivial kernel.

Next, we give an example showing that $\operatorname{coker}(\operatorname{sp}_{\overline{K},\overline{k}})$ is not always torsion-free.

Example 3.12. Let $\mathcal{O}$ be the maximal order of an imaginary quadratic field in which p splits. Let $\mathcal{O}'$ be the order of conductor p in $\mathcal{O}$. Over a finite extension R of $\mathbb{Z}_p$, there exists a p-isogeny $\psi \colon E \to E'$ between elliptic curves over R such that $\operatorname{End} E_{\overline{K}} \simeq \mathcal{O}$ and $\operatorname{End} E'_{\overline{K}} \simeq \mathcal{O}'$. Since p splits, E has good ordinary reduction and $\operatorname{End} E_{\overline{k}} \simeq \mathcal{O}$. But ψ must reduce to either Frobenius or Verschiebung, so $\operatorname{End} E'_{\overline{k}} \simeq \mathcal{O}$ too. Using that $\operatorname{coker}\left(\operatorname{End} E'_{\overline{K}} \to \operatorname{End} E'_{\overline{k}}\right)$ is of order p, one can show that the cokernel of $\operatorname{NS}((E' \times E')_{\overline{K}}) \to \operatorname{NS}((E' \times E')_{\overline{k}})$ contains nonzero elements of order p.

4. Convergent isocrystals and de Rham cohomology

We now begin work toward the p-adic proof of Theorem 1.1.

4.1. Goal of this section.

Definition 4.1. Assume Setup 1.6. Let $d = \dim B_K$. Let b be a smooth $\overline{K}$-point on B_K. If B is a closed subscheme of $\mathbb{A}^n$, a polydisk neighborhood of b is a neighborhood U of b in $B(\overline{K})$ in the analytic topology equipped with, for some $\epsilon > 0$, a bijection

$$\Delta_{d,\epsilon} := \{(z_1, \ldots, z_d) \in \overline{K}^d : |z_i| \leq \epsilon\} \to U$$

defined by an n-tuple of power series in $z_1, \dots, z_d$ with coefficients in some finite extension of K. (Such neighborhoods exist by the implicit function theorem. If we replace the embedding $B \hookrightarrow \mathbb{A}^n$, by a different one, $B \hookrightarrow \mathbb{A}^{n'}$, the notion of polydisk neighborhood of b changes, but the new *system* of polydisk neighborhoods of b is cofinal with the original one.) A polydisk neighborhood of b in an arbitrary B is a polydisk neighborhood of b in some affine open subscheme of B. Let $\mathscr{H}(U)$ be the subring of $\overline{K}[[z_1, \dots, z_d]]$ consisting of power series g with coefficients in some finite extension of K such that g converges on $\Delta_{d,\epsilon}$.

The goal of Section 4 is to prove the following:

Lemma 4.2. *Assume Setup 1.6. Let $b_0 \in B(\mathcal{O}_{\overline{K}}) \subseteq B(\overline{K})$ be such that $B_{\overline{K}}$ is smooth at b_0. Then there exists a polydisk neighborhood U of b_0 contained in $B(\mathcal{O}_{\overline{K}})$ and a finitely generated $\mathbb{Z}$-submodule $\Lambda \subseteq \mathscr{H}(U)^n$ for some n such that*

$$\{b \in U : \rho(\mathcal{X}_b) > \rho(\mathcal{X}_{\bar{\eta}})\} = \bigcup_{\substack{\lambda \in \Lambda \\ \lambda \neq 0}} (\text{zeros of } \lambda \text{ in } U).$$

Its proof will be completed in Section 4.7.

Remark 4.3. The analogue over $\mathbb{C}$ is a well-known consequence of the Lefschetz $(1,1)$ theorem, together with [Voi03, Lemma 5.13] and its proof. But the union will often be dense in $B(\mathbb{C})$, so this analogue is not useful for our purposes.

4.2. **Coherent sheaves on formal schemes.** Assume that K, $\mathcal{O}_K$, and k are as in Setup 1.6. In this section, we work with noetherian formal schemes over $\operatorname{Spf} \mathcal{O}_K$. If X is a noetherian $\mathcal{O}_K$-scheme, let $\widehat{X}$ be its completion with respect to the ideal sheaf $p\mathscr{O}_X$. Recall "formal GAGA":

Theorem 4.4.

(a) *If X is proper over $\mathcal{O}_K$, then the functor* $\operatorname{Coh}(X) \to \operatorname{Coh}(\widehat{X})$ *carrying $\mathscr{F}$ to its p-adic completion $\widehat{\mathscr{F}}$ is an equivalence of categories* [EGA III$_1$, Corollaire 5.1.6].

(b) *Under this equivalence, line bundles on X correspond to line bundles on $\widehat{X}$.*

(c) *If $f \colon \mathcal{X} \to B$ is a proper morphism of noetherian $\mathcal{O}_K$-schemes, $\widehat{f} \colon \widehat{\mathcal{X}} \to \widehat{B}$ is the induced morphism of formal schemes, $\mathscr{F} \in \operatorname{Coh}(\mathcal{X})$, and $q \geq 0$, then the natural morphism $\widehat{\mathrm{R}^q f_* \mathscr{F}} \to \mathrm{R}^q \widehat{f}_* \widehat{\mathscr{F}}$ in $\operatorname{Coh}(\widehat{B})$ is an isomorphism* [EGA III$_1$, Théorème 4.1.5]. *(For a construction of this morphism in a more general context, see Section 4.5.3.)*

We write $K \otimes (\cdot)$ as an abbreviation for $K \otimes_{\mathcal{O}_K} (\cdot)$. Similarly, $\overline{K} \otimes (\cdot)$ means $\overline{K} \otimes_{\mathcal{O}_K} (\cdot)$.

Definition 4.5 (cf. [Ogu84, Definition 1.1])**.** For any noetherian formal scheme T over $\operatorname{Spf} \mathcal{O}_K$, let $\operatorname{Coh}(K \otimes \mathscr{O}_T)$ denote the full subcategory of $(K \otimes \mathscr{O}_T)$-modules isomorphic to $K \otimes \mathscr{F}$ for some coherent $\mathscr{O}_T$-module $\mathscr{F}$. Equivalently, we could consider the category whose objects are coherent $\mathscr{O}_T$-modules but whose set of morphisms from $\mathscr{F}$ to $\mathscr{G}$ is $K \otimes \operatorname{Hom}(\mathscr{F}, \mathscr{G})$.

Definition 4.6. Similarly, if B is a noetherian $\mathcal{O}_K$-scheme, define $\operatorname{Coh}(K \otimes \mathscr{O}_B)$ to be the category whose objects are coherent $\mathscr{O}_B$-modules and whose set of morphisms from $\mathscr{F}$ to $\mathscr{G}$ is $K \otimes \operatorname{Hom}(\mathscr{F}, \mathscr{G})$.

Proposition 4.7. *Let B be a noetherian $\mathcal{O}_K$-scheme.*

(a) *The functor* $\mathrm{Coh}(K \otimes \mathscr{O}_B) \to \mathrm{Coh}(B_K)$ *sending* $\mathscr{F}$ *to* $\mathscr{F}|_{B_K}$ *is an equivalence of categories.*
(b) *There is a functor* $\mathrm{Coh}(K \otimes \mathscr{O}_B) \to \mathrm{Coh}(K \otimes \mathscr{O}_{\widehat{B}})$ *sending* $\mathscr{F}$ *to* $\widehat{\mathscr{F}}$*.*
(c) *The resulting functor* $\mathrm{Coh}(B_K) \to \mathrm{Coh}(K \otimes \mathscr{O}_{\widehat{B}})$ *is compatible with pullback by an* $\mathcal{O}_K$*-morphism* $B' \to B$ *on each side.*
(d) *If* $B = \operatorname{Spec} \mathcal{O}_K$*, then the functor in* (c) *is an equivalence of categories.*

Proof. This is all straightforward. In (d), both categories are equivalent to the category of finite-dimensional K-vector spaces. ☐

Remark 4.8. Let B be a separated finite-type $\mathcal{O}_K$-scheme. The rigid-analytic generic fiber of the formal scheme $\widehat{B}$ is open in the rigid analytification $(B_K)^{\mathrm{an}}$ of B_K. So, given a coherent $\mathscr{O}_B$-module $\mathscr{F}$ and $b_0 \in B(\mathcal{O}_K) \subseteq B(K)$ such that B_K is smooth at b_0, we have two routes to construct the "restriction" of $\mathscr{F}$ to a sheaf on a sufficiently small rigid-analytic neighborhood N of b_0: one route goes through $\widehat{\mathscr{F}}$, and the other goes through $\mathscr{F}_K$ on B_K. The resulting sheaves are canonically isomorphic on a sufficiently small N. In particular, if $\mathscr{F}_K$ is locally free of rank n on B_K, and U is a sufficiently small polydisk neighborhood of b_0, then a choice of local trivialization of $\mathscr{F}_K$ lets us "restrict" global sections of $\overline{K \otimes \widehat{\mathscr{F}}}$ to obtain elements of $\mathscr{H}(U)^n$. (Although we have used some language of rigid geometry in this remark, it is not needed anywhere else in this article.)

4.3. **Definition of convergent isocrystal.** In this subsection, we recall some key notions of [Ogu84], specialized slightly to the case we need. Assume Setup 1.6, but without the requirement that B is irreducible.

Definition 4.9 (cf. [Ogu84, §1])**.** A p-adic formal scheme over $\mathcal{O}_K$ is a noetherian formal scheme T of finite type over $\operatorname{Spf} \mathcal{O}_K$. (See [EGA I, §10] for basic definitions regarding formal schemes.)

Given a p-adic formal $\mathcal{O}_K$-scheme T, let T_1 be the closed subscheme defined by the ideal sheaf $p\mathscr{O}_T$, and let T_0 be the associated reduced subscheme $(T_1)_{\mathrm{red}}$.

Definition 4.10 (cf. [Ogu84, Definition 2.1])**.** An enlargement of B_{k} is a p-adic formal $\mathcal{O}_K$-scheme T equipped with a k-morphism $z\colon T_0 \to B_{\mathsf{k}}$. A morphism of enlargements $(T', z') \to (T, z)$ is an $\mathcal{O}_K$-morphism $T' \to T$ such that the induced k-morphism $T_0' \to T_0$ followed by z equals z'.

Example 4.11. Given $s \in B(\mathsf{k})$, let $\lceil s \rceil$ denote the enlargement $(\operatorname{Spf} \mathcal{O}_K, \operatorname{Spec} \mathsf{k} \xrightarrow{s} B_{\mathsf{k}})$ of B_{k}.

Example 4.12. If $\gamma\colon B' \to B$ is a morphism of $\mathcal{O}_K$-schemes of finite type, then we view $\widehat{B'}$ as an enlargement of B_{k} by equipping it with the k-morphism $\left(\widehat{B'}\right)_0 = (B'_{\mathsf{k}})_{\mathrm{red}} \to B_{\mathsf{k}}$ induced by γ.

