

A BRIEF SUMMARY OF THE STATEMENTS OF CLASS FIELD THEORY

BJORN POONEN

0. PROFINITE COMPLETIONS OF TOPOLOGICAL GROUPS

Let G be a topological group. The profinite completion of G is

$$\widehat{G} := \varprojlim_U \frac{G}{U},$$

where U ranges over the finite-index open normal subgroups of G . There is a natural continuous homomorphism $G \rightarrow \widehat{G}$ through which every other continuous homomorphism from G to a profinite group factors uniquely. If G is profinite already, then $G \rightarrow \widehat{G}$ is an isomorphism.

In general, $G \rightarrow \widehat{G}$ need not be injective or surjective. Nevertheless, we think of G as being almost isomorphic to $\widehat{G}$: The finite-index open subgroups of G are in bijection with those of $\widehat{G}$. And finite-index open subgroups of certain Galois groups are what we are interested in. . .

1. LOCAL CLASS FIELD THEORY

1.1. Notation associated to a discrete valuation ring.

$\mathcal{O}$: a complete discrete valuation ring

$K := \text{Frac}(\mathcal{O})$

v : the valuation $K^\times \rightarrow \mathbb{Z}$

$\mathfrak{p}$: the maximal ideal of $\mathcal{O}$

k : the residue field $\mathcal{O}/\mathfrak{p}$

K^s : a fixed separable closure of K

K^{ab} : the maximal abelian extension of K in K^s

K^{unr} : the maximal unramified extension of K in K^s

k^s : the residue field of K^{unr} , so k^s is a separable closure of k .

Equip K and its subsets with the topology coming from the absolute value $|x| := \exp(-v(x))$.

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1.2. Local fields.

Definition 1.1. A nonarchimedean local field is a complete discrete-valued field K as in Section 1.1 such that the residue field k is finite. An archimedean local field is $\mathbb{R}$ or $\mathbb{C}$.

Facts:

- A nonarchimedean local field of characteristic 0 is isomorphic to a finite extension of $\mathbb{Q}_p$.
- A (nonarchimedean) local field of characteristic $p > 0$ is isomorphic to $\mathbb{F}_q((t))$ for some power q of p .

1.3. The local Artin homomorphism. Let K be a local field. Local class field theory says that there is a homomorphism

$$\theta: K^\times \rightarrow \text{Gal}(K^{\text{ab}}/K)$$

that is almost an isomorphism. The homomorphism θ is called the **local Artin homomorphism**. It cannot be literally an isomorphism, because $\text{Gal}(K^{\text{ab}}/K)$ is a profinite group, hence compact, while $K^\times$ is not. What is true is that θ induces an isomorphism of topological groups $\widehat{K^\times} \rightarrow \text{Gal}(K^{\text{ab}}/K)$.

If K is archimedean, then $\theta: K^\times \rightarrow \text{Gal}(K^{\text{ab}}/K)$ is surjective and its kernel is the connected component of the identity in $K^\times$.

For the rest of Section 1.3, we assume that K is nonarchimedean. Then θ is injective: The choice of a uniformizer $\pi \in \mathcal{O}$ lets us write $K^\times = \mathcal{O}^\times \pi^\mathbb{Z} \simeq \mathcal{O}^\times \times \mathbb{Z}$, and $\mathcal{O}^\times$ is already profinite, so $\widehat{K^\times} \simeq \mathcal{O}^\times \times \hat{\mathbb{Z}}$. Thus local class field theory says that there is an isomorphism

$$\mathcal{O}^\times \times \hat{\mathbb{Z}} \rightarrow \text{Gal}(K^{\text{ab}}/K).$$

More canonically, without choosing π , the two horizontal exact sequences below are almost isomorphic:

$$(1) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}^\times & \longrightarrow & K^\times & \xrightarrow{v} & \mathbb{Z} \longrightarrow 0 \\ & & \parallel & & \downarrow \theta & & \downarrow \\ 0 & \longrightarrow & \text{Gal}(K^{\text{ab}}/K^{\text{unr}}) & \longrightarrow & \text{Gal}(K^{\text{ab}}/K) & \xrightarrow{\text{res}} & \text{Gal}(K^{\text{unr}}/K) \longrightarrow 0 \end{array}$$

