

THE BRAUER-MANIN OBSTRUCTION FOR SUBVARIETIES OF ABELIAN VARIETIES OVER FUNCTION FIELDS

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ABSTRACT. We prove that for a large class of subvarieties of abelian varieties over global function fields, the Brauer-Manin condition on adelic points cuts out exactly the rational points. This result is obtained from more general results concerning the intersection of the adelic points of a subvariety with the adelic closure of the group of rational points of the abelian variety.

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1. INTRODUCTION

The notation in this section remains in force throughout the paper, except in Section 3.3, and in Section 4 where we allow also the possibility that K is a number field.

Let k be a field. Let K be a finitely generated extension of k of transcendence degree 1. We assume that k is relatively algebraically closed in K , since the content of our theorems will be unaffected by this assumption. Let $\overline{K}$ be an algebraic closure of K . We will use this notation consistently for an algebraic closure, and we will choose algebraic closures compatibly whenever possible: thus $\overline{k}$ is the algebraic closure of k in $\overline{K}$. Let K^s be the separable closure of K in $\overline{K}$. Let Ω_{all} be the set of all nontrivial valuations of K that are trivial on k . Let Ω be a cofinite subset of Ω_{all} ; if k is finite, we may weaken the cofiniteness hypothesis to assume only that $\Omega \subseteq \Omega_{\text{all}}$ has Dirichlet density 1. For each $v \in \Omega$, let K_v be the completion of K at v , and let $\mathbf{F}_v$ be the residue field. Equip K_v with the v -adic topology. Define the ring of adèles $\mathbf{A}$ as the restricted direct product $\prod_{v \in \Omega} (K_v, \mathcal{O}_v)$ of the K_v with respect to their valuation subrings $\mathcal{O}_v$. Then $\mathbf{A}$ is a topological ring, in which $\prod_{v \in \Omega} \mathcal{O}_v$ is open and has the product topology.

If A is an abelian variety over K , then $A(K)$ embeds diagonally into $A(\mathbf{A}) \simeq \prod_v A(K_v)$. Define the adelic topology on $A(K)$ as the topology induced from $A(\mathbf{A})$. For any fixed v define the v -adic topology on $A(K)$ as the topology induced from $A(K_v)$. Let $\overline{A(K)}$ be the closure of $A(K)$ in $A(\mathbf{A})$.

For any extension of fields $F' \supset F$ and any F -variety X , let $X_{F'}$ be the base extension of X to F' . Call a K -variety X *constant* if $X \cong Y_K$ for some k -variety Y , and call X *isotrivial* if $X_{\overline{K}} \cong Y_{\overline{K}}$ for some variety Y defined over $\overline{k}$.

From now on, X is a closed K -subscheme of A . Call X *coset-free* if $X_{\overline{K}}$ does not contain a translate of a positive-dimensional abelian subvariety of $A_{\overline{K}}$.

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When k is finite and $\Omega = \Omega_{\text{all}}$, the intersection $X(\mathbf{A}) \cap \overline{A(K)} \subset A(\mathbf{A})$ is closely related to the Brauer-Manin obstruction to the Hasse principle for X/K : see Section 4. For curves over number fields, V. Scharaschkin and A. Skorobogatov independently raised the question of whether the Brauer-Manin obstruction is the only obstruction to the Hasse principle, and proved that this is so when the Jacobian has finite Mordell-Weil group and finite Shafarevich-Tate group. The connection with the adelic intersection is stated explicitly in [Sch99], and is based on global duality statements originating in the work of Cassels: see Remark 4.4. See also [Sko01, Fly04, Poo06, Sto07]: the last of these in particular contains many conjectures and theorems relating descent information, the method of Chabauty and Coleman, the Brauer-Manin obstruction, and Grothendieck's section conjecture.

In this paper we answer (most cases of) a generalization of the function field analogue of a question raised for curves over number fields in [Sch99], concerning whether the Brauer-Manin condition cuts out exactly the rational points: see Theorem D. This question is still wide open in the number field case. Along the way, we prove results about adelic intersections along the lines of the “adelic Mordell-Lang conjecture” suggested in [Sto07, Question 3.12]. Again, these are open in the number field case. In particular, we prove the following theorems.

Theorem A. *If $\text{char } k = 0$, then $X(K) = X(\mathbf{A}) \cap \overline{A(K)}$.*

Theorem B. *Suppose that $\text{char } k = p > 0$, that $A_{\overline{K}}$ has no nonzero isotrivial quotient, and that $A(K^s)[p^\infty]$ is finite. Suppose that X is coset-free. Then $X(K) = X(\mathbf{A}) \cap \overline{A(K)}$.*

Remark 1.1. The proposition in [Vol95] states that in the “general case” in which A is ordinary and the Kodaira-Spencer class of A/K has maximal rank, we have $A(K^s)[p^\infty] = 0$.

Conjecture C. For any closed K -subscheme X of any A , we have $\overline{X(K)} = X(\mathbf{A}) \cap \overline{A(K)}$, where $\overline{X(K)}$ is the closure of $X(K)$ in $X(\mathbf{A})$.

Remark 1.2. If $A_{\overline{K}}$ has no nonzero isotrivial quotient and X is coset-free, then $X(K)$ is finite [Hru96, Theorem 1.1] so $\overline{X(K)} = X(K)$. So Conjecture C predicts in particular that the hypothesis on $A(K^s)[p^\infty]$ in Theorem B is unnecessary.

Remark 1.3. Here is an example to show that the statement $X(K) = X(\mathbf{A}) \cap \overline{A(K)}$ can fail for a constant curve in its Jacobian. Let X be a curve of genus ≥ 2 over a finite field k . Choose a divisor of degree 1 on X to embed X in its Jacobian A . Let $F: A \rightarrow A$ be the k -Frobenius map. Let K be the function field of X . Let $P \in X(K)$ be the point corresponding to the identity map $X \rightarrow X$. Let $P_v \in X(\mathbf{F}_v)$ be the reduction of P at v .

For each v , the Teichmüller map $\mathbf{F}_v \rightarrow K_v$ identifies $\mathbf{F}_v$ with a subfield of K_v . Any $Q \in A(K_v)$ can be written as $Q = Q_0 + Q_1$ with $Q_0 \in A(\mathbf{F}_v)$ and Q_1 in the kernel of the reduction map $A(K_v) \rightarrow A(\mathbf{F}_v)$; then $\lim_{m \rightarrow \infty} F^m(Q_1) = 0$, so $\lim_{n \rightarrow \infty} F^{n!}(Q) = \lim_{n \rightarrow \infty} F^{n!}(Q_0) = Q_0$. In particular, taking $Q = P$, we find that $(F^{n!}(P))_{n \geq 1}$ converges in $A(\mathbf{A})$ to the point $(P_v) \in X(\mathbf{A}) = \prod_v X(K_v)$, where we have identified P_v with its image under the Teichmüller map $X(\mathbf{F}_v) \hookrightarrow X(K_v)$. If (P_v) were in $X(K)$, then in $X(K_v)$ we would have $P_v \in X(\mathbf{F}_v) \cap X(K) = X(k)$, which contradicts the definition of P_v if v is a place of degree greater than 1 over k . Thus (P_v) is in $X(\mathbf{A}) \cap \overline{A(K)}$ but not in $X(K)$.

In the final section of this paper, we restrict to the case of a global function field, and extend Theorem B to prove (under mild hypotheses) that for a subvariety of an abelian

variety, the Brauer-Manin condition cuts out exactly the rational points: see Section 4 for the definitions of $X(\mathbf{A})^{\text{Br}}$ and $\widehat{\text{Sel}}$. Our result is as follows.

Theorem D. *Suppose that K is a global function field of characteristic p , that $A_{\overline{K}}$ has no nonzero isotrivial quotient, and that $A(K^s)[p^\infty]$ is finite. Suppose that X is coset-free. Then $X(K) = X(\mathbf{A})^{\text{Br}} = X(\mathbf{A}) \cap \widehat{\text{Sel}}$.*

To our knowledge, Theorem D is the first result giving a wide class of varieties of general type such that the Brauer-Manin condition cuts out exactly the rational points.