Definition 4.13 (cf. [Ogu84, Definition 2.7])**.** A convergent isocrystal on B_{k} consists of the following data:

(a) For every enlargement (T, z) of B_{k}, a sheaf $E_T \in \mathrm{Coh}(K \otimes \mathscr{O}_T)$.
(b) For every morphism of enlargements $g\colon (T', z') \to (T, z)$, an isomorphism $\theta_g\colon g^*E_T \to E_{T'}$ in $\mathrm{Coh}(K \otimes \mathscr{O}_T)$. If $h\colon (T'', z'') \to (T', z')$ is another, the cocycle condition $\theta_h \circ h^*\theta_g = \theta_{g \circ h}$ is required, and $\theta_{\mathrm{id}} = \mathrm{id}$.

4.4. **Crystalline cohomology.**

Definition 4.14 (cf. [Gro68, §7] and [Ber74, III.1.1])**.** Let K and k be as in Setup 1.6, and let W be the Witt ring of k, so $\mathcal{O}_K$ is finite as a W-module. Given a smooth proper k-variety X and $q \in \mathbb{Z}_{\geq 0}$, we have the crystalline cohomology $\mathrm{H}^q_{\mathrm{cris}}(X/W)$, which is a finite W-module. Define

$$\mathrm{H}^q_{\mathrm{cris}}(X/K) := K \otimes_W \mathrm{H}^q_{\mathrm{cris}}(X/W).$$

There is a Chern class homomorphism [Gro68, §7.4]

$$c_1^{\mathrm{cris}} \colon \operatorname{Pic} X \to \mathrm{H}^2_{\mathrm{cris}}(X/K).$$

Remark 4.15. The fact that crystalline cohomology is a Weil cohomology [GM87] implies [Kle68, 1.2.1] that $c_1^{\mathrm{cris}}(\mathscr{L})$ depends only on the image of $\mathscr{L}$ in $\operatorname{NS} X_{\overline{\mathsf{k}}}$.

If instead of a single k-variety we have a family, then the K-vector space $\mathrm{H}^2_{\mathrm{cris}}(X/K)$ is replaced by a compatible system of sheaves, i.e., a convergent isocrystal:

Theorem 4.16 (cf. [Ogu84, Theorems 3.1 and 3.7])**.** *Assume Setup 1.6, but without the requirement that B is irreducible. For each $q \in \mathbb{Z}_{\geq 0}$, there exists a convergent isocrystal E on B_{k} with an isomorphism of K-vector spaces $E_{\lceil s \rceil} \simeq \mathrm{H}^q_{\mathrm{cris}}(\mathcal{X}_s/K)$ for each $s \in B(\mathsf{k})$.*

Following [Ogu84], we write $\mathrm{R}^q_{\mathrm{cris}} f_* \mathscr{O}_{\mathcal{X}/K}$ for the convergent isocrystal E whose construction is suggested by Theorem 4.16. (We added the subscript $_{\mathrm{cris}}$ for extra clarity.) Despite the notation, it depends only on the morphism $\mathcal{X}_{\mathsf{k}} \to B_{\mathsf{k}}$ (cf. [Ogu84, 3.9.3]).

4.5. **De Rham cohomology.**

4.5.1. *Algebraic de Rham cohomology.*

Definition 4.17. Let $f \colon \mathcal{X} \to B$ be a smooth proper morphism of noetherian schemes, and let $q \in \mathbb{Z}_{\geq 0}$. The q^{th} de Rham cohomology is defined as the coherent $\mathscr{O}_B$-module $\mathbb{R}^q f_* \Omega^\bullet_{\mathcal{X}/B}$ (cf. [Gro66]). It has a Hodge filtration in $\operatorname{Coh}(B)$ given by

$$\operatorname{Fil}^p(\mathbb{R}^q f_* \Omega^\bullet_{\mathcal{X}/B}) := \operatorname{im}\left(\mathbb{R}^q f_* \Omega^{\geq p}_{\mathcal{X}/B} \to \mathbb{R}^q f_* \Omega^\bullet_{\mathcal{X}/B}\right).$$

where $\Omega^{\geq p}$ is obtained from $\Omega^\bullet$ by replacing terms in degrees less than p by 0. Also define the coherent $\mathscr{O}_B$-module

$$\mathcal{H}^{02}(\mathcal{X}) := \frac{\mathbb{R}^2 f_* \Omega^\bullet_{\mathcal{X}/B}}{\operatorname{Fil}^1(\mathbb{R}^2 f_* \Omega^\bullet_{\mathcal{X}/B})}.$$

In the case $B = \operatorname{Spec} K$ for a field K, these $\mathscr{O}_B$-modules are K-vector spaces (we may then write H^{02} instead of $\mathcal{H}^{02}$), and there is a Chern class homomorphism (cf. [Har75, II.7.7])

$$c_1^{\mathrm{dR}} \colon \operatorname{Pic} \mathcal{X} \to \mathrm{H}^2_{\mathrm{dR}}(\mathcal{X}) := \mathbb{H}^2(\mathcal{X}, \Omega^\bullet_{\mathcal{X}/K}).$$

De Rham cohomology over varieties behaves well under pullback in characteristic 0:

Proposition 4.18. *Let K be a field of characteristic 0. Let B be a noetherian K-scheme. Let $f \colon \mathcal{X} \to B$ be a smooth proper morphism.*

(a) *The $\mathscr{O}_B$-module $\mathbb{R}^q f_* \Omega^\bullet_{\mathcal{X}/B}$ is locally free, and $\operatorname{Fil}^p(\mathbb{R}^q f_* \Omega^\bullet_{\mathcal{X}/B})$ is a* subbundle *for each p. In particular, $\mathcal{H}^{02}(\mathcal{X})$ is locally free.*

(b) *Let* $\alpha\colon B' \to B$ *be a morphism of noetherian* K*-schemes. Let* $f'\colon \mathcal{X}' \to B'$ *be the base extension of* f *by* α*. The natural map*

$$\alpha^* \mathbb{R}^q f_* \Omega^\bullet_{\mathcal{X}/B} \to \mathbb{R}^q f'_* \Omega^\bullet_{\mathcal{X}'/B'}$$

is an isomorphism. Moreover, it sends $\alpha^* \operatorname{Fil}^p(\mathbb{R}^q f_* \Omega^\bullet_{\mathcal{X}/B})$ *isomorphically to* $\operatorname{Fil}^p(\mathbb{R}^q f'_* \Omega^\bullet_{\mathcal{X}'/B'})$*. In particular, taking quotients yields*

$$\alpha^* \mathcal{H}^{02}(\mathcal{X}) \simeq \mathcal{H}^{02}(\mathcal{X}').$$

Proof. See [Del68, Théorème 5.5]. ☐

4.5.2. *Formal de Rham cohomology.* There is an analogous definition of de Rham cohomology for formal schemes:

Definition 4.19. Let $g\colon \mathcal{Y} \to T$ be a smooth proper morphism of noetherian formal schemes over $\operatorname{Spf} \mathcal{O}_K$ and let $q \in \mathbb{Z}_{\geq 0}$. The sheaf $K \otimes \mathbb{R}^q g_* \widehat{\Omega}^\bullet_{\mathcal{Y}/T} \in \operatorname{Coh}(K \otimes \mathscr{O}_T)$ has a Hodge filtration in $\operatorname{Coh}(K \otimes \mathscr{O}_T)$ given by

$$\operatorname{Fil}^p(K \otimes \mathbb{R}^q g_* \widehat{\Omega}^\bullet_{\mathcal{Y}/T}) := \operatorname{im}\left(K \otimes \mathbb{R}^q g_* \widehat{\Omega}^{\geq p}_{\mathcal{Y}/T} \to K \otimes \mathbb{R}^q g_* \widehat{\Omega}^\bullet_{\mathcal{Y}/T}\right).$$

Also define the sheaf

$$\mathcal{H}^{02}(\mathcal{Y}/K) := \frac{K \otimes \mathbb{R}^2 g_* \widehat{\Omega}^\bullet_{\mathcal{Y}/T}}{\operatorname{Fil}^1(K \otimes \mathbb{R}^2 g_* \widehat{\Omega}^\bullet_{\mathcal{Y}/T})}.$$

In the case $T = \operatorname{Spf} \mathcal{O}_K$, we define $\mathrm{H}^q_{\mathrm{dR}}(\mathcal{Y}/K) := K \otimes \mathbb{R}^q g_* \widehat{\Omega}^\bullet_{\mathcal{Y}/T}$ and we have

$$c_1^{\mathrm{dR}}\colon \operatorname{Pic} \mathcal{Y} \to \mathrm{H}^2_{\mathrm{dR}}(\mathcal{Y}/K).$$

4.5.3. *Comparing algebraic and formal de Rham cohomology.* Proposition 4.22 below shows that p-adically completing algebraically de Rham cohomology yields formal de Rham cohomology. In fact, the construction of the implied isomorphism uses very little about the de Rham complex. We thank Brian Conrad for suggesting the argument below.

Let $f\colon \mathcal{X} \to B$ be a smooth proper morphism of noetherian $\mathcal{O}_K$-schemes. Let $\mathscr{F}^\bullet$ be a bounded-below $f^{-1}\mathscr{O}_B$-linear complex of coherent $\mathscr{O}_{\mathcal{X}}$-modules (i.e., the morphisms $\mathscr{F}^i \to \mathscr{F}^{i+1}$ are $f^{-1}\mathscr{O}_B$-linear but not necessarily $\mathscr{O}_{\mathcal{X}}$-linear). Completing everything produces an $\widehat{f}^{-1}\mathscr{O}_{\widehat{B}}$-linear complex $\widehat{\mathscr{F}^\bullet}$ of coherent $\mathscr{O}_{\widehat{\mathcal{X}}}$-modules on the formal scheme $\widehat{\mathcal{X}}$ over $\widehat{B}$. For example, if $\mathscr{F}^\bullet = \Omega^\bullet_{\mathcal{X}/B}$, then $\widehat{\mathscr{F}^\bullet} \simeq \widehat{\Omega}^\bullet_{\widehat{\mathcal{X}}/\widehat{B}}$ naturally. Let $i\colon \widehat{\mathcal{X}} \to \mathcal{X}$ and $j\colon \widehat{B} \to B$ be the canonical morphisms of ringed spaces. In the derived category $D^+(f^{-1}\mathscr{O}_B)$, we have morphisms

$$\mathscr{F}^\bullet \to i_* \widehat{\mathscr{F}^\bullet} \to \mathbb{R} i_* \widehat{\mathscr{F}^\bullet}.$$

Apply $\mathbb{R} f_*$ to the composition and use $fi = j\widehat{f}$ to obtain a morphism

$$\mathbb{R} f_* \mathscr{F}^\bullet \to \mathbb{R} j_* \mathbb{R} \widehat{f}_* \widehat{\mathscr{F}^\bullet}. \tag{4.20}$$

in $D^+(\mathscr{O}_B)$. Since j is flat, $\mathbb{R} j_*$ is adjoint to the flat pullback j^* on bounded-below derived categories, so (4.20) yields a morphism

$$j^* \mathbb{R} f_* \mathscr{F}^\bullet \to \mathbb{R} \widehat{f}_* \widehat{\mathscr{F}^\bullet}$$

in $D^+(\mathscr{O}_{\widehat{B}})$. Since $j^*\mathscr{G} \simeq \widehat{\mathscr{G}}$ for any coherent $\mathscr{O}_B$-module $\mathscr{G}$, passing to homology sheaves yields $\mathscr{O}_{\widehat{B}}$-module morphisms

$$\widehat{\mathbb{R}^q f_* \mathscr{F}^\bullet} \to \mathbb{R}^q \widehat{f}_* \widehat{\mathscr{F}^\bullet} \tag{4.21}$$

for $q \geq 0$. The whole construction is functorial with respect to $f^{-1}\mathscr{O}_B$-linear morphisms $\mathscr{G}^\bullet \to \mathscr{F}^\bullet$. In particular, (4.21) for the truncation $\mathscr{F}^{\geq p}$ maps to (4.21) for $\mathscr{F}^\bullet$; taking the images on each side for all p makes (4.21) a morphism of *filtered* $\mathscr{O}_{\widehat{B}}$-modules.