With the identification of the group at lower right

$$\text{Gal}(K^{\text{unr}}/K) \simeq \text{Gal}(k^s/k) \simeq \hat{\mathbb{Z}}$$

mapping the Frobenius automorphism to $1 \in \hat{\mathbb{Z}}$, the right vertical map in (1) becomes the natural inclusion $\mathbb{Z} \hookrightarrow \hat{\mathbb{Z}}$. In other words, θ maps $K^\times$ isomorphically to the set of $\sigma \in \text{Gal}(K^{\text{ab}}/K)$ inducing an *integer* power of Frobenius on the residue field (as opposed to a $\hat{\mathbb{Z}}$ -power). The bottom row of (1) is simply the profinite completion of the top row.

Also from (1), one sees that $\theta(\mathcal{O}^\times)$ is the inertia subgroup $\text{Gal}(K^{\text{ab}}/K^{\text{unr}})$ of $\text{Gal}(K^{\text{ab}}/K)$, and that θ maps any uniformizer to a Frobenius automorphism in $\text{Gal}(K^{\text{ab}}/K)$. Moreover, the descending chain

$$\mathcal{O}^\times \supset 1 + \mathfrak{p} \supset 1 + \mathfrak{p}^2 \supset \dots$$

is mapped isomorphically by θ to the descending chain of ramification subgroups of $\text{Gal}(K^{\text{ab}}/K)$ in the upper numbering.

1.4. Functoriality. Let L be a finite extension of K . Let $N_{L/K}: L^\times \rightarrow K^\times$ be the norm map. Let θ_L, θ_K be the local Artin homomorphisms associated to L, K , respectively. Let $\text{res}: \text{Gal}(L^{\text{ab}}/L) \rightarrow \text{Gal}(K^{\text{ab}}/K)$ be the homomorphism mapping an automorphism σ of L^{ab} to its restriction $\sigma|_{K^{\text{ab}}}$. Then the square

$$\begin{array}{ccc} L^\times & \xrightarrow{\theta_L} & \text{Gal}(L^{\text{ab}}/L) \\ N_{L/K} \downarrow & & \downarrow \text{res} \\ K^\times & \xrightarrow{\theta_K} & \text{Gal}(K^{\text{ab}}/K) \end{array}$$

commutes.

1.5. Finite abelian extensions. Because θ is almost an isomorphism, and because of Galois theory, the following sets are in bijection:

- The finite-index open subgroups of $K^\times$.
- The (finite-index) open subgroups of $\text{Gal}(K^{\text{ab}}/K)$.
- The finite abelian extensions of K contained in K^{s} .

Going backwards, if L is a finite abelian extension of K in K^{s} , the corresponding subgroup of $K^\times$ is $N_{L/K}L^\times$. (This could be viewed as a consequence of the functoriality above.)

The composition

$$K^\times \rightarrow \text{Gal}(K^{\text{ab}}/K) \xrightarrow{\text{res}} \text{Gal}(L/K)$$

is surjective with kernel $N_{L/K}L^\times$, and $\mathcal{O}^\times$ maps to the inertia subgroup $I_{L/K} \trianglelefteq \text{Gal}(L/K)$, and any uniformizer π maps to a Frobenius element of $\text{Gal}(L/K)$.

2. GLOBAL CLASS FIELD THEORY (VIA IDELES)

2.1. Global fields.

Definition 2.1. A number field is a finite extension of $\mathbb{Q}$. A global function field is a finite extension of $\mathbb{F}_p(t)$ for some prime p , or equivalently is the function field of a geometrically integral curve over a finite field $\mathbb{F}_q$ (called the **constant field**), where q is a power of some prime p . A global field is a number field or a global function field.

Throughout Sections 2 and 3, K is a global field. If v is a nontrivial place of K (given by an absolute value on K), then the completion K_v is a local field. If v is nonarchimedean, let $\mathcal{O}_v$ be the valuation subring of K_v ; if v is archimedean, let $\mathcal{O}_v = K_v$.