2. CHARACTERISTIC 0

Throughout this section, we assume $\text{char } k = 0$. In this case, results follow rather easily.

Proposition 2.1. *For any v , the v -adic topology on $A(K)$ equals the discrete topology.*

Proof. The question is isogeny-invariant, so we reduce to the case where A is simple. Let $A(\mathbf{F}_v)$ denote the group of $\mathbf{F}_v$ -points on the Néron model of A over $\mathcal{O}_v$. Let $A^1(K_v)$ be the kernel of the reduction map $A(K_v) \rightarrow A(\mathbf{F}_v)$. The Lang-Néron theorem [LN59, Theorem 1] implies that either A is constant and $A(K)/A(k)$ is finitely generated, or A is non-constant and $A(K)$ itself is finitely generated. In either case, the subgroup $A^1(K) := A(K) \cap A^1(K_v)$ is finitely generated. By the theory of formal groups (cf. [Ser92, p. 118, Theorem 2]), $A^1(K_v)$ has a descending filtration by open subgroups in which the quotients of consecutive terms are torsion-free (this is where we use $\text{char } k = 0$), so the induced filtration on the finitely generated group $A^1(K)$ has only finitely many nonzero quotients. Thus $A^1(K)$ is discrete. Since $A^1(K_v)$ is open in $A(K_v)$, the group $A(K)$ is discrete in $A(K_v)$. $\square$

Remark 2.2. The literature contains results close to Proposition 2.1. It is mentioned in the third subsection of the introduction to [Man63] for elliptic curves with non-constant j -invariant, and it appears for abelian varieties with K/k -trace zero in [BV93].

Corollary 2.3. *The adelic topology on $A(K)$ equals the discrete topology.*

Proof. The adelic topology is at least as strong as (i.e., has at least as many open sets as) the v -adic topology for any v . $\square$

We can improve the result by imposing conditions in only the residue fields $\mathbf{F}_v$ instead of the completions K_v , that is, “flat” instead of “deep” information in the sense of [Fly04]. In fact, we have:

Proposition 2.4. *There exist $v, v' \in \Omega$ of good reduction for A such that $A(K) \rightarrow A(\mathbf{F}_v) \times A(\mathbf{F}_{v'})$ is injective.*

Proof. Let B be the K/k -trace of A . Pick any $v \in \Omega$ of good reduction. The kernel H of $A(K) \rightarrow A(\mathbf{F}_v)$ meets $B(k)$ trivially. By Silverman’s specialization theorem [Lan83, Chapter 12, Theorem 2.3], there exists $v' \in \Omega$ such that H injects under reduction modulo v' . $\square$

Proof of Theorem A. By Corollary 2.3, $X(\mathbf{A}) \cap \overline{A(K)} = X(\mathbf{A}) \cap A(K) = X(K)$. $\square$

3. CHARACTERISTIC p

Throughout this section, $\text{char } k = p$.

3.1. Field-theoretic lemmas.

Lemma 3.1. *For any v , if $\alpha \in K_v$ is algebraic over K , then α is separable over K .*

Proof. Replacing K by its relative separable closure in $L := K(\alpha)$, we may assume that L is purely inseparable over K . Then the valuation v on K admits a unique extension w to L , and the inclusion of completions $K_v \rightarrow L_w$ is an isomorphism. By [Ser79, I.§4, Proposition 10] (hypothesis (F) there holds for localizations of finitely generated algebras over a field), we have an “ $n = \sum e_i f_i$ ” result, which in our case says $[L : K] = [L_w : K_v] = 1$. So $\alpha \in K$. $\square$

If L is a finite extension of K , let $\mathbf{A}_L$ be the corresponding ring of adèles, defined as a restricted direct product over places of L . There is a natural inclusion $\mathbf{A} \hookrightarrow \mathbf{A}_L$.

Lemma 3.2. *Let L be a finite extension of K . Then in $\mathbf{A}_L$ we have $\mathbf{A} \cap L = K$.*

Proof. Fix $v \in \Omega$. By [Bou98, VI.§8.5, Corollary 3] and the fact that hypothesis (F) in the previous proof holds, the natural map $K_v \otimes_K L \rightarrow \prod_{w|v} L_w$ is an isomorphism. Hence in $\prod_{w|v} L_w$ we have $K_v \cap L = K$. The result follows. $\square$

3.2. Abelian varieties.

Lemma 3.3. *For any $n \in \mathbf{Z}_{\geq 1}$, the quotient $A(K_v)/nA(K_v)$ is Hausdorff.*

Proof. Equivalently, we must show that $nA(K_v)$ is closed in $A(K_v)$. Suppose (P_i) is a sequence in $nA(K_v)$ that converges to $P \in A(K_v)$. Write $P_i = nQ_i$ with $Q_i \in A(K_v)$. Then $n(Q_i - Q_{i+1}) \rightarrow 0$ as $i \rightarrow \infty$.

Let $\mathcal{O}_v$ be the valuation ring of K_v , and let $\mathcal{A}$ over be the Néron model of A over $\mathcal{O}_v$. Applying [Gre66, Corollary 1] to $\mathcal{A}[n]$ shows that for any sequence (R_i) in $A(K_v)$ with $nR_i \rightarrow 0$, the distance of R_i to the nearest point of $A(K_v)[n]$ tends to 0.

Thus by induction on i we may adjust each Q_i by a point in $A(K_v)[n]$ so that $Q_i - Q_{i+1} \rightarrow 0$ as $i \rightarrow \infty$. Since $A(K_v)$ is complete, (Q_i) converges to some $Q \in A(K_v)$, and $nQ = P$. Thus $nA(K_v)$ is closed. $\square$

Remark 3.4. In the case where k is finite, Lemma 3.3 is immediate since $A(K_v)$ is compact and its image under multiplication-by- n is closed.

The following is a slight generalization of [Vol95, Lemma 2], with a more elementary proof.

Proposition 3.5. *If $A(K^s)[p^\infty]$ is finite, then for any v , the v -adic topology on $A(K)$ is at least as strong as the topology induced by all subgroups of finite p -power index.*

Proof. For convenience choose algebraic closures $\overline{K}, \overline{K}_v$ of K, K_v such that $K^s \subseteq \overline{K} \subseteq \overline{K}_v$. As in the proof of Proposition 2.1, there is an open subgroup U of $A(K_v)$ such that $B := A(K) \cap U$ is finitely generated. It suffices to show that for every $e \in \mathbf{Z}_{\geq 0}$, there exists an open subgroup V of $A(K_v)$ such that $B \cap V \subseteq p^e A(K)$.

Choose m such that $p^m A(K^s)[p^\infty] = 0$. Let $M = e + m$. Then $B/p^M B$ is finite. By Lemma 3.3, $A(K_v)/p^M A(K_v)$ is Hausdorff, so the image of $B/p^M B$ in $A(K_v)/p^M A(K_v)$ is discrete. Hence there is an open subgroup V of $A(K_v)$ such that $B \cap V = \ker(B \rightarrow A(K_v)/p^M A(K_v))$.