Proposition 4.22. *The natural map* (4.21) *is an isomorphism of filtered $\mathscr{O}_{\widehat{B}}$-modules. In particular, for $\mathscr{F}^\bullet = \Omega^\bullet_{\mathcal{X}/B}$, the p-adic completion functor* $\operatorname{Coh}(B) \to \operatorname{Coh}(\widehat{B})$ *of Theorem 4.4*(a) *sends $\mathbb{R}^q f_* \Omega^\bullet_{\mathcal{X}/B}$ to $\mathbb{R}^q \widehat{f}_* \widehat{\Omega}^\bullet_{\widehat{\mathcal{X}}/\widehat{B}}$.*

Proof. It suffices to prove that (4.21) is an isomorphism of $\mathscr{O}_{\widehat{B}}$-modules, since then functoriality with respect to $\mathscr{F}^{\geq p} \to \mathscr{F}^\bullet$ gives the compatibility with filtrations. We can use functoriality and degree-shifting arguments to reduce to the case in which $\mathscr{F}^\bullet$ is a single coherent sheaf in degree 0. This case follows from Theorem 4.4(c). ☐

Corollary 4.23. *Under the hypotheses of Proposition 4.22, the functor of Proposition 4.7*(c) *sends $\mathbb{R}^q f_* \Omega^\bullet_{\mathcal{X}_K/B_K}$ to $K \otimes \mathbb{R}^q \widehat{f}_* \widehat{\Omega}^\bullet_{\widehat{\mathcal{X}}/\widehat{B}}$, and respects the Hodge filtrations.*

Proof. Apply $K \otimes (\cdot)$ to the conclusion of Proposition 4.22. ☐

4.5.4. *Fibers of de Rham cohomology.* For families arising as the p-adic completion of a smooth proper morphism of noetherian $\mathcal{O}_K$-schemes, we show that taking de Rham cohomology commutes with restriction to fibers:

Proposition 4.24. *Let $f\colon \mathcal{X} \to B$ be a smooth proper morphism of noetherian $\mathcal{O}_K$-schemes. Let $b \in B(\mathcal{O}_K)$, let $\mathcal{X}_b$ be the pullback of $\mathcal{X}$ by* $\operatorname{Spec} \mathcal{O}_K \xrightarrow{b} B$*, and let $\mathcal{X}_{b,K} = \mathcal{X}_b \times K$.*

(a) *For each $q \geq 0$, there are isomorphisms of filtered K-vector spaces*

$$K \otimes \mathbb{R}^q \widehat{f}_* \widehat{\Omega}^\bullet_{\widehat{\mathcal{X}}/\widehat{B}}\Big|_b \to \mathrm{H}^q_{\mathrm{dR}}(\widehat{\mathcal{X}_b}/K).$$

(b) *There is an isomorphism of K-vector spaces*

$$\mathcal{H}^{02}(\widehat{\mathcal{X}}/K)\Big|_b \to \mathrm{H}^{02}(\widehat{\mathcal{X}_b}/K).$$

Proof.

(a) The algebraic analogue of this isomorphism, namely

$$\left(\mathbb{R}^q f_* \Omega^\bullet_{\mathcal{X}_K/B_K}\right)\big|_{b_K} \to \mathrm{H}^q_{\mathrm{dR}}(\mathcal{X}_{b,K}), \tag{4.25}$$

is an isomorphism of filtered K-vector spaces by Proposition 4.18(b) with $\alpha = (b)_K\colon \operatorname{Spec} K \to B_K$. Apply the functor in Proposition 4.7(d) to both sides of (4.25): on the right side we obtain $\mathrm{H}^q_{\mathrm{dR}}(\widehat{\mathcal{X}_b}/K)$, by Corollary 4.23 for $\mathcal{X}_b \to \operatorname{Spec} \mathcal{O}_K$; on the left side we may apply the functor *before* restricting to b (by Proposition 4.7(c)), and then Corollary 4.23 for $\mathcal{X} \to B$ shows that we obtain $K \otimes \mathbb{R}^q \widehat{f}_* \widehat{\Omega}^\bullet_{\widehat{\mathcal{X}}/\widehat{B}}\Big|_b$.

(b) This follows from (a) for $q = 2$.

☐

4.6. **Comparison and the p-adic Lefschetz $(1,1)$ theorem.** The following result identifies crystalline and de Rham cohomologies, even in the family setting.

Theorem 4.26. *Assume Setup 1.6, but without the requirement that B is irreducible. Let (T,z) be an enlargement of B_{k}. Let $f_0\colon \mathscr{X}_0 \to T_0$ be obtained from $f\colon \mathscr{X} \to B$ by base change along $z\colon T_0 \to B_{\mathsf{k}} \hookrightarrow B$. Let $g\colon \mathscr{Y} \to T$ be a smooth proper lifting of f_0. Then for each $q \in \mathbb{Z}_{\geq 0}$ there is a canonical isomorphism*

$$\sigma_{\mathrm{cris},T}\colon K \otimes \mathbb{R}^q g_* \widehat{\Omega}^{\bullet}_{\mathscr{Y}/T} \to \left(\mathrm{R}^q_{\mathrm{cris}} f_* \mathscr{O}_{\mathscr{X}/K}\right)_T . \tag{4.27}$$

Moreover, if $t \in T(\mathcal{O}_K)$ and $s = z(t(\operatorname{Spec} \mathsf{k})) \in B(\mathsf{k})$, and g arises as the p-adic completion of a smooth proper morphism of finite-type $\mathcal{O}_K$-schemes, then the isomorphism $\sigma_{\mathrm{cris},t}$ induced by $\sigma_{\mathrm{cris},T}$ on the fibers above t fits in a commutative diagram

$$\begin{tikzcd}
\operatorname{Pic} \mathscr{Y}_t \arrow[r] \arrow[d, "c_1^{\mathrm{dR}}"'] & \operatorname{Pic} \mathscr{X}_s \arrow[d, "c_1^{\mathrm{cris}}"] \\
\mathrm{H}^2_{\mathrm{dR}}(\mathscr{Y}_t/K) \arrow[r, "\sigma_{\mathrm{cris},t}"] & \mathrm{H}^2_{\mathrm{cris}}(\mathscr{X}_s/K).
\end{tikzcd} \tag{4.28}$$

Proof. For (4.27), see [Ogu84, Theorem 3.8.2]. The bottom row of (4.28) is the case $T = \lceil s \rceil$ of (4.27), with the identifications provided by Proposition 4.24(a) and Theorem 4.16. For the commutativity of (4.28), see [BI70, 2.3] and [BO83, 3.4]. ☐

Finally, we have what one might call a p-adic analogue of the Lefschetz $(1,1)$ theorem:

Theorem 4.29 (cf. [BO83, Theorem 3.8])**.** *Let $X \xrightarrow{g} \operatorname{Spf} \mathcal{O}_K$ be a smooth proper p-adic formal scheme. Let $\mathscr{L}_{\mathsf{k}}$ be a line bundle on X_{k}. Then the following are equivalent:*

(a) *There exists m such that $\mathscr{L}_{\mathsf{k}}^{\otimes p^m}$ lifts to a line bundle on X.*

(b) *The element of $\mathrm{H}^2_{\mathrm{dR}}(X/K) := K \otimes \mathbb{R}^q g_* \widehat{\Omega}^{\bullet}_{X/\mathcal{O}_K}$ corresponding under $\sigma_{\mathrm{cris},\mathcal{O}_K}$ to $c_1(\mathscr{L}_{\mathsf{k}}) \in \mathrm{H}^2_{\mathrm{cris}}(X_{\mathsf{k}}/K)$ maps to 0 in the quotient $\mathrm{H}^{02}(X/K)$.*

All the above definitions and results are compatible with respect to base change from $\mathcal{O}_K$ to $\mathcal{O}_L$ for a finite extension L of K [Ogu84, 3.6, 3.9.1, 3.11.1].

4.7. **Proof of Lemma 4.2.** By smoothness, there is a unique irreducible component of $B_{\overline{K}}$ containing the point of $B(\overline{K})$ corresponding to b_0. Replace K by a finite extension so that b_0 and this component are defined over K, and replace B by the closure of this component. Then replace B by an open subscheme to assume that B is a closed subscheme of $\mathbb{A}^r = \operatorname{Spec} \mathcal{O}_K[x_1, \dots, x_r]$ for some r. Translate so that b_0 is the origin in $\mathbb{A}^r$. Let $s \in B(\mathsf{k})$ be the reduction of b_0, the origin in $\mathbb{A}^r(\mathsf{k})$.

Let $\wp\colon \mathbb{A}^r \to \mathbb{A}^r$ be the morphism induced by the $\mathcal{O}_K$-algebra homomorphism mapping each variable x_i to px_i. Let $B' = \wp^{-1}(B)$. Let $b_0' \in B'(\mathcal{O}_K)$ be the origin, so $\wp(b_0') = b_0$. Let $T = \widehat{B'}$. Pulling back $f\colon \mathscr{X} \to B$ yields a morphism of $\mathcal{O}_K$-schemes $\mathscr{X}' \to B'$. Completing yields a morphism of p-adic formal $\mathcal{O}_K$-schemes $\mathscr{X}_T \to T$. We write f for any of these.

Let E be the convergent isocrystal $\mathrm{R}^2_{\mathrm{cris}} f_* \mathscr{O}_{\mathscr{X}/K}$ on B_{k} given by Theorem 4.16. Because the special fiber of T maps to $s \in B(\mathsf{k})$, we have a morphism of enlargements $T \to \lceil s \rceil$, so the definition of convergent isocrystal gives an identification

$$E_T \simeq E_{\lceil s \rceil} \otimes_{\mathcal{O}_K} \mathscr{O}_T \simeq \mathrm{H}^2_{\mathrm{cris}}(\mathscr{X}_s/K) \otimes_{\mathcal{O}_K} \mathscr{O}_T,$$

and the latter is a globally free sheaf in $\operatorname{Coh}(K \otimes \mathscr{O}_T)$.