2.2. The adèle ring. The adèle ring of K is the restricted direct product

$$\mathbf{A} = \mathbf{A}_K := \prod'_v (K_v, \mathcal{O}_v) := \left\{ (a_v) \in \prod_v K_v : a_v \in \mathcal{O}_v \text{ for all but finitely many } v \right\}.$$

It is a topological ring: the topology is uniquely characterized by the condition that $\prod_v \mathcal{O}_v$ is open in $\mathbf{A}$ and has the product topology. The diagonal map $K \rightarrow \mathbf{A}$ is like $\mathbb{Z} \rightarrow \mathbb{R}$: it embeds K as a discrete co-compact subgroup of $\mathbf{A}$.

2.3. The idele group and idele class group. The idele group of K is

$$\mathbf{A}^\times = \prod'_v (K_v^\times, \mathcal{O}_v^\times) := \left\{ (a_v) \in \prod_v K_v^\times : a_v \in \mathcal{O}_v^\times \text{ for all but finitely many } v \right\}.$$

It is a topological group: the topology is uniquely characterized by the condition that $\prod_v \mathcal{O}_v^\times$ is open in $\mathbf{A}^\times$ and has the product topology.¹ The diagonal map $K^\times \rightarrow \mathbf{A}^\times$ is like $\mathbb{Z}^\times \rightarrow \mathbb{R}^\times$: it embeds $K^\times$ as a discrete subgroup of $\mathbf{A}^\times$, but the quotient $C = C_K := \mathbf{A}^\times / K^\times$ is not compact. The topological group C is called the **idele class group**.

2.4. The global Artin homomorphism. Let K^s be a fixed separable closure of K . Let K^{ab} be the maximal abelian extension of K contained in K^s .

The group C plays the role in global class field theory played by $K^\times$ in local class field theory. Namely, if K is a global field, there is a **global Artin homomorphism**

$$\theta: C \rightarrow \text{Gal}(K^{\text{ab}}/K)$$

that induces an isomorphism $\widehat{C} \xrightarrow{\sim} \text{Gal}(K^{\text{ab}}/K)$.

If K is a number field, then θ is surjective and its kernel is the connected component of the identity in C .

If K is a global function field with constant field k , then θ is injective and $\theta(C)$ equals the set of $\sigma \in \text{Gal}(K^{\text{ab}}/K)$ whose restriction in $\text{Gal}(k^s/k)$ is an *integer* power of the Frobenius generator.

2.5. Functoriality. Let L be a finite extension of K of degree n . Then $\mathbf{A}_L \simeq \mathbf{A}_K \otimes_K L$ is free of rank n over $\mathbf{A}_K$, so there is a norm map $N_{L/K}: \mathbf{A}_L \rightarrow \mathbf{A}_K$. We write $N_{L/K}$ also for the induced homomorphism $N_{L/K}: C_L \rightarrow C_K$. Then

$$\begin{array}{ccc} C_L & \xrightarrow{\theta_L} & \text{Gal}(L^{\text{ab}}/L) \\ N_{L/K} \downarrow & & \downarrow \text{res} \\ C_K & \xrightarrow{\theta_K} & \text{Gal}(K^{\text{ab}}/K) \end{array}$$

commutes.

2.6. Finite abelian extensions. The following sets are in bijection:

- The finite-index open subgroups of C .
- The finite-index open subgroups of $\text{Gal}(K^{\text{ab}}/K)$.
- The finite abelian extensions of K contained in K^s .

Going backwards, if L is a finite abelian extension of K in K^s , the corresponding subgroup of C is $N_{L/K}C_L$. The composition

$$C \rightarrow \text{Gal}(K^{\text{ab}}/K) \xrightarrow{\text{res}} \text{Gal}(L/K)$$

is surjective with kernel $N_{L/K}C_L$.

¹Alternatively, one can use the general recipe for getting the topology on the units of a topological ring R : not the subspace topology on $R^\times$ as a subset of R (this may fail to make the inverse map $R^\times \rightarrow R^\times$ continuous), but the subspace topology on the set of solutions to $xy = 1$ in $R \times R$ (this is what one gets if one expresses the multiplicative group scheme $\mathbb{G}_m$ as an affine variety).