Suppose $b \in B \cap V$. Then $b = p^M c$ for some $c \in A(K_v) \cap A(\overline{K})$. By Lemma 3.1, we obtain $c \in A(K^s)$. If $\sigma \in \text{Gal}(K^s/K)$, then $\sigma c - c \in A(K^s)[p^M]$ is killed by p^m . Thus $p^m c \in A(K)$. So $b = p^e p^m c \in p^e A(K)$. Hence $B \cap V \subseteq p^e A(K)$. $\square$

Proposition 3.6. *The adelic topology on $A(K)$ is at least as strong as the topology induced by all subgroups of finite index.*

Proof. As in the proof of Proposition 2.1, the Lang-Néron theorem implies that $A(\mathbf{A})$ has an open subgroup whose intersection with $A(K)$ is finitely generated. It suffices to study the topology induced on that finitely generated subgroup, so we may reduce to the case in which k is finitely generated over a finite field $\mathbf{F}_q$. This case is proved in [Mil72], which adapts and extends [Ser64] and [Ser71]. (The paper [Mil72] uses not the adelic topology as we have defined it, but the topology coming from the closed points of a finite-type $\mathbf{Z}$ -scheme with function field K . Since the adelic topology is stronger, [Mil72] contains what we want.) $\square$

Lemma 3.7. *Suppose that $A(K)$ is finitely generated. Then $\left(\overline{A(K)}\right)_{\text{tors}} = A(K)_{\text{tors}}$.*

Proof. Let

$$T := \ker \left(A(K) \rightarrow \prod_{v \in \Omega} \frac{A(\mathbf{F}_v)}{A(\mathbf{F}_v)_{\text{tors}}} \right),$$

where $A(\mathbf{F}_v)$ is the group of $\mathbf{F}_v$ -points on the Néron model of A . Since $A(K)$ is finitely generated and the groups $A(\mathbf{F}_v)/A(\mathbf{F}_v)_{\text{tors}}$ are torsion-free, there is a finite subset $S \subset \Omega$ such that $T = A(K) \cap U$ for the open subgroup

$$U := \ker \left(A(\mathbf{A}) \rightarrow \prod_{v \in S} \frac{A(\mathbf{F}_v)}{A(\mathbf{F}_v)_{\text{tors}}} \right)$$

of $A(\mathbf{A})$. The finitely generated group $A(K)/T$ is contained in the torsion-free group $\prod_{v \in S} \frac{A(\mathbf{F}_v)}{A(\mathbf{F}_v)_{\text{tors}}}$, so $A(K)/T$ is free, and we have $A(K) \cong T \oplus F$ as topological groups, where F is a discrete free abelian group of finite rank.

We claim that the topology of T is that induced by the subgroups nT for $n \geq 1$. For $n \geq 1$, the subgroup nT is open in T by Proposition 3.6. If $t \in T$, then some positive multiple of t is in the kernel of $A(K_v) \rightarrow A(\mathbf{F}_v)$, and then p -power multiples of this multiple tend to 0. Applying this to a finite set of generators of T , we see that any open neighborhood of 0 in T contains nT for some $n \in \mathbf{Z}_{>0}$.

It follows that $\overline{T} \cong T \otimes \hat{\mathbf{Z}}$. Now

$$\left(\overline{A(K)}\right)_{\text{tors}} = (\overline{T} \oplus F)_{\text{tors}} = \overline{T}_{\text{tors}} \cong (T \otimes \hat{\mathbf{Z}})_{\text{tors}} \cong T_{\text{tors}}.$$

$\square$

Remark 3.8. When k is finite, an easier proof of Lemma 3.7 is possible: combined with the fact that $A(\mathbf{A})$ is profinite, Proposition 3.6 implies that $\overline{A(K)} \cong A(K) \otimes \hat{\mathbf{Z}}$; the torsion subgroup of the latter equals $A(K)_{\text{tors}}$.

The following proposition is a function field analogue of [Sto07, Proposition 3.6]. Our proof must be somewhat different, however, since [Sto07] made use of strong “image of Galois” theorems whose function field analogues have recently been disproved [Zar06].

Proposition 3.9. *Suppose that $A(K^s)[p^\infty]$ is finite. Let Z be a finite K -subscheme of A . Then $Z(\mathbf{A}) \cap \overline{A(K)} = Z(K)$.*

Proof. In this first paragraph we show that replacing K by a finite extension L does not destroy the hypothesis that $A(K^s)[p^\infty]$ is finite. This is obvious if L is separable over K , so assume that L is purely inseparable. Choose $n \in \mathbf{Z}_{\geq 0}$ with $L^{p^n} \subseteq K$. Then $(L^s)^{p^n} \subseteq K^s$, so $p^n A(L^s)[p^\infty] \subseteq A(K^s)[p^\infty]$. Thus $p^n A(L^s)[p^\infty]$ is finite. But multiplication-by- p^n has finite fibers, so $A(L^s)[p^\infty]$ itself is finite.

Next we claim that if we prove the conclusion after base extension to a finite extension L , then the desired conclusion over K holds. Namely, suppose that we prove $Z(\mathbf{A}_L) \cap \overline{A(L)} = Z(L)$. Then

$$Z(\mathbf{A}) \cap \overline{A(K)} \subseteq Z(\mathbf{A}_L) \cap \overline{A(L)} = Z(L)$$

so

$$Z(\mathbf{A}) \cap \overline{A(K)} \subseteq Z(\mathbf{A}) \cap Z(L) = Z(K),$$

where the last equality uses Lemma 3.2.

Thus we may replace K by a finite extension to assume that Z consists of a finite set of K -points of A . (The same idea was used in [Sto07].) A point $P \in \overline{A(K)}$ is represented by a sequence $(P_n)_{n \geq 1}$ in $A(K)$ such that for every v , the limit $\lim_{n \rightarrow \infty} P_n$ exists in $A(K_v)$. If in addition $P \in Z(\mathbf{A})$, then there is a point $Q_v \in Z(K)$ whose image in $Z(K_v)$ equals $\lim_{n \rightarrow \infty} P_n \in A(K_v)$. The $P_n - Q_v$ are eventually contained in the kernel of $A(K) \rightarrow A(\mathbf{F}_v)$, which is finitely generated, so there are finitely generated subfields $k_0 \subseteq k$, $K_0 \subseteq K$ with K_0/k_0 a function field such that all the P_n and the points of $Z(K)$ are in $A(K_0)$. By Proposition 3.5, the sequence $(P_n - Q_v)_{n \geq 1}$ is eventually divisible in $A(K_0)$ by an arbitrarily high power of p . For any other $v' \in \Omega$, then the same is true of $(P_n - Q_{v'})_{n \geq 1}$. Then $Q_{v'} - Q_v \in A(K_0)$ is divisible by every power of p . Since $A(K_0)$ is finitely generated, $Q_{v'} - Q_v$ is a torsion point in $A(K_0)$. This holds for every $v' \in \Omega$, and $A(K_0)_{\text{tors}}$ is finite, so $R := P - Q_v \in \overline{A(K_0)}$ is a torsion point in $\overline{A(K_0)}$. Lemma 3.7 applied to K_0 yields $R \in A(K_0)_{\text{tors}}$. Thus $P = R + Q_v \in A(K)$. So $P \in Z(\mathbf{A}) \cap A(K) = Z(K)$. $\square$

Lemma 3.10. *Fix $v \in \Omega$. Let Γ_v be the closure of $A(K)$ in $A(K_v)$. Then for every $e \in \mathbf{Z}_{\geq 0}$, the map $A(K)/p^e A(K) \rightarrow \Gamma_v/p^e \Gamma_v$ is surjective.*

Proof. Let $\mathcal{O}_v$ be the valuation ring of K_v , and let $\mathfrak{m}_v$ be its maximal ideal. Let $\mathcal{A}$ over $\mathcal{O}_v$ be the Néron model. For $r \in \mathbf{Z}_{\geq 1}$, let G_r be the kernel of $A(K_v) = \mathcal{A}(\mathcal{O}_v) \rightarrow \mathcal{A}(\mathcal{O}_v/\mathfrak{m}_v^r)$. It follows from [Ser92, p. 118, Theorem 2] that G_r/G_{r+1} is isomorphic to $(\mathcal{O}_v/\mathfrak{m}_v)^{\dim A}$, which is killed by p , so each G_r is an abelian pro- p -group, and hence a topological $\mathbf{Z}_p$ -module. There are only finitely many points of order p in $A(K_v)$, and $\bigcap_{r \geq 1} G_r = \{0\}$, so some G_r contains no nontrivial p -torsion points, and hence is torsion-free. In particular, $A(K_v)$ has an open subgroup $A^\circ(K_v)$ that is a torsion-free topological $\mathbf{Z}_p$ -module, and we may choose $A^\circ(K_v)$ so that $A^\circ(K) := A(K) \cap A^\circ(K_v)$ is finitely generated.