Let $\mathscr{L}_{\mathsf{k}}$ be a line bundle on $\mathcal{X}_s$. Then $c_1^{\text{cris}}(\mathscr{L}_{\mathsf{k}}) \in \mathrm{H}^2_{\text{cris}}(X_s/K)$ gives rise to a constant section $\gamma_{\text{cris}}(\mathscr{L}_{\mathsf{k}}) := c_1^{\text{cris}}(\mathscr{L}_{\mathsf{k}}) \otimes 1 \in \mathrm{H}^2_{\text{cris}}(\mathcal{X}_s/K) \otimes_{\mathcal{O}_K} \mathscr{O}_T \simeq E_T$. Applying $\sigma_{\text{cris},T}^{-1}$ yields a section $\gamma_{\text{dR}}(\mathscr{L}_{\mathsf{k}})$ of $K \otimes \mathbb{R}^2 f_* \widehat{\Omega}^\bullet_{\mathcal{X}_T/T}$, which can be mapped to a section $\gamma_{02}(\mathscr{L}_{\mathsf{k}})$ of the quotient sheaf $\mathcal{H}^{02}(\mathcal{X}_T/K)$.

Let $b' \in B'(\mathcal{O}_K)$. Let $\mathcal{X}_{b'}$ be the $\mathcal{O}_K$-scheme obtained by pulling back $\mathcal{X} \to B$ by the composition $\operatorname{Spec} \mathcal{O}_K \xrightarrow{b'} B' \to B$. Let $\mathcal{X}_{b',K} = \mathcal{X}_{b'} \times K$. We can "evaluate" $\gamma_{\text{cris}}(\mathscr{L}_{\mathsf{k}})$, $\gamma_{\text{dR}}(\mathscr{L}_{\mathsf{k}})$, and $\gamma_{02}(\mathscr{L}_{\mathsf{k}})$ at b' by pulling back via $\operatorname{Spf} \mathcal{O}_K \xrightarrow{b'} T$ to obtain values in K-vector spaces

$$\begin{aligned}
\gamma_{\text{cris}}(\mathscr{L}_{\mathsf{k}}, b') &\in \mathrm{H}^2_{\text{cris}}(\mathcal{X}_s/K), \\
\gamma_{\text{dR}}(\mathscr{L}_{\mathsf{k}}, b') &\in \mathrm{H}^2_{\text{dR}}(\widehat{\mathcal{X}_{b'}}/K), \text{ and} \\
\gamma_{02}(\mathscr{L}_{\mathsf{k}}, b') &\in \mathrm{H}^{02}(\widehat{\mathcal{X}_{b'}}/K).
\end{aligned}$$

Because the composition of enlargements $\widehat{b'} \to T \to \lceil s \rceil$ is the identity, the cocycle condition in Definition 4.13 yields $\gamma_{\text{cris}}(\mathscr{L}_{\mathsf{k}}, b') = c_1^{\text{cris}}(\mathscr{L}_{\mathsf{k}})$.

Everything so far has been compatible with base extension from $\mathcal{O}_K$ to $\mathcal{O}_L$ for a finite extension L of K, and we may take direct limits to adapt the definitions and results above to $\mathcal{O}_{\overline{K}}$.

Proposition 3.6(b) gives an injective homomorphism

$$\operatorname{sp}_{\bar{b}',\bar{s}} \colon (\operatorname{NS} \mathcal{X}_{b',\overline{K}})_{\mathbb{Q}} \hookrightarrow (\operatorname{NS} \mathcal{X}_{\bar{s}})_{\mathbb{Q}}.$$

Claim 4.30. The class $[\mathscr{L}_{\mathsf{k}}] \in (\operatorname{NS} \mathcal{X}_{\bar{s}})_{\mathbb{Q}}$ is in the image of $\operatorname{sp}_{\bar{b}',\bar{s}}$ if and only if $\gamma_{02}(\mathscr{L}_{\mathsf{k}}, b') = 0$.

Proof. Suppose that $[\mathscr{L}_{\mathsf{k}}]$ is in the image of $\operatorname{sp}_{\bar{b}',\bar{s}}$. After replacing $\mathscr{L}_{\mathsf{k}}$ by a tensor power, replacing K by a finite extension, and tensoring $\mathscr{L}_{\mathsf{k}}$ with a line bundle algebraically equivalent to 0 (which, by Remark 4.15, does not change any of the sections and values $\gamma_\cdot(\cdot)$), we may assume that the isomorphism class of $\mathscr{L}_{\mathsf{k}}$ in $\operatorname{Pic} \mathcal{X}_s$ is the specialization of the isomorphism class of some line bundle $\mathscr{L}_K$ on $\mathcal{X}_{b',K}$. Lift $\mathscr{L}_K$ to a line bundle $\mathscr{L}$ on the $\mathcal{O}_K$-scheme $\mathcal{X}_{b'}$. Completing yields $\widehat{\mathscr{L}} \in \operatorname{Pic} \widehat{\mathcal{X}_{b'}}$. The commutative diagram in Theorem 4.26 shows that the element $c_1^{\text{dR}}(\widehat{\mathscr{L}}) \in \mathrm{H}^2_{\text{dR}}(\widehat{\mathcal{X}_{b'}}/K)$ is mapped by $\sigma_{\text{cris},b'}$ to $c_1^{\text{cris}}(\mathscr{L}_{\mathsf{k}})$. Then Theorem 4.29 applied to $\widehat{\mathcal{X}_{b'}}$ shows that $\gamma_{02}(\mathscr{L}_{\mathsf{k}}, b') = 0$.

Conversely, suppose that $\gamma_{02}(\mathscr{L}_{\mathsf{k}}, b') = 0$. Theorem 4.29 applied to $\widehat{\mathcal{X}_{b'}}$ shows that after raising $\mathscr{L}_{\mathsf{k}}$ to a power of p, we have that $\mathscr{L}_{\mathsf{k}}$ comes from some $\widehat{\mathscr{L}}$ on $\widehat{\mathcal{X}_{b'}}$. By Theorem 4.4(b), $\widehat{\mathscr{L}}$ comes from some $\mathscr{L}$ on $\mathcal{X}_{b'}$. Then $[\mathscr{L}_{\mathsf{k}}] = \operatorname{sp}_{\bar{b}',\bar{s}}([\overline{K} \otimes \mathscr{L}])$. This completes the proof of Claim 4.30. ☐

Because of Remark 4.15, γ_{02} on $\operatorname{Pic} \mathcal{X}_{\bar{s}}$ induces a homomorphism from $\operatorname{NS} \mathcal{X}_{\bar{s}}$ to the space of sections of the sheaf $\overline{K} \otimes_K \mathcal{H}^{02}(\mathcal{X}_T/K)$ on T. Let Λ_T be the image, so Λ_T is a finitely generated $\mathbb{Z}$-module. For any $b' \in B'(\mathcal{O}_{\overline{K}})$, evaluation at b' as in Proposition 4.24(b) defines a homomorphism $\operatorname{ev}_{b'}$ from Λ_T (or $(\Lambda_T)_{\mathbb{Q}}$) to $\mathrm{H}^{02}(\widehat{\mathcal{X}_{b'}}/\overline{K})$.

Applying Claim 4.30 over all finite extensions of $\mathcal{O}_K$ yields

Corollary 4.31.

(a) *For any $b' \in B'(\mathcal{O}_{\overline{K}})$, $\rho(\mathcal{X}_{b'})$ is the rank of the kernel of the composition*

$$\begin{tikzcd}(\operatorname{NS}\mathcal{X}_{\bar{s}})_{\mathbb{Q}} \arrow[r, two heads, "\gamma_{02}"] \arrow[rr, bend right, "\gamma_{02}(-{,}b')"'] & (\Lambda_T)_{\mathbb{Q}} \arrow[r, "\mathrm{ev}_{b'}"] & \mathrm{H}^{02}(\widehat{\mathcal{X}_{b'}}/\overline{K}).\end{tikzcd}$$

(b) *In particular,*

$$\rho(\mathcal{X}_{b'}) \geq \operatorname{rk}\ker\gamma_{02}, \tag{4.32}$$

with equality if and only if $\mathrm{ev}_{b'} \colon \Lambda_T \to \mathrm{H}^{02}(\widehat{\mathcal{X}_{b'}}/\overline{K})$ *is injective.*

Proposition 4.18(a) lets us apply Remark 4.8 to $\mathscr{F} := \mathcal{H}^{02}(\mathcal{X}')$ on B' to obtain a polydisk neighborhood U' of b'_0 in $B'(\mathcal{O}_{\overline{K}})$ such that the subgroup Λ_T of global sections of $\overline{K} \otimes \mathcal{H}^{02}(\mathcal{X}_T/K)$ is expressed on U' as a subgroup Λ' of $\mathscr{H}(U')^n$: in fact, if K is enlarged so that all elements of $\operatorname{NS}(\mathcal{X}_{\bar{s}})$ are defined over the residue field k, then the coefficients of the elements in Λ' lie in K. For $b' \in U'$, we may interpret $\mathrm{ev}_{b'}$ concretely in terms of evaluation of power series in Λ'.

We will prove

$$\operatorname{rk}\ker\gamma_{02} = \rho(\mathcal{X}_{\bar{\eta}}) \tag{4.33}$$

by comparing both with $\rho(\mathcal{X}_\beta)$ for a "very general" $\beta \in B'(\mathscr{O}_{\overline{K}})$.

Lemma 4.34. *Let Z be a finite-type K-scheme that is smooth of pure dimension n. Fix $z_0 \in Z(K)$. Then no countable union of subschemes of Z of dimension less than n can contain a neighborhood of z_0 in $Z(K)$.*

Proof. Shrink Z so that there is an étale morphism $\pi \colon Z \to \mathbb{A}^n$. By the definition of étale morphism given in [Mum99, III.§5, Definition 1] and the nonarchimedean implicit function theorem [Igu00, Theorem 2.1.1], π maps any sufficiently small neighborhood of z_0 in $Z(K)$ bijectively to a neighborhood of $\pi(z_0)$ in $\mathbb{A}^n(K)$. Also, the scheme-theoretic image in $\mathbb{A}^n$ of any subscheme in Z of dimension less than n is of dimension less than n. So we may reduce to the case $Z = \mathbb{A}^n$. This case follows by induction on n by projecting onto $\mathbb{A}^{n-1}$ and using the uncountability of K. ☐

Corollary 3.9 and Lemma 4.34 show that there exists $\beta \in B'(\mathscr{O}_{\overline{K}})$ near b'_0 such that $\rho(\mathcal{X}_\beta) = \rho(\mathcal{X}_{\bar{\eta}})$. For any $b' \in B'(\mathscr{O}_{\overline{K}})$, we have

$$\operatorname{rk}\ker\gamma_{02} \leq \rho(\mathcal{X}_\beta) = \rho(\mathcal{X}_{\bar{\eta}}) \leq \rho(\mathcal{X}_{b'}) \tag{4.35}$$

by (4.32), the choice of β, and Proposition 3.6(i), respectively.

Claim 4.36. For some $b' \in B'(\mathscr{O}_{\overline{K}})$, we have $\operatorname{rk}\ker\gamma_{02} = \rho(\mathcal{X}_{b'})$.

Proof. Applying Proposition 5.1[1] to Λ' gives a nonempty open subset V of the polydisk in C^d such that ev_u is injective for $u \in V$. Since $\overline{K}$ is dense in C, the set V contains a point in $\overline{K}^d$, and we let b' be its image in $B'(\mathcal{O}_{\overline{K}})$, so $\mathrm{ev}_{b'}$ is injective. Apply Corollary 4.31(b) to b'. ☐

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[1] The proofs in Section 5 do not rely on any results in this section.

Applying (4.35) with b' as in Claim 4.36 proves (4.33).

