ERRATA FOR "RATIONAL POINTS ON VARIETIES"

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This is an errata list for the book

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The name in parentheses is the discoverer of the error.

- Definition 1.3.1: "direct product" should be "finite direct product".
- Section 1.5.7.4, first sentence: "a another" should be "another". (Francesc Fité)
- Example 2.3.10, last sentence: $\mathbb{A}^n$ should be $\mathbb{A}^{n-3}$. (Douglas Ulmer)
- Theorem 2.5.1(b): p is the characteristic of k. (Francesc Fité)
- Exercise 2.6: The morphism $\operatorname{Spec} k \to S$ is unnecessary. (Douglas Ulmer)
- Example 3.1.5: Exercise should be Example. (Francesc Fité)
- Remark 3.5.13: After "regular local ring", add "of dimension r".
- Section 3.5.15: The sentence "But X'_R need not be smooth." is correct, but it would be more to the point to say "But X'_R need not be regular."
- Proof of Proposition 3.5.19: $\frac{\partial g_i}{\partial t_j}(x)$ should be $\frac{\partial g_i}{\partial t_j}$. (Francesc Fité)
- Proof of Lemma 3.5.57: The first period should be a comma. (Francesc Fité)
- Remark 3.5.62: "Theorem 3.5.59" should be "Definition 3.5.59".
- Line before Example 4.1.3: $\mathscr{F}_{S_{ij}}$ should be $\mathscr{F}|_{S_{ij}}$. (Francesc Fité)
- Proof of Proposition 5.2.7(a): In the last reference, 11.7 should be 11.17.
- Proof of Theorem 5.3.1: The actions were not specified clearly. The *left* translation action on G induces a right G-action on A, which can be turned into a left G-action on A (in which g acts as right action of g^{-1}). It is this left G-action on A and the induced contragredient left G-action on A^* that are used in Step 2.
- Remark 5.6.24: One should assume that G is a connected algebraic group, and that $\operatorname{char} k = 0$ or G is reductive. (Alex Youcis)
- Section 5.6.6: The definition of simple algebraic group is too restrictive: no positive-dimensional algebraic group in characteristic p would be simple by this definition, because the kernel of Frobenius would be a normal subgroup scheme.
- The references to [**Wit10**] at the beginning of Section 5.7.2 and in the proof of Theorem 5.7.13 should be to [**Wit08**]. (Borys Kadets.)
- Section 5.11: The terminology needs to be corrected to reflect standard usage, which is as follows. Let $\operatorname{Inn} G_{k_s}$ be the group of inner automorphisms of $G(k_s)$. The homomorphisms $G(k_s) \to \operatorname{Inn} G_{k_s} \to \operatorname{Aut} G_{k_s}$ induce maps $\mathrm{H}^1(k, G) \to \mathrm{H}^1(k, \operatorname{Inn} G_{k_s}) \to \mathrm{H}^1(k, \operatorname{Aut} G_{k_s})$. The algebraic groups corresponding to elements in the image of the second map (resp. the composition) are called inner forms (resp. pure inner forms).

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Researchers working on the Langlands program equip these with rigidifying data. They define an inner twist to be a pair (H, ξ), where H is an algebraic group over k and $\xi\colon G_{k_s} \to H_{k_s}$ is an isomorphism such that $\xi^{-1}({}^\sigma\xi) \in \operatorname{Inn} G_{k_s}$ for all $\sigma \in \mathfrak{G}_k$; then $\sigma \mapsto \xi^{-1}({}^\sigma\xi)$ is a 1-cocycle representing an element of $\mathrm{H}^1(k, \operatorname{Inn} G_{k_s})$. They then define a pure inner twist (pure rational form in [Vog93, Definition 2.6]) to be (H, ξ, z), where (H, ξ) is as above and $z\colon \mathfrak{G}_k \to G(k_s))$ is a 1-cocycle such that $\xi^{-1}({}^\sigma\xi)$ equals the inner automorphism $\operatorname{inn}_{z_\sigma}$ for all $\sigma \in \mathfrak{G}_k$; then z represents an element of $\mathrm{H}^1(k, G)$. See also [Kal16, §2.3]. (Alex Youcis)