The group $\Gamma_v^\circ := \Gamma_v \cap A^\circ(K_v)$ is the closure of $A^\circ(K)$, so there is an isomorphism of topological groups $\Gamma_v^\circ \cong \mathbf{Z}_p^{\oplus m}$ for some $m \in \mathbf{Z}_{\geq 0}$. In particular, for any $e \in \mathbf{Z}_{\geq 0}$, the group $p^e \Gamma_v^\circ$ is open in Γ_v° , which is open in Γ_v . So the larger group $p^e \Gamma_v$ also is open in Γ_v . But the image of $A(K)$ in the discrete group $\Gamma_v/p^e \Gamma_v$ is dense, so the map $A(K)/p^e A(K) \rightarrow \Gamma_v/p^e \Gamma_v$ is surjective. $\square$

3.3. A uniform Mordell-Lang conjecture. We thank Zoé Chatzidakis, Françoise Delon, and Thomas Scanlon for many of the ideas used in this section. See [Del98] for the definitions of *separable*, *p-basis*, *p-free*, *p-components*, etc. By *iterated p-components* we mean *p-components* of *p-components* of $\dots$ of *p-components* (all with respect to a given *p-basis*).

The goal of this section is to deduce a uniform version (Proposition 3.16) of the function field Mordell-Lang conjecture from a version in [Hru96]. Under some hypotheses, the uniform version asserts the finiteness of the intersection of a subvariety X of an abelian variety A with any coset of a subgroup $p^e A(F)$ of $A(F)$, where F is allowed to range over p -basis-preserving extensions of an initial ground field K .

Remark 3.11. The p -basis condition on F , or something like it, is necessary for the truth of Proposition 3.16: with no condition, F might be algebraically closed, and then $p^e A(F) = A(F)$, so the desired finiteness would fail assuming $\dim X > 0$. The p -basis condition is used in the proof of Proposition 3.16 to imply separability of F over K , which guarantees that a non-isotriviality hypothesis on A over K is preserved by base extension to F : see Lemma 3.13 and its proof.

Lemma 3.12. *Let $\mathcal{B}$ be a p -basis for a field K of characteristic p . Let L be an extension of K such that $\mathcal{B}$ is also a p -basis for L . Suppose that c is an element of L that is not algebraic over K . Then there exists a separably closed extension F of L such that $\mathcal{B}$ is a p -basis of F and the $\text{Aut}(F/K)$ -orbit of c is infinite.*

Proof. Fix a transcendence basis T for L/K . Let Ω be an algebraically closed extension of K such that the transcendence basis of Ω/K is identified with the set $\mathbf{Z} \times T$. Identify L with a subfield of Ω in such a way that each $t \in T$ maps to the transcendence basis element for Ω/K labelled by $(0, t) \in \mathbf{Z} \times T$. The map of sets $\mathbf{Z} \times T \rightarrow \mathbf{Z} \times T$ mapping (i, t) to $(i+1, t)$ extends to an automorphism $\sigma \in \text{Aut}(\Omega/K)$. For $i \in \mathbf{Z}$, let $L_i = \sigma^i(L)$. Let L_∞ be the compositum of the L_i in Ω . Then $\sigma(L_\infty) = L_\infty$. Let F be the separable closure of L_∞ in Ω . So $\sigma(F) = F$. The σ -orbit of c is infinite, since $L_i \cap L_j$ is algebraic over K whenever $i \neq j$.

The p -basis hypothesis implies that L is separable over K . Applying σ^i shows that L_i is separable over K . Moreover, the L_i are algebraically disjoint over K , so their compositum L_∞ is separable over K , by the last corollary in [Bou03, V.§16.7] and Proposition 3(b) in [Bou03, V.§15.2]. Thus $\mathcal{B}$ is p -free in L_∞ .

The p -basis hypothesis implies also that $L = L^p(\mathcal{B})$. Thus $L_i = L_i^p(\mathcal{B})$ for all $i \in \mathbf{Z}$, and $L_\infty = L_\infty^p(\mathcal{B})$.

Combining the previous two paragraphs shows that $\mathcal{B}$ is a p -basis for L_∞ , and hence also for F . $\square$

Lemma 3.13. *Let $K \subseteq F$ be a separable extension such that the field $K^{p^\infty} := \bigcap_{n \geq 1} K^{p^n}$ is algebraically closed. Let A be an abelian variety over K such that no nonzero quotient of $A_{\bar{K}}$ is the base extension of an abelian variety over K^{p^∞} . Then no nonzero quotient of $A_{\bar{F}}$ is the base extension of an abelian variety over F^{p^∞} .*

Proof. Suppose not. Thus there exists a nonzero abelian variety B over F^{p^∞} and a surjective homomorphism $\phi: A_{\bar{F}} \rightarrow B_{\bar{F}}$. Choose a finitely generated extension F_0 of K^{p^∞} over which B is defined. Whenever A and B are abelian varieties over a field L , any homomorphism $A \rightarrow B$ is definable over $L(A[\ell^\infty], B[\ell^\infty])$ for any prime $\ell \neq \text{char } L$; thus, in our situation, ϕ is definable over a finite separable extension F_1 of KF_0 . Since K^{p^∞} is algebraically closed, we may choose a place $F_0 \dashrightarrow K^{p^\infty}$ extending the identity on K^{p^∞} , such that B has good reduction at this place. By [Del98, Fact 1.4], K and F^{p^∞} are linearly disjoint over K^{p^∞} , so the place $F_0 \dashrightarrow K^{p^\infty}$ extends to a place $KF_0 \dashrightarrow K$ that is the identity on K , and then to a place $F_1 \dashrightarrow K_1$ for some finite extension K_1 of K . Reduction of $\phi: A_{F_1} \rightarrow B_{F_1}$ yields

a homomorphism $A_{K_1} \rightarrow B_{K_1}$, and since the place restricted to F_0 has values in K^{p^∞} , the abelian variety B_{K_1} is the base extension of an abelian variety over K^{p^∞} . This contradicts the hypothesis on A . $\square$

Lemma 3.14 (a version of the Mordell-Lang conjecture). *Let F be a separably closed field of characteristic p . Suppose that A is an abelian variety over F such that no nonzero quotient of $A_{\overline{F}}$ is the base extension of an abelian variety over F^{p^∞} . Suppose that X is a coset-free closed F -subvariety of A . Then there exists $e \in \mathbf{Z}_{\geq 1}$ such that for every $a \in A(F)$, the intersection $X(F) \cap (a + p^e A(F))$ is finite.*

Proof. This is a special case of [Hru96, Lemma 6.2]. $\square$

Using Lemma 3.14 and the compactness theorem in model theory, we can deduce a version of the Mordell-Lang conjecture that is more uniform as we vary the ground field, Proposition 3.16.

But first we introduce some notation. Suppose that K is a field of characteristic p such that $[K : K^p] < \infty$. Let F be any field extension with the same p -basis. Suppose that A is an abelian variety over K . Let R_n be the restriction of scalars of A from K to K^{p^n} , so R_n is an abelian variety over K^{p^n} . Let A_n be the base extension of R_n by the isomorphism $K^{p^n} \rightarrow K$ that takes an element to its p^n -th root, so $A_n(F) = R_n(F^{p^n}) = A(F)$. For $n \in \mathbf{Z}_{\geq 0}$, we have a natural morphism $A_{n+1} \rightarrow A_n$ such that $A_{n+1}(F) \rightarrow A_n(F)$ is compatible with the identifications $A_{n+1}(F) = A(F)$ and $A_n(F) = A(F)$.

Suppose moreover that X is a K -variety. Let $\mathcal{Y}_n$ be the set of (not necessarily closed) K -subvarieties $Y \subseteq A_n \times X$ such that the projection from Y to A_n has finite fibers. For such Y , use the identification $A_n(F) = A(F)$ to view $Y(F)$ as a subset of $A(F) \times X(F)$. Taking inverse images under $A_{n+1} \times X \rightarrow A_n \times X$ defines a map of sets $\mathcal{Y}_n \rightarrow \mathcal{Y}_{n+1}$. Let $\mathcal{Y} = \varinjlim \mathcal{Y}_n$. For $Y \in \mathcal{Y}$, the set $Y(F) \subseteq A(F) \times X(F)$ is independent of which $\mathcal{Y}_n$ we consider Y as coming from. Each $\mathcal{Y}_n$ is closed under taking finite unions of elements, so the same is true of $\mathcal{Y}$.