Next, $\wp$ maps U' isomorphically to a polydisk neighborhood U of b_0 in $B(\mathcal{O}_{\overline{K}})$, and Λ' corresponds to some $\Lambda \subseteq \mathscr{H}(U)^n$. Substituting (4.33) into Corollary 4.31(b), expressed on U in terms of Λ, shows that for $b \in U$,

$$\rho(\mathcal{X}_b) \geq \rho(\mathcal{X}_{\bar{\eta}}),$$

with equality if and only if $\lambda(b) \neq 0$ for every nonzero $\lambda \in \Lambda$. This completes the proof of Lemma 4.2.

Remark 4.37. We conjecture that Lemma 4.2 holds with $B(\mathcal{O}_{\overline{K}})$ replaced by $B(\overline{K})$, but crystalline methods do not suffice to prove this. This would imply that Theorem 1.7 holds with $B(C)$ in place of $B(\mathcal{O}_C)$.

5. UNIONS OF ZERO LOCI OF POWER SERIES

To understand the nature of the following statement, the reader is urged to consider the $r = 1$ case first.

Proposition 5.1. *Let C be as in Setup 1.6. Let $D = \{(z_1, \ldots, z_d) \in C^d : v(z_i) \leq \epsilon \text{ for all } i\}$ for some $\epsilon > 0$. Let R be the subring of $C[[z_1, \ldots, z_d]]$ consisting of power series that converge on D. Let r be a nonnegative integer. Let Λ be a finite-dimensional $\mathbb{Q}_p$-subspace of R^r. Then there exists a nonempty analytic open subset U of D such that for all $u \in U$, the evaluation-at-u map*

$$\begin{aligned} \operatorname{ev}_u \colon \Lambda &\to C^r \\ (f_1, \ldots, f_r) &\mapsto (f_1(u), \ldots, f_r(u)) \end{aligned}$$

is injective.

Remark 5.2. The archimedean analogue of Proposition 5.1 is false. For example, if Λ is the $\mathbb{R}$-span of $1, x, x^2$, then there is no $u \in \mathbb{C}$ such that the evaluation-at-u map $\Lambda \to \mathbb{C}$ is injective. Even if we consider only the $\mathbb{Z}$-span of $1, x, x^2$, the evaluation-at-u map fails to be injective on a dense subset of $\mathbb{C}$.

The rest of this section is devoted to the proof of Proposition 5.1. The proof is by induction on r. Because the base case $r = 1$ is rather involved, we begin by explaining the inductive step.

Suppose that $r > 1$, and that Proposition 5.1 is known for $r' < r$. Let $\pi \colon R^r \to R^{r-1}$ be the projection to the first $r - 1$ coordinates. Let $\Lambda^{(r)}$ and Λ_{r-1} be the kernel and image of $\pi|_\Lambda$. View $\Lambda^{(r)}$ as a subgroup of R. For any $u \in D$, we have a commutative diagram with exact rows

(5.3)
$$\begin{tikzcd} 0 \arrow[r] & \Lambda^{(r)} \arrow[r] \arrow[d, "\operatorname{ev}_u"] & \Lambda \arrow[r] \arrow[d, "\operatorname{ev}_u"] & \Lambda_{r-1} \arrow[r] \arrow[d, "\operatorname{ev}_u"] & 0 \\ 0 \arrow[r] & C \arrow[r] & C^r \arrow[r] & C^{r-1} \arrow[r] & 0. \end{tikzcd}$$

The inductive hypothesis applied to $\Lambda_{r-1} \subseteq R^{r-1}$ gives a closed polydisk $D' \subseteq D$ such that the right vertical map in (5.3) is injective for all $u \in D'$. The inductive hypothesis applied to $\Lambda^{(r)} \subseteq R$ gives a closed polydisk $D'' \subseteq D'$ such that the left vertical map in (5.3) is injective for all $u \in D''$. Then for $u \in D''$, the middle vertical map in (5.3) is injective. This completes the proof of the inductive step.

Before discussing the base case $r = 1$, we prove another lemma. Let $v \colon C \to \mathbb{Q} \cup \{+\infty\}$ be the valuation on C, normalized by $v(p) = 1$. If $\vec{t} = (t_1, \dots, t_s) \in C^s$ for some s, define $v(\vec{t}) := \min_j v(t_j)$.

Lemma 5.4.

(a) *If* $\vec{t_1}, \dots, \vec{t_n} \in C^s$ *are* $\mathbb{Q}_p$*-independent, then* $\left\{v\left(\sum a_i\vec{t_i}\right) : (a_1, \dots, a_n) \in (\mathbb{Z}_p)^n - (p\mathbb{Z}_p)^n\right\}$ *is finite.*

(b) *If* $t_1, \dots, t_n \in C$*, then* $\{v(t) : t = \sum a_i t_i \neq 0 \text{ for some } a_i \in \mathbb{Q}_p\}$ *has finite image in* $\mathbb{Q}/\mathbb{Z}$*.*

Proof.

(a) The function

$$\begin{aligned}(\mathbb{Z}_p)^n - (p\mathbb{Z}_p)^n &\to \mathbb{Q}\\ (a_1, \dots, a_n) &\mapsto v\left(\sum a_i\vec{t_i}\right)\end{aligned}$$

is continuous for the p-adic topology on the left and the discrete topology on the right, so by compactness its image is finite.

(b) By replacing the t_i with a basis for their $\mathbb{Q}_p$-span, we reduce to the $s = 1$ case of (a). ☐

From now on, we assume $r = 1$; i.e., $\Lambda \subseteq R$. Choose a $\mathbb{Q}_p$-basis $f_1, \dots, f_n$ of Λ. We may assume that D is the unit polydisk, so the coefficients of each f_i tend to 0. Multiply all the f_i by a single power of p to assume that $f_i \in \mathcal{O}_C[[z_1, \dots, z_d]]$. For some m, the images of f_i in $C[[z_1, \dots, z_d]]/(z_1, \dots, z_d)^m$ are $\mathbb{Q}_p$-independent, because a descending sequence of vector spaces in Λ with zero intersection must be eventually zero. Fix such an m.

Let M be the set of monomials μ in the z_i whose total degree $\deg \mu$ is less than m. For each μ, let $c_i^\mu \in C$ be the coefficient of μ in f_i. For each $\mu \in M$, apply Lemma 5.4(b) to $c_1^\mu, \dots, c_n^\mu$ to obtain a finite subset S_μ of $\mathbb{Q}/\mathbb{Z}$. Let $S = \bigcup_{\mu \in M} S_\mu$. Let $q_1, \dots, q_d$ be distinct primes greater than m that do not appear in the denominators of rational numbers representing elements of S. For $i = 1, \dots, n$, let $\vec{t_i} \in C^{\#M}$ be the vector whose coordinates are the c_i^μ for $\mu \in M$. By choice of m, the $\vec{t_i}$ are $\mathbb{Q}_p$-independent. By Lemma 5.4(a), the set

$$\left\{v\left(\sum a_i\vec{t_i}\right) : (a_1, \dots, a_n) \in (\mathbb{Z}_p)^n - (p\mathbb{Z}_p)^n\right\}$$

is finite; choose $A \in \mathbb{Q}$ larger than all its elements. Choose a positive integer N such that

$$mN > (m-1)(N + 1/q_i) + A \tag{5.5}$$

for all i. Let

$$U := \{(z_1, \dots, z_n) \in C^n : v(z_i) = N + 1/q_i \text{ for all } i\},$$

so U is open in D.

Consider an arbitrary nonzero element $f = \sum_{\text{all }\mu} c^\mu \mu$ of Λ. So $f = \sum_{i=1}^n a_i f_i$ for some $(a_1, \dots, a_n) \in \mathbb{Q}_p^n - \{\vec{0}\}$. Let $u = (u_1, \dots, u_d) \in U$. It remains to show that $f(u) \neq 0$.

By multiplying f by a power of p, we may assume that $(a_1, \dots, a_n) \in (\mathbb{Z}_p)^n - (p\mathbb{Z}_p)^n$. If $\mu = z_1^{e_1} \cdots z_d^{e_d}$ and $\deg \mu \geq m$, then

$$v(c^\mu \mu(u)) \geq 0 + e_1 N + \cdots + e_d N \geq mN \tag{5.6}$$

On the other hand, the definition of A yields $\xi = z_1^{e_1} \cdots z_d^{e_d} \in M$ such that $v(c^\xi) < A$, so that

$$v(c^\xi \xi(u)) < A + \sum_{i=1}^{d} e_i(N + 1/q_i) \le mN \tag{5.7}$$

by (5.5).

To show that $f(u) \neq 0$, it remains to show that the valuations $v(c^\mu \mu(u))$ for $\mu \in M$ such that $c^\mu \neq 0$ are distinct, since then the minimum of these is finite and equals $v(f(u))$, by (5.6) and (5.7). In fact, if $\mu = z_1^{e_1} \cdots z_d^{e_d}$ and $\deg \mu < m$ and $c^\mu \neq 0$, then for each i the choice of the q_i implies $v(c^\mu) \in \mathbb{Z}_{(q_i)}$ (i.e., q_i does not divide the denominator of $v(c^\mu)$), so

$$v(c^\mu \mu(u)) \in \frac{e_i}{q_i} + \mathbb{Z}_{(q_i)};$$

moreover $e_i \le \deg \mu < m < q_i$, so e_i is determined by $v(c^\mu \mu(u))$ whenever $c^\mu \neq 0$. This completes the proof of Proposition 5.1.

6. PROOF OF THEOREMS 1.7 AND 1.1

Lemma 6.1. *Let $k \subseteq k'$ be an extension of algebraically closed valued fields such that k is dense in k'. Then for any finite-type k-scheme B, the set $B(k)$ is dense in $B(k')$ with respect to the analytic topology.*

Proof. Let $b' \in B(k')$ be a point to be approximated by k-points. We may replace B by the Zariski closure of the image of b' under $B_{k'} \to B$. Then b' is a smooth point. We may shrink B to assume that there is a finite étale morphism $\pi \colon B \to U$ of some degree $d \ge 1$ for some open subscheme U of $\mathbb{A}^n$ for some n. Since k is dense in k', we can approximate $\pi(b') \in (k')^n$ arbitrarily well by points $u \in U(k) \subset k^n$. By "continuity of the set of roots of a polynomial as a function of the coefficients", above each u we can choose one geometric point b of $\pi^{-1}(u)$ such that $b \to b'$ as $u \to \pi(b')$. Since k is algebraically closed, $b \in B(k)$. ☐

Proof of Theorem 1.7. Let $b_0 \in B(\mathcal{O}_C)$. We need to show that any neighborhood U_0 of b_0 in $B(\mathcal{O}_C)$ contains a nonempty open set V that does not meet $B(\mathcal{O}_C)_{\text{jumping}}$. By Lemma 6.1 we may assume that $b_0 \in B(\mathcal{O}_{\overline{K}})$. Lemma 4.2 gives a smaller neighborhood U_1 of b_0 in $B(\mathcal{O}_C)$ in which the jumping locus is described explicitly in terms of power series, and Proposition 5.1 gives a nonempty open subset V of U_1 such that $\rho(\mathcal{X}_b) = \rho(\mathcal{X}_{\bar\eta})$ for all $b \in V \cap B(\mathcal{O}_{\overline{K}})$.