- Remark 5.12.7: In the second paragraph, one should assume that X is reduced. (Alex Youcis)
- Theorem 5.12.24(c): $A(k)$ should be $A^t(k)$, where A^t is the dual abelian variety (Olivier de Gaay Fortman). Also, Tate's proof is for finite extensions of $\mathbb{Q}_p$; the case $k = \mathbb{R}$ is related to older results of Witt [Wit34] (see [Sch94, p. 221]), and the characteristic p case is proved in [Mil70].
- Warning 5.12.26: In the second sentence, k should be nonarchimedean. (Olivier de Gaay Fortman)
- Warning 5.12.27: In the last sentence, delete "the group".
- Definition 6.2.7(1): "collection" should be "the collection". (Francesc Fité)
- Definition 6.3.21: "equipped" should be "equipped with". (Francesc Fité)
- Proof of Proposition 6.3.22: "such" should be "such that". (Francesc Fité)
- Definition 6.5.1: "An" should be "an". (Francesc Fité)
- Proposition 6.5.9: "a isomorphism" should be "an isomorphism". (Francesc Fité)
- Proof of Proposition 6.5.9: In the displayed isomorphism, $\mathscr{F}$ should be G. (Long Liu)
- Section 6.5.6.4: The second sentence should say "Let $\tau \in \mathrm{H}^1(S, G)$ be the class of T." (Francesc Fité)
- Two lines before Proposition 6.7.1, it should say "$E_\infty^{n,0}$ is a subobject of L^n." (Anthony Várilly-Alvarado)
- Paragraph after Proposition 6.8.1: Proposition 1.3.15(iii) should be Proposition 1.5.13(iii). (Anthony Várilly-Alvarado)
- Thanks to the proof of the purity conjecture [Česnavičius19], some simplifications are possible:
  - In Theorem 6.8.3, the "caveat" can be simplified to "the caveat that one must exclude the p-primary part of all the groups if there exists $x \in X^{(1)}$ such that $\mathbf{k}(x)$ is imperfect of characteristic p".
  - In Corollaries 6.8.5 and 6.8.7, the caveats are unnecessary.
  - In the proof of Proposition 6.9.10, the $\operatorname{char} k = 0$ proof then works in arbitrary characteristic.
- Warning 6.8.4: $\operatorname{Br} k(X)$ should be $\operatorname{Br} \mathbf{k}(X)$.
- Proof of Lemma 6.9.8: In the first sentence, the claims are true, but the logic is not presented correctly. The commutative square involving s^* initially involves the *downward* homomorphism on the left induced by the restriction of s to $\operatorname{Spec} \mathbf{k}(B)$. It is only after knowing that the two vertical homomorphisms on the left are inverses that one can use s^* to deduce $\operatorname{Br} X \subset \operatorname{Br} B$. (Anthony Várilly-Alvarado and Ken Zheng)
- Proof of Lemma 6.9.8: Where Theorem 6.9.7 is invoked, Corollary 6.7.8 should be mentioned too.

- Section 7.3.3, first sentence: Change "over a" to "of". (Francesc Fité)
- Proposition 7.5.17: $1 \otimes \sigma$ should be $1 \times \sigma$. (Francesc Fité)
- Remark 7.5.19: The Grothendieck–Lefschetz trace formula is not true in the generality suggested (it fails for the morphism $\mathbb{A}^1 \to \mathbb{A}^1$ sending x to $x+1$; see Milne, *Lectures on étale cohomology*, 2013–03–22, Example 29.1), but Grothendieck did prove it for the Frobenius morphism of a variety over a finite field. See this preprint of Yakov Varshavsky for a generalization. Also, the reference to Deligne is for the Poincaré duality statement. (Kaloyan Slavov)
- Proof of Proposition 7.6.1(b): "subscheme" should be "subschemes". (Francesc Fité)
- Theorem 8.4.10 and Corollary 8.4.11: It is necessary to add the hypothesis "If $\operatorname{char} k = p$, assume that X is proper." In the proof of Theorem 8.4.10, change the sentence starting "For any nonarchimedean $v \in S$" to "For any nonarchimedean $v \in S$, there are only finitely many possibilities for the k_v-scheme $F^{-1}(x_v)$ as x_v ranges over $X(k_v)$: when $\operatorname{char} k = 0$, this follows since k_v has only finitely many extensions of each degree; when $\operatorname{char} k = p$, use Krasner's lemma (Proposition 3.5.74) and compactness of $X(k_v)$." (Fei Xu)
- Remark 8.4.12: Change "irrational" to "rational", and "dominant morphism $\mathbb{P}^1 \twoheadrightarrow X$" to "morphism $\mathbb{P}^1 \to X$ inducing a surjection $\mathbb{P}^1(\mathbf{A}^S) \to X(\mathbf{A}^S)$".
- Paragraph after Definition 9.4.1: [Kol96, Corollary III.2.3.5.2] should be [Kol96, Corollary III.3.2.5.2]. (Osami Yasukura)
- Section A.4: The set $\{x, \{y\}\}$ does not necessarily determine x and y. It should be changed to Kuratowski's definition $\{\{x\}, \{x, y\}\}$. (Juan Climent Vidal)
- Section C.3, Table 3: In the "affine, Quotient" entry, 11.7 should be 11.17. (Long Liu)

Acknowledgments

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