Remark 3.15. Alternatively, one can think of Y as a “variety” defined by not only polynomial equations and inequations in the coordinates on A and X , but by polynomial equations and inequations in the iterated p -components of coordinates on A and usual coordinates on X .

Proposition 3.16 (more uniform version of the Mordell-Lang conjecture). *Let k be an algebraically closed field of characteristic p . Let K be a finitely generated extension of k . Fix a (finite) p -basis $\mathcal{B}$ of K . Suppose that A is an abelian variety over K such that no nonzero quotient of $A_{\overline{K}}$ is the base extension of an abelian variety over k . Suppose that X is a coset-free closed K -subscheme of A . Define $\mathcal{Y}$ as in the preceding paragraph. Then there exists $e \in \mathbf{Z}_{\geq 1}$ and $Y \in \mathcal{Y}$ such that for every field extension $F \supseteq K$ having $\mathcal{B}$ as p -basis, if $a \in A(F)$ and $c \in X(F) \cap (a + p^e A(F))$, then $(a, c) \in Y(F)$.*

Proof. Consider the language of fields augmented by a constant symbol α_κ for each element $\kappa \in K$ and by additional finite tuples of constant symbols a and c (to represent coordinates of points on A and X , respectively). We construct a theory $\mathcal{T}$ in this language. Start with the field axioms, and the arithmetic sentences involving the α_κ that hold in K . Include a sentence saying that $\mathcal{B}$ is a p -basis for the universe field F . Include sentences saying that $a, c \in A(F)$ (for some fixed representation of A as a definable set). For each $e \in \mathbf{Z}_{\geq 1}$, include

a sentence saying that

$$c \in X(F) \cap (a + p^e A(F)).$$

For each $Y \in \mathcal{Y}$, include a sentence saying that $(a, c) \notin Y(F)$; such a sentence may be constructed since iterated p -components admit a first-order definition.

Suppose that the conclusion of Proposition 3.16 fails. Then we claim that $\mathcal{T}$ is consistent. To prove this, we show that every finite subset $\mathcal{T}_0$ of $\mathcal{T}$ has a model. Given $\mathcal{T}_0$, let $e_{\max}$ be the maximum e that occurs in the sentences, and let $Y_{\max} \in \mathcal{Y}$ be the union of the Y 's that occur. The negation of Proposition 3.16 implies that there exists a field extension $F \supseteq K$ having $\mathcal{B}$ as a p -basis, together with $a \in A(F)$ and $c \in X(F) \cap (a + p^{e_{\max}} A(F))$, such that $(a, c) \notin Y_{\max}(F)$. This (F, a, c) is a model for $\mathcal{T}_0$.

By the compactness theorem, there is a model (F, a, c) for $\mathcal{T}$. So F is a field extension of K having $\mathcal{B}$ as a p -basis, and

$$c \in X(F) \cap (a + p^e A(F)).$$

for all $e \in \mathbf{Z}_{\geq 1}$ and for every $Y \in \mathcal{Y}$ we have $(a, c) \notin Y(F)$. Let $\widetilde{K(a)}$ be the smallest extension of $K(a)$ in F that is closed under taking p -components with respect to $\mathcal{B}$, so $\widetilde{K(a)}$ too has $\mathcal{B}$ as p -basis. Then $K(c)$ is not algebraic over $\widetilde{K(a)}$, because an algebraic relation could be used to define a $Y \in \mathcal{Y}$ containing (a, c) . By Lemma 3.12 applied to $F/\widetilde{K(a)}$, it is possible to enlarge F to assume that F is separably closed and the $\text{Aut}(F/\widetilde{K(a)})$ -orbit of c is infinite while $\mathcal{B}$ is still a p -basis for F . This infinite orbit is contained in the set $X(F) \cap (a + p^e A(F))$, so the latter is infinite too, for every $e \in \mathbf{Z}_{\geq 1}$. By Lemma 3.13 applied to $K \subseteq F$, the abelian variety $A_{\overline{F}}$ has no nonzero quotient defined over F^{p^∞} . Also, the fact that X is coset-free implies that X_F is coset-free (if $X_{\overline{K}}$ acquires a coset over some extension of $\overline{K}$, it acquires one over some finitely generated extension of $\overline{K}$, and then a standard specialization argument shows that it contains a coset already over $\overline{K}$). Thus $X_F \subseteq A_F$ is a counterexample to Lemma 3.14. $\square$

Remark 3.17. The proof of Lemma 3.14 involves more model theory than we have used elsewhere in this section. On the other hand, probably the proof of [AV92, Theorem A(3)] could be adapted to give a purely algebraic proof of Lemma 3.14 in the case where A is ordinary, and this would suffice to prove Proposition 3.16 in the case where A is ordinary.

3.4. Subvarieties of abelian varieties. We now return to our situation in which K is the function field of a curve over k .

Lemma 3.18. *Suppose that $A_{\overline{K}}$ has no nonzero isotrivial quotient, and that X is coset-free. Then there exists a finite K -subscheme Z of X such that $X(\mathbf{A}) \cap \overline{A(K)} \subseteq Z(\mathbf{A})$.*

Proof. By replacing K with its compositum with $\overline{k}$ in $\overline{K}$, we may reduce to the case in which k is algebraically closed.

We have $[K : K^p] = p$. Choose $\alpha \in K - K^p$. Then $\{\alpha\}$ is a p -basis for K . For any v , the field K_v is generated by K and K_v^p , so $[K_v : K_v^p] \leq p$. Moreover, Lemma 3.1 implies $\alpha \notin K_v^p$, so $\{\alpha\}$ is a p -basis for K_v .

Let e and Y be as in Proposition 3.16. We think of Y as a subvariety of $A_n \times X$ for some n . For each $a \in A(K)$, let Y_a be the fiber of $Y \rightarrow A_n$ above the point of $A_n(K)$ corresponding to a . View Y_a as a finite subscheme of X .

Since $A_{\overline{K}}$ has no nonzero isotrivial quotient, $A(K)$ is finitely generated. Choose a (finite) set of representatives $\mathcal{A} \subseteq A(K)$ for $A(K)/p^e A(K)$. Let $Z = \bigcup_{a \in \mathcal{A}} Y_a$, so Z is a finite subscheme of X . The conclusion of Proposition 3.16 applied to $F = K_v$ says that

$$X(K_v) \cap (a + p^e A(K_v)) \subseteq Y_a(K_v)$$

for each $a \in \mathcal{A}$. Let Γ_v be the closure of $A(K)$ in $A(K_v)$. Then

$$X(K_v) \cap (a + p^e \Gamma_v) \subseteq Y_a(K_v).$$

Taking the union over $a \in \mathcal{A}$ yields

$$X(K_v) \cap \Gamma_v \subseteq Z(K_v),$$

since by Lemma 3.10, the $a \in \mathcal{A}$ represent all classes in $\Gamma_v/p^e \Gamma_v$. This holds for all v , so $X(\mathbf{A}) \cap \overline{A(K)} \subseteq Z(\mathbf{A})$. $\square$

Remark 3.19. In the number field case, one cannot expect finiteness of $X(K_v) \cap \Gamma_v$ in general. Consider the case in which $K_v = \mathbf{Q}_p$. If the “Chabauty condition” $\text{rank } A(K) < \dim A$ fails, it may happen that Γ_v is an open subgroup of $A(K_v)$, in which case $X(K_v) \cap \Gamma_v$ will be infinite if nonempty.

3.5. End of proof of Theorem B. We now prove Theorem B. Let Z be as in Lemma 3.18. By Lemma 3.18 and then Proposition 3.9,

$$X(\mathbf{A}) \cap \overline{A(K)} \subseteq Z(\mathbf{A}) \cap \overline{A(K)} = Z(K) \subseteq X(K).$$

The opposite inclusion $X(K) \subseteq X(\mathbf{A}) \cap \overline{A(K)}$ is trivial. This completes the proof.