2.7. Connection between the global and local Artin homomorphisms. Let v be a place of K . Identify $K_v^\times$ with a subgroup of $\mathbf{A}^\times$ by mapping $\alpha \in K_v^\times$ to the idele with α in the v -th position and 1 in every other position. The composition $K_v^\times \hookrightarrow \mathbf{A}^\times \twoheadrightarrow C$ is injective. Let θ_v be the local Artin homomorphism for K_v . Then the diagram

$$\begin{array}{ccc} K_v^\times & \xrightarrow{\theta_v} & \text{Gal}(K_v^{\text{ab}}/K_v) \\ \downarrow & & \downarrow \text{res} \\ C & \xrightarrow{\theta} & \text{Gal}(K^{\text{ab}}/K) \end{array}$$

commutes. Thus θ determines θ_v .

Conversely, if one knows θ_v for all v , one can construct θ as follows. Let L be a finite abelian extension of K contained in K^{s} . Define

$$\begin{aligned} \mathbf{A}^\times &\rightarrow \text{Gal}(L/K) \\ (a_v) &\mapsto \prod_v \theta_v(a_v); \end{aligned}$$

if v is unramified in L/K , and $a_v \in \mathcal{O}_v^\times$, then $\theta_v(a_v) = 1$, so all but finitely many terms in the infinite product are 1, and the product makes sense. Take the inverse limit over all possible L to get

$$\mathbf{A}^\times \rightarrow \text{Gal}(K^{\text{ab}}/K).$$

The idelic version of the Artin reciprocity law says that $K^\times$ is in the kernel, so we get a homomorphism

$$C \rightarrow \text{Gal}(K^{\text{ab}}/K),$$

which is θ .

2.8. Moduli.

Definition 2.2. A **modulus** is a formal product $\mathfrak{m} = \prod_v v^{e_v}$ where $e_v \in \mathbb{Z}_{\geq 0}$, all but finitely many e_v equal 0, and $e_v \in \{0, 1\}$ for real v , and $e_v = 0$ for complex v . The **support** $\text{supp } \mathfrak{m}$ is the (finite) set of *nonarchimedean* places v such that $e_v \neq 0$.

If K is a number field, then a modulus can be viewed as a pair consisting of

- (1) an integral ideal of the ring of integers $\mathcal{O}_K$, and
- (2) a subset of the real places.

If K is the function field of a smooth projective curve X over a finite field, then a modulus is the same thing as an effective divisor on X .

2.9. Ray class groups and ray class fields. In this section we assume that K is a number field. Fix a modulus $\mathfrak{m} = \prod_v v^{e_v}$. We will define a finite-index open subgroup $U_{\mathfrak{m},v} \subseteq \mathcal{O}_v^\times$ for each v . If $e_v = 0$, define $U_{\mathfrak{m},v} := \mathcal{O}_v^\times$. If $e_v > 0$ and v is nonarchimedean, define $U_{\mathfrak{m},v} := 1 + \mathfrak{p}_v^{e_v}$, where $\mathfrak{p}_v$ is the maximal ideal of $\mathcal{O}_v$. If $e_v > 0$ and v is real, define $U_{\mathfrak{m},v}$ as $\mathbb{R}_{>0} \subseteq \mathbb{R}^\times \simeq K_v^\times$. Define $U_{\mathfrak{m}} := \prod_v U_{\mathfrak{m},v} \subseteq \mathbf{A}^\times$. The image of $U_{\mathfrak{m}}$ under $\mathbf{A}^\times \twoheadrightarrow C$ is a finite-index open subgroup $U'_{\mathfrak{m}}$ of C (this is equivalent to finiteness of the class number of K , as we will see in Section 3.4). The corresponding finite abelian extension $R_{\mathfrak{m}}$ of K is called

the ray class field of modulus $\mathfrak{m}$, and $R_{\mathfrak{m}}$ over K is unramified at all v with $e_v = 0$. The ray class group of modulus $\mathfrak{m}$ is

$$\frac{C}{U'_{\mathfrak{m}}} = \frac{\mathbf{A}^{\times}}{U_{\mathfrak{m}}K^{\times}},$$

which is isomorphic to $\text{Gal}(R_{\mathfrak{m}}/K)$ via the global Artin homomorphism.