Suppose that $b \in V \cap B(\mathcal{O}_C)_{\text{jumping}}$. By Corollary 3.9, b is contained in a $\overline{K}$-variety Z such that $Z(C) \subseteq B(C)_{\text{jumping}}$. Lemma 6.1 gives $b' \in V \cap Z(\overline{K}) \subseteq B(\mathscr{O}_{\overline{K}})_{\text{jumping}}$, which contradicts the definition of V. ☐

Proof that Theorem 1.7 implies Theorem 1.1. Assume that k and $\mathcal{X} \to B$ are as in Theorem 1.1. Replacing B by a dense open subscheme, we may assume that B is smooth over k. Choose a finitely generated $\mathbb{Z}$-algebra A in k such that $\mathcal{X} \to B$ is the base extension of a morphism $\mathcal{X}_A \to B_A$ of A-schemes. By localizing A, we may assume that $\mathcal{X}_A \to B_A$ is a smooth proper morphism, and that $B_A \to \operatorname{Spec} A$ is smooth with geometrically irreducible fibers [EGA IV$_3$, 8.10.5(xii), 9.7.7(i)], [EGA IV$_4$, 17.7.8(ii)]. Because of Proposition 3.1, we may replace k by the algebraic closure of $\operatorname{Frac}(A)$ in k.

By [Cas86, Chapter 5, Theorem 1.1], there exists an embedding $A \hookrightarrow \mathbb{Z}_p$ for some prime p. Extend it to an embedding $k \hookrightarrow \mathbb{C}_p$. Base extend by $A \hookrightarrow \mathbb{Z}_p$ to obtain $\mathcal{X}_{\mathbb{Z}_p} \to B_{\mathbb{Z}_p}$. Since

$B_{\mathbb{Z}_p} \to \operatorname{Spec} \mathbb{Z}_p$ is smooth with geometrically irreducible special fiber, the set $B_{\mathbb{Z}_p}(\mathcal{O}_{\mathbb{C}_p})$ is nonempty. Apply Theorem 1.7 to $\mathcal{X}_{\mathbb{Z}_p} \to B_{\mathbb{Z}_p}$ to find a nonempty open subset U of $B_{\mathbb{Z}_p}(\mathcal{O}_{\mathbb{C}_p}) \subset B_{\mathbb{Z}_p}(\mathbb{C}_p) = B(\mathbb{C}_p)$ in the analytic topology such that $\rho(\mathcal{X}_b) = \rho(\mathcal{X}_{\bar{\eta}})$ for all $b \in U$. The field k is dense in $\mathbb{C}_p$ since even $\overline{\mathbb{Q}}$ is dense in $\mathbb{C}_p$, so Lemma 6.1 shows that U contains a point b of $B(k)$; this b is as required in Theorem 1.1, because of Proposition 3.1. ☐

7. Projective vs. proper

Theorem 7.1. *Let B be a variety over a field k of characteristic 0. Let $\mathcal{X} \to B$ be a smooth proper morphism such that all closed fibers are projective. Then there exists a Zariski dense open subvariety U of B such that $\mathcal{X}_U \to U$ is projective.*

Proof. We may assume that B is irreducible, say with generic point η. Theorem 1.1 yields a closed point $b \in B$ such that a specialization map $\operatorname{NS} \mathcal{X}_{\bar{\eta}} \to \operatorname{NS} \mathcal{X}_{\bar{b}}$ is an isomorphism. By assumption, $\mathcal{X}_b$ is projective, so we may choose an ample line bundle $\mathscr{L}_{\bar{b}}$ on $\mathcal{X}_{\bar{b}}$. By construction of b, the class $[\mathscr{L}_{\bar{b}}] \in \operatorname{NS} \mathcal{X}_{\bar{b}}$ comes from the class in $\operatorname{NS} \mathcal{X}_{\bar{\eta}}$ of some line bundle $\mathscr{L}$ on $\mathcal{X}_{\bar{\eta}}$. By Proposition 3.6(c), $\mathscr{L}$ is ample. Then $\mathscr{L}$ comes from a line bundle on $\mathcal{X}_L$ for some finite extension L of $\kappa(B)$, so $\mathcal{X}_L$ is projective, so $\mathcal{X}_\eta$ is projective [EGA II, 6.6.5]. Finally, spreading out shows that $\mathcal{X}_U \to U$ is projective for some dense open subscheme U of B [EGA IV$_3$, 8.10.5(xiii)]. ☐

Remark 7.2. Under the hypotheses of Theorem 7.1, we may also deduce that *every* fiber of $\mathcal{X} \to B$ is projective: just apply Theorem 7.1 to each irreducible subvariety of B. But we *cannot* deduce that $\mathcal{X} \to B$ is projective, or even that B can be covered by open sets U such that $\mathcal{X}_U \to U$ is projective, as the following example of Raynaud shows.

Example 7.3 ([Ray70, XII.2])**.** Let A be a nonzero abelian variety over a field k. Let σ be the automorphism $(x, y) \mapsto (x, x + y)$ of $A \times A$. Let B be a rational curve whose singular locus is a single node, and let $\tilde{B}$ be its normalization. Then one can construct an abelian scheme $\mathcal{A} \to B$ whose base extension to $\tilde{B}$ is simply $(A \times A) \times \tilde{B}$ with the fibers above the two preimages of the node identified via σ. Moreover, for any neighborhood U of the node, $\mathcal{A}_U \to U$ is not projective.

Question 7.4. Can one construct a counterexample to Theorem 7.1 over $k = \overline{\mathbb{F}}_p$?

Example 7.5. With notation as in Example 7.3, one can show that the first projection $A \times A \times \tilde{B} \to A$ factors through a morphism $\mathcal{A} \to A$ whose fiber above a given $a \in A(k)$ is projective if and only if a is a torsion point. This provides a counterexample over $\overline{\mathbb{F}}_p$ except for the fact that $\mathcal{A} \to A$ is not smooth.

8. Comparison with the Hodge-theoretic approach

Both André's Hodge-theoretic proof of Theorem 1.1 and our proof give information about the jumping locus beyond the mere existence of a non-jumping point. In this section, we compare these refinements.

8.1. Refinement in terms of sparse sets.

To state the more precise information about the jumping locus that André's approach yields, we introduce the notion of "sparse", which is a version for closed points of the notion "thin" defined in [Ser97, §9.1] in the context of Hilbert irreducibility.

Definition 8.1. Let F be a finitely generated field over $\mathbb{Q}$. Let B be an F-variety. Call a subset S of $|B|$ sparse if there exists a dominant and generically finite morphism $\pi\colon B' \to B$ of irreducible F-varieties such that for each $s \in S$, the fiber $\pi^{-1}(s)$ is either empty or contains *more* than one closed point.

Remark 8.2. In the context of Definition 8.1, call a subset of $B(\overline{F})$ sparse if and only if the associated set of closed points is sparse.

André's approach yields the following:

Theorem 8.3. *Let F be a field that is finitely generated over $\mathbb{Q}$. Let B be an irreducible F-variety, and let $f\colon \mathcal{X} \to B$ be a smooth projective morphism. Then the set*

$$B_{\mathrm{jumping}} := \{b \in |B| : \rho(\mathcal{X}_{\bar{b}}) > \rho(\mathcal{X}_{\bar{\eta}})\}.$$

is sparse.

Remark 8.4. Although Theorem 8.3 has a restriction on F and assumes that f is projective (not just proper), an easy argument shows that it still implies the general case of Theorem 1.1. The extension to the proper case uses Chow's lemma and weak factorization of birational maps in characteristic 0 (see the first sentence of Remark 2 following Theorem 0.3.1 in [AKMW02]).

The strategy of the proof of Theorem 8.3 is to use Deligne's global invariant cycle theorem ([Del71] or [Voi03, 4.3.3]) and the semisimplicity of the category of polarized Hodge structures to decompose the Hodge structure on the Betti cohomology $\mathrm{H}^2(\mathcal{X}_b^{\mathrm{an}}, \mathbb{Q})$ as $H \oplus H^{\perp}$, where H consists of the classes invariant under some finite-index subgroup of the geometric monodromy group. The space H carries a Hodge structure independent of b; after an étale base change, H comes from the cohomology of a smooth completion of the total space of the family. On the other hand, the "essentially varying part" $H^{\perp}$ is susceptible to Terasoma's argument ([Ter85] and [GGP04, Lemma 8]), which shows that the corresponding subspace of étale cohomology has no nonzero Tate classes for b outside a sparse set.

For a complete proof, see [And96, §5.2]. For an exposition of the Hodge-theoretic aspects of the proof, see also [Voi10].

8.2. **Sparse versus p-adically nowhere dense.** The purpose of this subsection is to develop basic properties of sparse sets, and to discuss the difference between the two notions of smallness appearing in Theorems 8.3 and 1.7.

Proposition 8.5.

(a) *For any dense open subscheme U of B and any $S \subseteq |B|$, the following are equivalent:*
   (i) *S is sparse,*
   (ii) *$S \cap U$ is sparse as a subset of $|B|$,*
   (iii) *$S \cap U$ is sparse as a subset of $|U|$.*
   In particular, for any closed subscheme $Z \subsetneq B$, the subset $|Z|$ is sparse in $|B|$.

(b) *A finite union of sparse sets is sparse.*

(c) *A dominant and generically finite morphism of irreducible varieties maps sparse sets to sparse sets.*

(d) *If S is a sparse subset of $|\mathbb{A}^n|$, then $S \cap \mathbb{A}^n(F)$ is a thin set.*

(e) *If $S \subseteq |B|$ is sparse, then $|B| - S$ is Zariski dense in B.*

Proof.

(a) Take $B' = B - Z$.

(b) Let V be an irreducible component dominating B of the fiber product of the varieties B' over B. By (a), we may replace B by a dense open subscheme U, and the other schemes by the corresponding preimages of U. This lets us assume that each morphism $V \to B'$ and $B' \to B$ is surjective. If s is in the union of the sparse sets, then its preimage in some B' contains more than one closed point, and so does its preimage in V.

(c) Let $f \colon C \to B$ be the morphism, and let S be a sparse subset of $|C|$, witnessed by $C' \to C$. By (a), we may shrink B to assume that the morphisms $C' \to C \to B$ are surjective. If $s \in S$, then the preimage of $f(s)$ in C' contains more than one closed point.

(d) By definition.

(e) If not, then by (a) there is a dense open subscheme U of B such that $|U|$ is sparse in $|U|$. After shrinking U, there is a finite surjective morphism from U to a nonempty open subscheme V of some $\mathbb{A}^n$. Then $|V|$ is sparse in $|\mathbb{A}^n|$ by (c), so $V(F)$ is a thin set by (d). This contradicts the fact that a finitely generated field F over $\mathbb{Q}$ is Hilbertian. $\square$

The following proposition shows that neither of the properties of being sparse or p-adically nowhere dense implies the other.

Proposition 8.6.

(i) *Let F be a number field. Fix embeddings $F \subseteq \overline{\mathbb{Q}} \subseteq \mathbb{C}_p$. Let $\pi \colon B' \to B$ be any dominant and generically finite morphism of irreducible F-varieties such that $\dim B > 0$ and $\deg \pi > 1$. Let S be the preimage of*

$$\{b \in |B| : \pi^{-1}(b) \text{ is not a single closed point}\}.$$

in $B(\overline{\mathbb{Q}})$. Then S is dense in $B(\mathbb{C}_p)$.