Remark 3.20. Our proof of Theorem B required both the flat information (from residue fields) implicit in Proposition 3.6 and the deep information (from each K_v) in Proposition 3.5. But it seems possible that the flat information suffices, even for Conjecture C. Specifically, if Ω is a cofinite subset of the set of places of good reduction for X/K , then it is possible that $\overline{X(K)} = X(\prod_{v \in \Omega} \mathbf{F}_v) \cap \overline{A(K)}$ always holds, where the closures are now taken in $A(\prod_{v \in \Omega} \mathbf{F}_v)$.

4. THE BRAUER-MANIN OBSTRUCTION

In this section we assume that K is either a global function field (i.e., as before, but with k finite) or a number field. We assume also that Ω is the set of all nontrivial places of K , and define the ring of adèles $\mathbf{A}$ in the usual way.

The purpose of this section is to relate the adelic intersections we have been considering to the Brauer-Manin obstruction. This relationship was discovered by Scharaschkin [Sch99] in the number field case. We will give a proof similar in spirit to his, and verify that it works in the function field case.

For convenience, if X is any topological space, let $X_{\bullet}$ denote the set of connected components, and equip $X_{\bullet}$ with the quotient topology: this will be needed to avoid annoyances with the archimedean places in the number field case.

Let A be an abelian variety over K . By the adelic topology on $A(K)$, we mean the topology induced by $A(\mathbf{A})_{\bullet}$; in the function field case this agrees with our terminology in Section 1.

Proposition 4.1. *The adelic topology on $A(K)$ equals the topology induced by subgroups of finite index.*

Proof. Proposition 3.6 gives one inclusion. The other follows from the fact that $A(\mathbf{A})_\bullet = \prod_{v \in \Omega} A(K_v)_\bullet$ is a profinite group. $\square$

4.1. Global duality for abelian varieties. All cohomology below is fppf cohomology. As usual, for any $n \in \mathbf{Z}_{\geq 1}$, define

$$\begin{aligned} \text{Sel}^n &:= \ker \left(H^1(K, A[n]) \rightarrow \prod_{v \in \Omega} H^1(K_v, A) \right) \\ \text{III} &:= \ker \left(H^1(K, A) \rightarrow \prod_{v \in \Omega} H^1(K_v, A) \right). \end{aligned}$$

Following the notation of [Sto07, §2], define

$$\begin{aligned} \widehat{A(K)} &:= \varprojlim \frac{A(K)}{nA(K)} \\ \widehat{\text{Sel}} &:= \varprojlim \text{Sel}^n \\ T\text{III} &:= \varprojlim \text{III}[n], \end{aligned}$$

where each inverse limit is over positive integers n ordered by divisibility. Taking inverse limits of the usual descent sequence yields

$$(1) \quad 0 \rightarrow \widehat{A(K)} \rightarrow \widehat{\text{Sel}} \rightarrow T\text{III} \rightarrow 0.$$

For any abelian profinite group G and any prime ℓ , let $G^{(\ell)}$ be the maximal pro- ℓ quotient of G , so $G \cong \prod_\ell G^{(\ell)} \cong \varprojlim G/nG$, where ℓ ranges over primes, and n ranges over positive integers ordered by divisibility.

Lemma 4.2. *For each prime ℓ , each of $\widehat{A(K)}^{(\ell)}$, $\widehat{\text{Sel}}^{(\ell)}$, $T\text{III}^{(\ell)}$ is a finitely generated $\mathbf{Z}_\ell$ -module.*

Proof. For $\widehat{A(K)}^{(\ell)}$ it holds simply because $A(K)$ is finitely generated as a $\mathbf{Z}$ -module. For $T\text{III}^{(\ell)}$, it follows from the finiteness of $\text{III}[\ell]$, which is proved in [Mil70]. Now the result for $\widehat{\text{Sel}}^{(\ell)}$ follows from (1). $\square$

By definition, Sel^n maps to

$$\ker \left(\prod_{v \in \Omega} H^1(K_v, A[n]) \rightarrow \prod_{v \in \Omega} H^1(K_v, A) \right) = \prod_{v \in \Omega} \frac{A(K_v)}{nA(K_v)},$$

so as in [Sto07, §2] we obtain a map

$$\widehat{\text{Sel}} \rightarrow A(\mathbf{A})_\bullet.$$

For each v we have a pairing

$$A(K_v)_\bullet \times H^1(K_v, A^\vee) \rightarrow H^2(K_v, \mathbf{G}_m) \cong \mathbf{Q}/\mathbf{Z},$$

and in fact Tate local duality holds: each of the first two groups is identified with the Pontryagin dual of the other [Mil06, Theorem III.7.8]. An element of $H^1(K, A^\vee)$ maps to 0

in $H^1(K_v, A^\vee)$ for all but finitely many v , so summing the Tate local duality pairing over v defines a pairing

$$A(\mathbf{A})_\bullet \times H^1(K, A^\vee) \rightarrow \mathbf{Q}/\mathbf{Z},$$

or equivalently a homomorphism

$$A(\mathbf{A})_\bullet \xrightarrow{\text{Tate}} H^1(K, A^\vee)^D,$$

where the superscript D denotes Pontryagin dual.

The following global duality statement is a version of the ‘‘Cassels dual exact sequence’’ in which finiteness of **III** is not assumed:

Proposition 4.3 (Cassels dual exact sequence). *The sequence*

$$0 \longrightarrow \widehat{\text{Sel}} \longrightarrow A(\mathbf{A})_\bullet \xrightarrow{\text{Tate}} H^1(K, A^\vee)^D$$

is exact.

Proof. This is part of the main theorem of [GAT06]. □

Remark 4.4. Proposition 4.3 has a long history. For an elliptic curve A over a number field, Proposition 4.3 was proved by Cassels [Cas62, Cas64] under the assumption that **III**(A) is finite, using a result from [Ser64] to obtain exactness on the left. This, except for the exactness on the left, was generalized by Tate to abelian varieties over number fields with finite Shafarevich-Tate group, at around the same time, though the first published proof appeared much later [Mil06, Theorem I.6.13]. The latter reference also proved the global function field analogue except for the p -part in characteristic p . The $\widehat{\text{Sel}}$ version (not requiring finiteness of **III**(A)), except for exactness on the left and the p -part, is implicit in the right three terms of the exact sequence in the middle of page 104 of [Mil06] if there we make the substitution

$$T\text{III} \cong \widehat{\text{Sel}}/\widehat{A(K)}$$

from (1). Exactness on the left for the prime-to- p part is implicit in [Ser64, Ser71] and is mentioned explicitly in [Mil06, Corollary 6.23]. The full statement was proved in [GAT06].

4.2. Review of the Brauer-Manin obstruction. Let X be a projective K -scheme. Let $X_s := X \times_K K^s$. Define $\text{Br } X := H^2(X, \mathbf{G}_m)$, where here we may use étale cohomology since it gives the same result as fppf cohomology for smooth quasi-projective commutative group schemes [Mil80, Theorem III.3.9]. For simplicity, let $\frac{\text{Br } X}{\text{Br } K}$ denote the cokernel of $\text{Br } K \rightarrow \text{Br } X$ even if this map is not injective. The Hochschild-Serre spectral sequence in étale cohomology gives an exact sequence

$$\text{Br } K \rightarrow \ker(\text{Br } X \rightarrow \text{Br } X_s) \rightarrow H^1(K, \text{Pic } X_s) \rightarrow H^3(K, \mathbf{G}_m),$$

and the last term is 0 for any global field: see [Mil06, Corollary 4.21], for instance. From this we extract an injection

$$H^1(K, \text{Pic } X_s) \hookrightarrow \frac{\text{Br } X}{\text{Br } K};$$

if $\text{Br } X_s = 0$, then this is an isomorphism. Composing with the map induced by $\text{Pic}^0 X_s \hookrightarrow \text{Pic } X_s$, we obtain

$$(2) \quad H^1(K, \text{Pic}^0 X_s) \rightarrow \frac{\text{Br } X}{\text{Br } K}.$$