Every finite-index open subgroup of $\mathbf{A}^{\times}$ contains $U_{\mathfrak{m}}$ for some $\mathfrak{m}$, so every finite abelian extension of K is contained in $R_{\mathfrak{m}}$ for some $\mathfrak{m}$.

3. GLOBAL CLASS FIELD THEORY (VIA IDEALS)

In this section we assume that K is a number field.

3.1. Classical ray class groups. Let I be the group of fractional ideals of K , or equivalently, the free abelian group on the nonarchimedean places of K . Let P be the subgroup of principal ideals. The **class group** is $\text{Cl } \mathcal{O}_K := I/P$.

We now generalize to an arbitrary modulus $\mathfrak{m} = \prod_v v^{e_v}$. Let $I_{\mathfrak{m}}$ be the subgroup of fractional ideals that do not involve the primes dividing $\mathfrak{m}$; i.e., $I_{\mathfrak{m}}$ is the free abelian group on the nonarchimedean places v satisfying $e_v = 0$. For $a \in K^{\times}$, the notation $a \equiv 1 \pmod{\times \mathfrak{m}}$ means that $a \in U_{\mathfrak{m},v}$ for every v satisfying $e_v > 0$. The group $P_{\mathfrak{m}} \subseteq I_{\mathfrak{m}}$ is the group of principal ideals generated by some $a \in K^{\times}$ with $a \equiv 1 \pmod{\times \mathfrak{m}}$. The **classical ray class group of modulus $\mathfrak{m}$** is $\text{Cl}_{\mathfrak{m}} \mathcal{O}_K := I_{\mathfrak{m}}/P_{\mathfrak{m}}$. Section 3.4 will prove that this is isomorphic to the ray class group $C/U'_{\mathfrak{m}}$ defined in Section 2.9.

3.2. The classical Artin homomorphism. Let L/K be a finite abelian extension of number fields. Let S be a finite set of finite primes of K such that S contains every prime that ramifies in L . Let I_S be the group of fractional ideals that do not involve the primes in S . The classical Artin homomorphism is the map

$$\Theta: I_S \rightarrow \text{Gal}(L/K)$$

sending each prime ideal $\mathfrak{p} \notin S$ to the Frobenius element $\text{Frob}_{\mathfrak{p}} \in \text{Gal}(L/K)$.

3.3. The main theorems. The Artin reciprocity law states that there exists a modulus $\mathfrak{m}$ (depending on L/K) with $\text{supp } \mathfrak{m} = S$ such that the subgroup $P_{\mathfrak{m}} \subseteq I_{\mathfrak{m}} = I_S$ is contained in $\ker \Theta$. The existence theorem states that given a modulus $\mathfrak{m}$ and group H with $P_{\mathfrak{m}} \subseteq H \subseteq I_{\mathfrak{m}}$ there exists an abelian extension L of K unramified outside $\text{supp } \mathfrak{m}$ such that the kernel of Θ for L/K equals H .

3.4. Comparison of ideal groups and idele groups. Consider the trivial modulus $\mathfrak{m} = 1$ (with $e_v = 0$ for all v). Taking the restricted direct product of the valuation maps $v: K_v^{\times} \rightarrow \mathbb{Z}$ gives a surjective homomorphism

$$\mathbf{A}^{\times} \rightarrow I$$

that discards the archimedean components of its input, and its kernel is $U_1 = \prod_v \mathcal{O}_v^{\times}$. Thus $\frac{\mathbf{A}^{\times}}{U_1} \simeq I$. If we take the quotient by the image of $K^{\times}$ on both sides, we find that the ray class group $\frac{\mathbf{A}^{\times}}{U_1 K^{\times}}$ of modulus 1 is isomorphic to the class group $I/P = \text{Cl } \mathcal{O}_K$. The ray class field R_1 of modulus 1 is called the **Hilbert class field**, which can be characterized also as the

maximal abelian extension of K in K^s that is unramified at all places of K (including the archimedean ones). We get

$$\frac{C}{U'_1} = \frac{\mathbf{A}^\times}{U_1 K^\times} \simeq \frac{I}{P} = \text{Cl } \mathcal{O}_K \simeq \text{Gal}(R_1/K).$$