(ii) *There exists a non-sparse subset S of $|\mathbb{A}^1_{\mathbb{Q}}|$ such that the associated subset of $\mathbb{A}^1(\overline{\mathbb{Q}})$ is p-adically nowhere dense in $\mathbb{A}^1(\mathbb{C}_p)$.*

Proof.

(i) Let $T := \{b' \in B'(\overline{\mathbb{Q}}) : F(b') = F(\pi(b'))\}$. By replacing B' and B by dense open subschemes, we may assume that $B' \to B$ is finite étale, so $B'(\mathbb{C}_p) \to B(\mathbb{C}_p)$ is surjective. Then it suffices to prove that T is p-adically dense in $B'(\mathbb{C}_p)$, or equivalently, p-adically dense in $B'(\overline{\mathbb{Q}})$. To approximate a point P in $B'(\overline{\mathbb{Q}})$ by a point in T, choose an irreducible F-curve C in B through $\pi(P)$ (maybe not geometrically irreducible), and let C' be the irreducible component of $\pi^{-1}(C)$ containing P. By replacing $B' \to B$ by $C' \to C$, we may assume that $\dim B = 1$.

After replacing B' and B by dense open subschemes, we can compose with a finite étale morphism $B \to \mathbb{A}^1$ to reduce to the case that B is a dense open subscheme of $\mathbb{A}^1$. By the primitive element theorem, after shrinking again, we may find a dominant morphism $\rho \colon B' \to \mathbb{A}^1$ such that $(\pi, \rho) \colon B' \to B \times \mathbb{A}^1$ is an immersion.

Let $u_0 \in B'(\overline{\mathbb{Q}})$. We need to p-adically approximate u_0 by some $u \in T$. Let $E := F(u_0)$. Let $d := [E : F]$. Let $\sigma_1, \ldots, \sigma_d$ be the F-embeddings $E \to \overline{\mathbb{Q}}$. Let Z be the Zariski closure of the subscheme of $(\mathbb{A}^1)^d$ whose geometric points are $(Q_1, \ldots, Q_d)$ such that for some $i \neq j$, we have $\pi(\rho^{-1}(Q_i)) \cap \pi(\rho^{-1}(Q_j)) \neq \emptyset$. Since $Z \neq (\mathbb{A}^1)^d$, Lemma 8.7 below gives $Q \in \mathbb{A}^1(E)$ p-adically close to $\rho(u_0)$ such that $(\sigma_1(Q), \ldots, \sigma_d(Q)) \notin Z$.

For sufficiently close Q, there exists $u \in \rho^{-1}(Q) \subseteq B'(\overline{\mathbb{Q}})$ p-adically close to u_0. By definition of Z, the $\operatorname{Gal}(\overline{\mathbb{Q}}/F)$-conjugates of u map to distinct points in $B(\overline{\mathbb{Q}})$, so $u \in T$.

(ii) The set $\mathbb{Q}$ is p-adically nowhere dense in $\mathbb{C}_p$. On the other hand, $\mathbb{Q}$ is not thin in $\mathbb{Q}$, so Proposition 8.5(d) proves that $\mathbb{Q}$ is not sparse.

□

Lemma 8.7. *Let $E \supseteq F$ be an extension of number fields. Fix inclusions $F \subseteq E \subseteq \overline{\mathbb{Q}} \subseteq \mathbb{C}_p$. Let $\sigma_1, \ldots, \sigma_d$ be the F-embeddings $E \to \overline{\mathbb{Q}}$. If $Z \subsetneq (\mathbb{A}^1_{\overline{\mathbb{Q}}})^d$ is a closed subscheme, then the set $W := \{Q \in \mathbb{A}^1(E) : (\sigma_1(Q), \ldots, \sigma_d(Q)) \notin Z(\overline{\mathbb{Q}})\}$ is p-adically dense in $\mathbb{A}^1(E)$.*

Proof. Let R be the restriction of scalars $\operatorname{Res}_{E/F} \mathbb{A}^1$, which is isomorphic to $\mathbb{A}^d_F$. The bijection $\iota \colon R(F) \to \mathbb{A}^1(E)$ is p-adically continuous for the topologies induced by $F \subseteq E \subseteq \mathbb{C}_p$ (though its inverse need not be, since the topology on $\mathbb{A}^1(E)$ is induced by just *one* of the valuations on E extending the p-adic valuation on F). The embeddings $\sigma_1, \ldots, \sigma_d$ define an isomorphism $s \colon R_{\overline{\mathbb{Q}}} \to (\mathbb{A}^1)^d_{\overline{\mathbb{Q}}}$. The complement of $s^{-1}(Z)$ in $R(F)$ is p-adically dense in $R(F)$, so its image under ι is p-adically dense in $\mathbb{A}^1(E)$. □

Remark 8.8. In the setting of Theorem 1.1, but over a field k of characteristic 0 that need not be algebraically closed, either Theorem 8.3 or Theorem 1.7 can be used to show that there exists $d \geq 1$ such that

$$\{b \in B \mid [\kappa(b) : k] \leq d \text{ and } \rho(\mathcal{X}_{\bar{b}}) = \rho(\mathcal{X}_{\bar{\eta}})\}$$

is Zariski dense in B. In fact, Theorem 8.3 (with Proposition 8.5(c,d)) proves the stronger result that the degree of any generically finite rational map $B \dashrightarrow \mathbb{P}^n$ can serve as d.

9. CYCLES OF HIGHER CODIMENSION

Many of our arguments apply to specialization of cycles of codimension greater than 1, but the results are conditional. André obtains a generalization at the cost of replacing algebraic cycles by "motivated cycles", which are the same if the Hodge conjecture is true. In the p-adic approach, we would need a higher-codimension analogue (Conjecture 9.2) of the p-adic Lefschetz $(1,1)$ theorem (Theorem 4.29).

9.1. **Cycle class maps.** To state the generalizations, we recall some standard definitions. For a smooth proper variety X over a field, let $\mathscr{Z}^r(X)_{\mathbb{Q}}$ be the vector space of codimension-r cycles with $\mathbb{Q}$ coefficients.

Definition 9.1. Let X be a smooth proper variety over an algebraically closed field k of characteristic 0. Fix a prime ℓ. Define $\rho^r(X)$ as the dimension of the $\mathbb{Q}_\ell$-span of the image of the ℓ-adic cycle class map

$$\operatorname{cl}_{\text{ét}} \colon \mathscr{Z}^r(X)_{\mathbb{Q}} \to \operatorname{H}^{2r}_{\text{ét}}(X, \mathbb{Q}_\ell(r)).$$

The homomorphism $\operatorname{cl}_{\text{ét}}$ factors through the vector space $\mathscr{Z}^r(X)_{\mathbb{Q}}/\text{alg}$ of cycles modulo algebraic equivalence. Suppose that k' is an algebraically closed field extension of k. Then every element of $\mathscr{Z}^r(X_{k'})_{\mathbb{Q}}/\text{alg}$ comes from a cycle over $\kappa(V) \subseteq k'$ for some k-variety V, and can be spread out over some dense open subvariety in V, and hence is algebraically equivalent to the k'-base extension of the resulting cycle above any k-point of V. So $\mathscr{Z}^r(X)_{\mathbb{Q}}/\text{alg} \to \mathscr{Z}^r(X_{k'})_{\mathbb{Q}}/\text{alg}$ is surjective (it is injective too), and $\rho^r(X) = \rho^r(X_{k'})$. This, together with standard comparison theorems, shows that $\rho^r(X)$ equals its analogue defined using the Betti cycle class map

$$\mathrm{cl}_{\mathrm{Betti}} \colon \mathscr{Z}^r(X)_{\mathbb{Q}} \to \mathrm{H}^{2r}(X^{\mathrm{an}}, \mathbb{Q})$$

when $k = \mathbb{C}$, and also its analogue defined using de Rham cohomology (see [Har75, II.7.8]). The comparison with Betti cohomology shows that $\rho^r(X)$ equals $\dim_{\mathbb{Q}} \mathrm{cl}_{\mathrm{Betti}}(\mathscr{Z}^r(X)_{\mathbb{Q}})$, which is independent of ℓ.

As follows from [SGA 6, X App 7, especially §7.14], all the facts in Section 3.2 (other than facts not needed for the proofs of Theorem 1.1, namely the claims about ampleness and about the cokernel of specialization being torsion-free) have analogues with $\mathrm{NS}\, X$ replaced by $\mathrm{cl}_{\text{ét}}(\mathscr{Z}^r(X)_{\mathbb{Q}})$. In particular, in a smooth proper family, $\rho^r(X)$ can only increase under specialization, and the jumping locus for ρ^r is a countable union of Zariski closed subsets.

9.2. **The p-adic approach.** Now assume that K, $\mathcal{O}_K$, k are as in Setup 1.6. Let $X \to \operatorname{Spec} \mathcal{O}_K$ be a smooth proper morphism. We have a specialization map sp, cycle class maps $\mathrm{cl}_{\mathrm{dR}}$ and $\mathrm{cl}_{\mathrm{cris}}$ ([Har75, II.7.8] and [GM87], respectively), and a comparison isomorphism σ_{cris} (cf. [Ber74, V.2.3.2]) making the following diagram commute:

$$\begin{tikzcd}
\mathscr{Z}^r(X_K)_{\mathbb{Q}} \arrow[r, "\mathrm{sp}"] \arrow[d, "\mathrm{cl}_{\mathrm{dR}}"'] & \mathscr{Z}^r(X_k)_{\mathbb{Q}} \arrow[d, "\mathrm{cl}_{\mathrm{cris}}"] \\
\mathrm{H}^{2r}_{\mathrm{dR}}(X_K/K) \arrow[r, "\sigma_{\mathrm{cris}}"] & \mathrm{H}^{2r}_{\mathrm{cris}}(X_k/K).
\end{tikzcd}$$

Conjecture 9.2 (p-adic variational Hodge conjecture). *Let notation be as above. Let $Z_k \in \mathscr{Z}^r(X_k)_{\mathbb{Q}}$. If $\sigma_{\mathrm{cris}}^{-1}(\mathrm{cl}_{\mathrm{cris}}(Z_k)) \in \mathrm{Fil}^r \mathrm{H}^{2r}_{\mathrm{dR}}(X_K/K)$, then there exists $Z \in \mathscr{Z}^r(X_K)_{\mathbb{Q}}$ such that $\mathrm{cl}_{\mathrm{dR}}(Z) = \sigma_{\mathrm{cris}}^{-1}(\mathrm{cl}_{\mathrm{cris}}(Z_k))$.*

Remark 9.3. Conjecture 9.2 seems to be well-known to experts; it is a variant of [Eme04, Conjecture 2.2], which is where we learned about it.

Remark 9.4. The $r = 1$ case of Conjecture 9.2 is a weak form of Theorem 4.29.

Repeating the p-adic proof of Theorem 1.7 yields the following.