Each $A \in \text{Br } X$ induces evaluation maps ev_A fitting in a commutative diagram

$$\begin{array}{ccccccc} X(K) & \longrightarrow & X(\mathbf{A})_{\bullet} & & & & \\ \text{ev}_A \downarrow & & \text{ev}_A \downarrow & & \searrow & & \\ 0 \longrightarrow & \text{Br } K & \longrightarrow & \bigoplus_{v \in \Omega} \text{Br}(K_v) & \longrightarrow & \mathbf{Q}/\mathbf{Z} & \longrightarrow 0. \end{array}$$

The diagonal arrows give rise to a pairing

$$X(\mathbf{A})_{\bullet} \times \frac{\text{Br } X}{\text{Br } K} \rightarrow \mathbf{Q}/\mathbf{Z}$$

which induces a map

$$X(\mathbf{A})_{\bullet} \xrightarrow{\text{BM}} \left(\frac{\text{Br } X}{\text{Br } K} \right)^D.$$

The kernel (inverse image of 0) is called the *Brauer set*, and is denoted $X(\mathbf{A})_{\bullet}^{\text{Br}}$: it contains $X(K)$ and hence also its closure $\overline{X(K)}$ in $X(\mathbf{A})_{\bullet}$. One says that the Brauer-Manin obstruction to the Hasse principle is the only one for X if the implication

$$X(\mathbf{A})_{\bullet}^{\text{Br}} \neq \emptyset \implies X(K) \neq \emptyset$$

holds. (This is equivalent to the usual notion without the $\bullet$.)

4.3. The Brauer set for abelian varieties.

Theorem E. *We have*

$$\widehat{A(K)} = \overline{A(K)} \subseteq A(\mathbf{A})_{\bullet}^{\text{Br}} \subseteq \widehat{\text{Sel.}}$$

in $A(\mathbf{A})$.

Proof. Proposition 4.1 shows that $A(K) \rightarrow A(\mathbf{A})_{\bullet}$ induces an isomorphism $\widehat{A(K)} \rightarrow \overline{A(K)}$, and the first inclusion is automatic from the previous subsection.

So it remains to prove the second inclusion. Taking the dual of (2) for A yields the vertical arrow in

$$\begin{array}{ccc} 0 \longrightarrow \widehat{\text{Sel}} \longrightarrow A(\mathbf{A})_{\bullet} & \xrightarrow{\text{Tate}} & H^1(K, \text{Pic}^0 A_s)^D \\ & \searrow \text{BM} & \uparrow \\ & & \left(\frac{\text{Br } A}{\text{Br } K} \right)^D \end{array}$$

Applying [Man71, Proposition 8] to each K_v shows that the triangle commutes (the hypothesis there that the ground field be perfect is not used). The diagram now gives $\ker(\text{BM}) \subseteq \ker(\text{Tate})$, or equivalently $A(\mathbf{A})_{\bullet}^{\text{Br}} \subseteq \widehat{\text{Sel.}}$ $\square$

Remark 4.5. If III is finite (or more generally, if its maximal divisible part III_{div} is 0), then $T\text{III} = 0$ and the injection $\widehat{A(K)} \rightarrow \widehat{\text{Sel}}$ is an isomorphism; in this case, the conclusion of Theorem E simplifies to

$$\widehat{A(K)} = \overline{A(K)} = A(\mathbf{A})_{\bullet}^{\text{Br}} = \widehat{\text{Sel.}}$$

Closely related results in the number field case can be found in [Wan96].

4.4. The Brauer set for subvarieties of abelian varieties.

Proposition 4.6. *Let $i: X \rightarrow A$ be a morphism from a projective K -scheme to an abelian variety over K . So i induces a map $X(\mathbf{A})_{\bullet} \rightarrow A(\mathbf{A})_{\bullet}$.*

- (a) *We have $i(X(\mathbf{A})_{\bullet}^{\text{Br}}) \subseteq \widehat{\text{Sel}}$.*
- (b) *Suppose that*
 - *X is a smooth projective geometrically integral curve,*
 - *A is the Jacobian of X ,*
 - *i is an Albanese map (that is, i sends a point P to the class of $P - D$, where $D \in \text{Div } A_s$ is a fixed divisor of degree 1 whose class is $\text{Gal}(K^s/K)$ -invariant), and*
 - *III is finite (or at least $\text{III}_{\text{div}} = 0$).*

Then $X(\mathbf{A})_{\bullet}^{\text{Br}} = i^{-1}(\overline{A(K)})$.

Proof. (a) This follows immediately from Theorem E, since i maps $X(\mathbf{A})_{\bullet}^{\text{Br}}$ to $A(\mathbf{A})_{\bullet}^{\text{Br}}$.

(b) We use the diagram

$$\begin{array}{ccccccc}
 & & & & \left(\frac{\text{Br } X}{\text{Br } K} \right)^D & & \\
 & & & & \downarrow & & \\
 & & \text{BM}_X & & H^1(K, \text{Pic } X_s)^D & & \\
 & & \nearrow & & \downarrow & & \\
 X(\mathbf{A})_{\bullet} & & & & H^1(K, \text{Pic}^0 X_s)^D & & \\
 \downarrow i & & & & \downarrow & & \\
 0 \longrightarrow \overline{A(K)} \longrightarrow A(\mathbf{A})_{\bullet} & \xrightarrow{\text{Tate}} & H^1(K, \text{Pic}^0 A_s)^D & & & & \\
 & \searrow \text{BM}_A & \uparrow & & & & \\
 & & \left(\frac{\text{Br } A}{\text{Br } K} \right)^D & & & &
 \end{array}$$

in which the horizontal sequence is the exact sequence of Proposition 4.3 in which we have used the finiteness of III to replace $\widehat{\text{Sel}}$ with $\overline{A(K)}$. The lower triangle commutes: as mentioned earlier, this follows from a result of Manin. The “pentagon” at the far right also commutes, since the homomorphism (2) is functorial in X . Thus the whole diagram commutes.

We next claim that the three downward vertical arrows at the right are isomorphisms. The first is an isomorphism because $\text{Br } X_s = 0$ [Gro68, Corollaire 5.8]. The second is an isomorphism because $\text{Pic } X_s \cong \text{Pic}^0 X_s \oplus \mathbf{Z}$ (where the $\mathbf{Z}$ is generated by the class of D) and $H^1(K, \mathbf{Z}) = 0$. The third is an isomorphism because an Albanese map i induces an isomorphism $i^*: \text{Pic}^0 A_s \rightarrow \text{Pic}^0 X_s$.

The commutativity of the upper left “hexagon” now implies

$$X(\mathbf{A})_{\bullet}^{\text{Br}} := \ker(\text{BM}_X) = \ker(\text{Tate} \circ i) = i^{-1}(\ker(\text{Tate})) = i^{-1}(\overline{A(K)}).$$

□

Remark 4.7. Part (b) was originally proved in the number field case in [Sch99] using a slightly different proof. For yet another proof of this case, see [Sto07, Corollary 7.4].

Remark 4.8. If K is a global function field, the conclusion of (b) can be written as $X(\mathbf{A})^{\text{Br}} = X(\mathbf{A}) \cap \overline{A(K)}$ in $A(\mathbf{A})$.

5. INTERSECTIONS WITH $\widehat{\text{Sel}}$

From now on, we assume that K is a global function field.

The proof of Theorem D will follow that of Theorem B, with $\widehat{\text{Sel}}$ playing the role of $\overline{A(K)}$. We begin by proving $\widehat{\text{Sel}}$ -versions of several of the lemmas and propositions of Section 3.2. The following is an analogue of Lemma 3.7.

Lemma 5.1. *The maps $A(K)_{\text{tors}} \rightarrow \widehat{A(K)}_{\text{tors}} \rightarrow \widehat{\text{Sel}}_{\text{tors}}$ are isomorphisms.*

Proof. Proposition 4.1 yields $\widehat{A(K)} \cong \overline{A(K)}$, so the first map is an isomorphism by Lemma 3.7. The second map is an isomorphism by (1), since $TIII$ is torsion-free by definition. □

The following is an analogue of Proposition 3.5.