This can be generalized to an arbitrary modulus $\mathfrak{m} = \prod v^{e_v}$ as follows. Let $\mathbf{A}^\mathfrak{m} \subseteq \mathbf{A}^\times$ be the subgroup consisting of (a_v) with $a_v \in U_{\mathfrak{m},v}$ for every v with $e_v > 0$. Let $K^\mathfrak{m} = \mathbf{A}^\mathfrak{m} \cap K^\times$. We have an isomorphism

$$\frac{\mathbf{A}^\mathfrak{m}}{U_\mathfrak{m}} \xrightarrow{\sim} I_\mathfrak{m}.$$

Dividing by the image of $K^\mathfrak{m}$ on both sides gives

$$(2) \quad \frac{\mathbf{A}^\mathfrak{m}}{U_\mathfrak{m} K^\mathfrak{m}} \xrightarrow{\sim} \frac{I_\mathfrak{m}}{P_\mathfrak{m}}.$$

On the other hand, $\mathbf{A}^\times = \mathbf{A}^\mathfrak{m} K^\times$, so there is an isomorphism

$$\frac{\mathbf{A}^\mathfrak{m}}{K^\mathfrak{m}} \xrightarrow{\sim} \frac{\mathbf{A}^\times}{K^\times} = C.$$

Dividing by the image of $U_\mathfrak{m}$ on both sides, and combining with (2), we get isomorphisms

$$\frac{C}{U'_\mathfrak{m}} = \frac{\mathbf{A}^\times}{U_\mathfrak{m} K^\times} \simeq \frac{I_\mathfrak{m}}{P_\mathfrak{m}} = \text{Cl}_\mathfrak{m} \mathcal{O}_K \simeq \text{Gal}(R_\mathfrak{m}/K).$$

4. AN INTRODUCTION TO AN INTRODUCTION TO THE LANGLANDS PROGRAM

Let K be a local or global field. Every 1-dimensional character (continuous homomorphism)

$$\text{Gal}(K^s/K) \rightarrow \mathbb{C}^\times$$

factors through $\text{Gal}(K^{\text{ab}}/K)$ and has finite image. These characters form a discrete abelian group, the Pontryagin dual of the profinite group $\text{Gal}(K^{\text{ab}}/K)$. It follows that the problem of classifying finite abelian extensions of K is more or less the same as the problem of describing all these characters.

The Langlands program is an attempt to understand $\text{Gal}(K^s/K)$ more completely by describing its higher-dimensional representations: the group $\mathbb{C}^\times = \text{GL}_1(\mathbb{C})$ is replaced by $\text{GL}_n(\mathbb{C})$, or even $G(\mathbb{C})$ for other linear algebraic groups G . The continuous homomorphisms

$$\text{Gal}(K^s/K) \rightarrow G(\mathbb{C})$$

are conjectured to correspond to certain “automorphic” objects defined intrinsically in terms of K , just as class field theory gives a description of the group $\text{Gal}(K^{\text{ab}}/K)$ (which is defined in terms of extrinsic objects such as finite abelian extensions, which are initially mysterious) in terms of intrinsic objects ($K^\times$ or C) obtained directly from K .

Ultimately, the program would give information about nonabelian algebraic extensions of K .

5. FURTHER READING

For basics on profinite groups, see [Ser02, I.§1] and [Gru86]. The latter discusses infinite Galois theory as well.

For local class field theory, see [Ser86]. For the approach to global class field theory via cohomology of ideles, see [Tat86]. For a treatment of global class field theory via ideals, see [Jan96]. All these topics are covered also in [Neu99].

For an introduction to the Langlands program, see [BG03].

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DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA 02139-4307, USA

Email address: `poonen@math.mit.edu`

URL: `http://math.mit.edu/~poonen/`