Theorem 9.5. *Assume Conjecture 9.2. Let notation be as in Setup 1.6. Let r be a nonnegative integer. Then the set*

$$B(\mathcal{O}_C)_{\mathrm{jumping}} := \{b \in B(\mathcal{O}_C) : \rho^r(\mathcal{X}_b) > \rho^r(\mathcal{X}_{\bar{\eta}})\}$$

is nowhere dense in $B(\mathcal{O}_C)$ for the analytic topology.

9.3. **The Hodge-theoretic approach.** To obtain a result using André's approach requires either replacing algebraic cycles with motivated cycles as in [And96, §5.2], or else assuming the variational Hodge conjecture, which we now state:

Conjecture 9.6 (Variational Hodge conjecture). *Let $f \colon \mathcal{X} \to B$ be a smooth projective morphism between smooth quasi-projective complex varieties with B irreducible, and let r be a nonnegative integer. Let $b \in B(\mathbb{C})$ and let $\alpha_b \in \mathrm{cl}_{\mathrm{Betti}}(\mathscr{Z}^r(\mathcal{X}_b)_{\mathbb{Q}})$. If α_b is the restriction of a class $\alpha \in \mathrm{H}^{2r}(\mathcal{X}^{\mathrm{an}}, \mathbb{Q})$ (or equivalently, by Deligne's global invariant cycle theorem, α_b is invariant under the monodromy action of $\pi_1(B^{\mathrm{an}}, b)$), then there is a cycle class $\alpha' \in \mathrm{cl}_{\mathrm{Betti}}(\mathscr{Z}^r(\mathcal{X})_{\mathbb{Q}})$ such that α_b is the restriction of α'.*

Remark 9.7. The variational Hodge conjecture is a consequence of the Hodge conjecture because Deligne's global invariant cycle theorem ([Del71] or [Voi03, 4.3.3]) shows that the class α_b above is the restriction of a Hodge class $\overline{\alpha}$ on some smooth completion of $\mathcal{X}$. On the other hand, it is a priori a much weaker statement, since the Hodge class $\overline{\alpha}$ is an "absolute Hodge class" in the sense of [DMOS82, p. 28].

Assuming Conjecture 9.6, André's argument yields the following extension of Theorem 8.3:

Theorem 9.8. *Assume that k is an algebraic closure of a field k_f that is finitely generated over $\mathbb{Q}$. Let B be an irreducible k_f-variety. Let $\mathcal{X} \to B$ be a smooth proper morphism. Let r be a nonnegative integer. If Conjecture 9.6 holds for r, then the set of $b \in B(k)$ such that $\rho^r(\mathcal{X}_b) > \rho^r(\mathcal{X}_{\bar{\eta}})$ is sparse.*

Remark 9.9. In fact, André works with motivic Galois groups instead of spaces of cycle classes, and hence proves more, namely that one can consider not only $\mathcal{X} \to B$ but also all its fibered powers simultaneously.

9.4. **An implication.** We introduced above two different variational forms of the Hodge conjecture, in order to extend the results on Néron-Severi groups to higher-degree classes. We now show that the p-adic version is stronger.

Theorem 9.10. *The p-adic variational Hodge conjecture (Conjecture 9.2) implies the variational Hodge conjecture (Conjecture 9.6).*

The proof will use the following statement.

Lemma 9.11. *Let K be any field of characteristic* 0*. Let B be a K-variety. Let $f \colon \mathcal{X} \to B$ be a smooth projective morphism. Let α be a relative de Rham class, i.e., an element of $\mathrm{H}^0(B, \mathbb{R}^{2r} f_* \Omega^\bullet_{\mathcal{X}/B})$. Then there exists a countable set of closed subschemes V_i of B such that for any field extension $L \supseteq K$ and any $b \in B(L)$, the restriction $\alpha_b \in \mathrm{H}^{2r}_{\mathrm{dR}}(\mathcal{X}_b/L)$ is algebraic (i.e., in $\mathrm{cl}_{\mathrm{dR}}(\mathscr{Z}^r(\mathcal{X}_b)_{\mathbb{Q}})$) if and only if $b \in \bigcup_i V_i(L)$.*

Proof. This is a standard consequence of the fact that the codimension-r subschemes in the fibers of f are parametrized by countably many relative Hilbert schemes H_j, each of which is projective over B. Let E be $\mathbb{R}^{2r} f_* \Omega^\bullet_{\mathcal{X}/B}$ viewed as a geometric vector bundle on B. The relative cycle class map is a B-morphism $H_j \to \operatorname{Fil}^r E$; its image I_j is projective over B and points on I_j over an algebraically closed field parametrize the classes of effective cycles given by H_j. Taking $\mathbb{Q}$-linear combinations of these classes yields countably many closed subvarieties W_i of $\operatorname{Fil}^r E$ that together parametrize all de Rham cycle classes in fibers of f. The desired subschemes V_i are the inverse images of the W_i under the section $B \to \operatorname{Fil}^r E$ given by α. ☐

Proof of Theorem 9.10. Let $\mathcal{X}$, B, f, b, α_b, and α be as in Conjecture 9.6. Let $Z \in \mathscr{Z}^r(\mathcal{X}_b)_{\mathbb{Q}}$ be such that $\mathrm{cl}_{\mathrm{Betti}}(Z) = \alpha_b$. Let $\overline{\mathcal{X}}$ be a smooth projective completion of $\mathcal{X}$. As in Remark 9.7, we note that $\alpha_b = \bar{\alpha}|_{\mathcal{X}_b}$, where $\bar{\alpha} \in \mathrm{H}^{2r}(\overline{\mathcal{X}}^{\mathrm{an}}, \mathbb{Q})$ is a Hodge class. In the Leray spectral sequence, α maps to a relative Betti class in $\mathrm{H}^0(B^{\mathrm{an}}, \mathrm{R}^{2r} f_* \mathbb{C})$, and the relative Grothendieck comparison isomorphism identifies this with a relative de Rham class $\alpha_{\mathrm{dR}} \in \mathrm{H}^0(B, E)$, where E is the sheaf $\mathbb{R}^{2r} f_* \Omega^\bullet_{\mathcal{X}/B}$ equipped with the Hodge filtration $\operatorname{Fil}^\bullet$ and Gauss-Manin connection. Since $\bar{\alpha}$ is Hodge and restricts to the same section of E as α, α_{dR} is a horizontal section lying in $\mathrm{H}^0(B, \operatorname{Fil}^r E)$.

There is a finitely generated subring $A \subset \mathbb{C}$ such that $\mathcal{X}$, B, f, b, Z, E, and α_{dR} are obtained via base change from corresponding objects over A, which we next base extend by an embedding $A \hookrightarrow \mathbb{Z}_p$ provided by [Cas86, Chapter 5, Theorem 1.1]. To ease notation, from now on we use the notation of Setup 1.6 with $\mathcal{O}_K := \mathbb{Z}_p$, and reuse the symbols above to denote the corresponding objects over $\mathcal{O}_K$. Let $s \in B(k)$ be the reduction of $b \in B(\mathcal{O}_K)$. We may assume that the constituents of Z are flat over $\mathcal{O}_K$, so that the special fiber Z_s is the specialization of Z_K.

As in the proof of Lemma 4.2, we use $\mathrm{cl}_{\mathrm{cris}}(Z_s) \in \mathrm{H}^{2r}_{\mathrm{cris}}(\mathcal{X}_s/K)$ to construct a horizontal section $\gamma_{\mathrm{dR}}(Z_s)$ of $E|_D$ for some closed polydisk neighborhood D of b inside the residue disk in $B(\mathcal{O}_K)$ corresponding to s. (This time we need only the $\mathcal{O}_K$-points, not the $\mathcal{O}_{\overline{K}}$-points.) Since $\gamma_{\mathrm{dR}}(Z_s)$ and $\alpha_{\mathrm{dR},K}|_D$ are represented by convergent power series on D, it suffices to check that they are equal in the formal completion of E at b. The corresponding formal sections are horizontal and take the same value at b (namely, $\mathrm{cl}_{\mathrm{dR}}(Z_K)$), so they are equal. In particular, for any $b' \in D \subset B(\mathcal{O}_K)$, the value of $\gamma_{\mathrm{dR}}(Z_s)$ at the generic point b'_K of b' is in $\mathrm{Fil}^r E_{b'_K}$.

So Conjecture 9.2 applied to the $\mathcal{O}_K$-morphism $\mathcal{X}_{b'} \to B_{b'}$ implies that there exists $Z' \in \mathscr{Z}^r(\mathcal{X}_{b'_K})_{\mathbb{Q}}$ such that

$$\sigma_{\mathrm{cris}}(\mathrm{cl}_{\mathrm{dR}}(Z')) = \mathrm{cl}_{\mathrm{cris}}(Z_s).$$

The isomorphism σ_{cris} also maps $\alpha_{\mathrm{dR},K}|_{b'_K} = \gamma_{\mathrm{dR}}(Z_s)|_{b'_K}$ to $\mathrm{cl}_{\mathrm{cris}}(Z_s) \in \mathrm{H}^2_{\mathrm{cris}}(\mathcal{X}_s/K)$, so

$$\mathrm{cl}_{\mathrm{dR}}(Z') = \alpha_{\mathrm{dR},K}|_{b'_K}.$$

So the value of $\alpha_{\mathrm{dR},K}$ on any K-point in D is an algebraic class.

Applying Lemma 9.11 to $\mathcal{X}_K \to B_K$ and $\alpha_{\mathrm{dR},K}$ yields a countable set of closed subschemes V_i of B_K. The previous paragraph shows that $D \subseteq \bigcup V_i(K)$. By Lemma 4.34, we have $\dim V_i = \dim B_K$ for some i. This V_i contains the generic point η of B_K. So $\alpha_{\mathrm{dR},K}|_\eta = \mathrm{cl}_{\mathrm{dR}}(Y_\eta)$ for some $Y_\eta \in \mathscr{Z}^r(\mathcal{X}_\eta)_{\mathbb{Q}}$. Taking the closure in $\mathcal{X}_K$ of the constituents of Y_η defines some $Y_K \in \mathscr{Z}^r(\mathcal{X}_K)_{\mathbb{Q}}$. To Y_K we associate a relative de Rham class $\alpha'_{\mathrm{dR},K} \in \mathrm{H}^0(B_K, \mathbb{R}^{2r} f_* \Omega^\bullet_{\mathcal{X}_K/B_K})$ and a Betti class $\alpha' := \mathrm{cl}_{\mathrm{Betti}}((Y_K)^{\mathrm{an}}) \in \mathrm{H}^{2r}(\mathcal{X}^{\mathrm{an}}, \mathbb{Q})$, where the complex analytic spaces are obtained by fixing an A-algebra homomorphism $K \hookrightarrow \mathbb{C}$. The set $\{q \in B_K : \alpha'_{\mathrm{dR},K}|_q = \alpha_{\mathrm{dR},K}|_q\}$ is closed and contains η, so it contains b, which we now view as a K-point. Under the comparison isomorphisms, the equal de Rham classes $\alpha'_{\mathrm{dR},K}|_b$ and $\alpha_{\mathrm{dR},K}|_b$ correspond to the Betti classes α'_b and α_b, so $\alpha'_b = \alpha_b$. ☐

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DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA 02139-4307, USA
E-mail address: `dmaulik@math.mit.edu`

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA 02139-4307, USA
E-mail address: `poonen@math.mit.edu`
URL: `http://math.mit.edu/~poonen/`