Proposition 5.2. *If $A(K^s)[p^\infty]$ is finite, then for any v , the map $\widehat{\text{Sel}}^{(p)} \rightarrow A(K_v)^{(p)}$ is injective.*

Proof. Let $K'_v \subseteq K^s$ be the Henselization of K at v . Define $\text{Sel}^{n'}$ and $\widehat{\text{Sel}}'$ in the same way as Sel^n and $\widehat{\text{Sel}}$, but using K'_v in place of its completion K_v . By [Mil06, I.3.10(a)(ii)], the natural maps $\text{Sel}^{n'} \rightarrow \text{Sel}^n$ and $\widehat{\text{Sel}}' \rightarrow \widehat{\text{Sel}}$ are isomorphisms. Similarly, by [Mil06, I.3.10(a)(i)] we may replace $A(K_v)^{(p)}$ by $A(K'_v)^{(p)} := \varprojlim A(K'_v)/p^n A(K'_v)$. Hence it suffices to prove injectivity of $(\widehat{\text{Sel}}')^{(p)} \rightarrow A(K'_v)^{(p)}$.

Choose m such that $p^m A(K^s)[p^\infty] = 0$. Suppose $b \in \ker \left((\widehat{\text{Sel}}')^{(p)} \rightarrow A(K'_v)^{(p)} \right)$. For each $M \in \mathbf{Z}_{\geq 0}$, let b_M be the image of b in $\text{Sel}^{p^M} \subseteq H^1(K, A[p^M])$. Then the image of b_M under

$$\text{Sel}^{p^M} \rightarrow \frac{A(K'_v)}{p^M A(K'_v)} \subseteq H^1(K'_v, A[p^M]) \rightarrow H^1(K^s, A[p^M])$$

is 0. The Hochschild-Serre spectral sequence (see [Mil80, III.2.21]) gives an exact sequence

$$0 \rightarrow H^1(\text{Gal}(K^s/K), A(K^s)[p^M]) \rightarrow H^1(K, A[p^M]) \rightarrow H^1(K^s, A[p^M])$$

so b_M comes from an element of $H^1(\text{Gal}(K^s/K), A(K^s)[p^M])$, which is killed by p^m . Thus $p^m b_M = 0$ for all M , so $p^m b = 0$.

By Lemma 5.1, b comes from a point in $A(K)[p^\infty]$, which injects into $A(K'_v)^{(p)}$. □

The next result is an analogue of Proposition 3.9.

Proposition 5.3. *Suppose that $A(K^s)[p^\infty]$ is finite. Let Z be a finite K -subscheme of A . Then $Z(\mathbf{A}) \cap \widehat{\text{Sel}} = Z(K)$.*

Proof. One inclusion is easy: $Z(K) \subseteq Z(\mathbf{A})$ and $Z(K) \subseteq A(K) \subseteq \widehat{A(K)} \subseteq \widehat{\text{Sel}}$. So we focus on the other inclusion.

As in the proof of Proposition 3.9, we may replace K by a finite extension to assume that Z consists of a finite set of K -points of A . Suppose $P \in Z(\mathbf{A}) \cap \widehat{\text{Sel}}$. The v -adic component of P in $Z(K_v)$ equals the image of a point $Q_v \in Z(K)$. Then $P - Q_v$ maps to 0 in $A(K_v)$, and in particular in $A(K_v)^{(p)}$, so by Proposition 5.2 the image of $P - Q_v$ in $\widehat{\text{Sel}}^{(p)}$ is 0. This holds for every v , so if v' is another place, then $Q_{v'} - Q_v$ maps to 0 in $\widehat{\text{Sel}}^{(p)}$. The kernel of $A(K) \rightarrow \widehat{A(K)}^{(p)} \hookrightarrow \widehat{\text{Sel}}^{(p)}$ is the prime-to- p torsion of $A(K)$, so $Q_{v'} - Q_v \in A(K)_{\text{tors}}$. This holds for all v' , so the point $R := P - Q_v$ belongs to $\widehat{\text{Sel}}_{\text{tors}}$. By Lemma 5.1, $R \in A(K)_{\text{tors}}$. Thus $P = R + Q_v \in A(K)$. So $P \in Z(\mathbf{A}) \cap A(K) = Z(K)$. $\square$

Each $\tau \in \text{Sel}^{p^e}$ may be represented by a “covering space”: an fppf torsor T under A equipped with a morphism $\phi_T: T \rightarrow A$ that after base extension to K^s becomes isomorphic to the base extension of $[p^e]: A \rightarrow A$. (Passing to K^s is enough to trivialize T since A is smooth.)

Lemma 5.4. *We have $\widehat{\text{Sel}} \subseteq \bigcup_{\tau \in \text{Sel}^{p^e}} \phi_T(T(\mathbf{A}))$.*

Proof. In the commutative diagram

$$\begin{array}{ccc} & & T(\mathbf{A}) \\ & & \downarrow \phi_T \\ \widehat{\text{Sel}} & \xrightarrow{\quad} & A(\mathbf{A}) \\ \downarrow & & \downarrow \\ \text{Sel}^{p^e} & \xrightarrow{\quad} & \frac{A(\mathbf{A})}{p^e A(\mathbf{A})} \end{array}$$

the set $\phi_T(T(\mathbf{A}))$ is a coset of $p^e A(\mathbf{A})$ in $A(\mathbf{A})$, and its image in $\frac{A(\mathbf{A})}{p^e A(\mathbf{A})}$ equals the image of τ under the bottom horizontal map, by the definition of this bottom map. Thus any element of $\widehat{\text{Sel}}$ mapping to τ in Sel^{p^e} belongs to the corresponding $\phi_T(T(\mathbf{A}))$. $\square$

Proof of Theorem D. We have $X(K) \subseteq X(\mathbf{A})^{\text{Br}} \subseteq X(\mathbf{A}) \cap \widehat{\text{Sel}}$, by Proposition 4.6(a), so it will suffice to show that $X(\mathbf{A}) \cap \widehat{\text{Sel}}$ consists of K -rational points.

Let e and Y be as in Proposition 3.16. For each $\tau \in \text{Sel}^{p^e}$, choose $\phi_T: T \rightarrow A$ as above, choose $t \in T(K^s)$, let $a = \phi_T(t) \in A(K^s)$, and let Y_a be the fiber of $Y \rightarrow A_n$ above the point corresponding to a . View Y_a as a finite subscheme of X_{K^s} . It follows from [Mil70] that Sel^{p^e} is finite, so we can choose a finite K -subscheme Z of X such that Z_{K^s} contains all the Y_a as τ ranges over Sel^{p^e} .

For each v , choose a separable closure K_v^s of K_v containing K^s . By the proof of Lemma 3.18, any p -basis of K is also a p -basis for K_v , and hence for K_v^s , so the conclusion of Proposition 3.16 may be applied with $F = K_v^s$ to yield

$$X(K_v^s) \cap (a + p^e A(K_v^s)) \subseteq Y_a(K_v^s)$$

for each a . By definition of ϕ_T , we have $\phi_T(T(K_v^s)) = a + p^e A(K_v^s)$. Thus

$$X(K_v) \cap \phi_T(T(K_v)) \subseteq X(K_v^s) \cap (a + p^e A(K_v^s)) \subseteq Y_a(K_v^s) \subseteq Z(K_v^s).$$

Hence

$$X(K_v) \cap \phi_T(T(K_v)) \subseteq X(K_v) \cap Z(K_v^s) = Z(K_v).$$

This holds for all v , so $X(\mathbf{A}) \cap \phi_T(T(\mathbf{A})) \subseteq Z(\mathbf{A})$. Taking the union over $\tau \in \text{Sel}^{p^e}$, and applying Lemma 5.4, we obtain $X(\mathbf{A}) \cap \widehat{\text{Sel}} \subseteq Z(\mathbf{A})$. Thus $X(\mathbf{A}) \cap \widehat{\text{Sel}} \subseteq Z(\mathbf{A}) \cap \widehat{\text{Sel}}$, and the latter equals $Z(K)$ by Proposition 5.3, so we are done. $\square$

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