EXISTENCE OF RATIONAL POINTS ON SMOOTH PROJECTIVE VARIETIES

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to Jean-Louis Colliot-Thelene on his 60 birthday

Abstract. Fix a number field k. We prove that if there is an algorithm for deciding whether a smooth projective geometrically integral k-variety has a k-point, then there is an algorithm for deciding whether an arbitrary k-variety has a k-point and also an algorithm for computing $X(k)$ for any k-variety X for which $X(k)$ is finite. The proof involves the construction of a one-parameter algebraic family of Châtelet surfaces such that exactly one of the surfaces fails to have a k-point.

1. Statement of results

Given a field k, a k-variety is a separated scheme of finite type over k. We will consider algorithms (Turing machines) accepting as input k-varieties where k is a number field. Each such variety may be presented by a finite number of affine open patches together with gluing data, so it admits a finite description suitable for input into a Turing machine. We do not require algorithms to run in polynomial time or any other specified time, but they must terminate with an answer for each allowable input.

Theorem 1.1. *Fix a number field k. Suppose that there exists an algorithm for deciding whether a regular projective geometrically integral k-variety has a k-point. Then*

(i) *there is an algorithm for deciding whether an arbitrary k-variety has a k-point, and*
(ii) *there is an algorithm for computing $X(k)$ for any k-variety X for which $X(k)$ is finite.*

Remark 1.2.

(a) For a field k of characteristic 0, a k-variety is regular if and only if it is smooth over k. Nevertheless, we have two reasons for sometimes using the adjective "regular":
- In some situations, for instance when speaking of families of varieties, it helps to distinguish the absolute notion (regular) from the relative notion (smooth).
- In Section 11, we say what can be said about the analogue for global function fields.

(b) For regular proper integral k-varieties, the property of having a k-point is a birational invariant, equivalent to the existence of a (not necessarily rank 1) valuation v on the

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function field such that v is trivial on k and k maps isomorphically to the residue field: this follows from [Nis55] and also is close to [Lan54, Theorem 3]; see also [CTCS80, Lemme 3.1.1]. Thus one might wonder whether the decision problem is easier for regular projective geometrically integral varieties than for arbitrary ones. But Theorem 1.1(i) says that in fact the two problems are equivalent.

(c) For $k = \mathbb{Q}$, Theorem 1.1(i) was more or less known: it is easily deduced from a result of R. Robinson [Smo91, §II.7] that the problem of deciding the existence of a rational zero of a polynomial over $\mathbb{Q}$ is equivalent to the problem of deciding the existence of a nontrivial rational zero of a *homogeneous* polynomial over $\mathbb{Q}$. Robinson's argument generalizes easily to number fields with a real place.

(d) Theorem 1.1 becomes virtually trivial if the word "projective" is changed to "affine". On the other hand, there are related statements for affine varieties that are nontrivial: for instance, if there is an algorithm for deciding whether any irreducible affine plane curve of geometric genus at least 2 has a rational point, then there is algorithm for determining the set of rational points on any such curve [Kim03].

(e) By restriction of scalars, if we have an algorithm for deciding whether a regular projective geometrically integral $\mathbb{Q}$-variety has a rational point, then we have an analogous algorithm over any number field. But there is no number field for which the existence of such algorithms is known.

(f) Remark 8.2 will imply that to have algorithms as in (i) and (ii) of Theorem 1.1 for *curves*, it would suffice to be able to decide the existence of rational points on regular projective geometrically integral 3-*folds*. (If over $\mathbb{Q}$ one uses Robinson's reduction instead, one would need an algorithm for 9-folds!)

Theorem 1.1 will be deduced in Section 10 from the following:

Theorem 1.3. *Let k be a number field. Let X be a projective k-variety. Let $U \subseteq X$ be an open subvariety. Then there exists a regular projective variety Y and a morphism $\pi : Y \to X$ such that $\pi(Y(k)) = U(k)$. Moreover, there exists an algorithm for constructing (Y, π) given (k, X, U).*

The key special case, from which all others will be derived, is the case where $U = \mathbb{A}^1$ and $X = \mathbb{P}^1$. In this case we can arrange also for $\pi^{-1}(t)$ to be smooth and geometrically integral for all $t \in \mathbb{P}^1(k)$: see Proposition 7.2. Thus we will have a family of smooth projective geometrically integral varieties in which every rational fiber but one has a rational point, an extreme example of geometry *not* controlling arithmetic!

Remark 1.4. Theorem 1.3 fails for many fields k that are not number fields, even for those that have a complicated arithmetic. Proposition 7.3 implies that it fails for the function field of any $\mathbb{C}$-variety, for instance.

2. Notation

Let k be a number field. Let $\mathcal{O}_k$ be the ring of integers in k. Let Ω_k be the set of places of k. If $v \in \Omega_k$, let k_v be the completion of k at v. If v is nonarchimedean, let $\mathbb{F}_v$ be the residue field. Call v odd if it is nonarchimedean and $\#\mathbb{F}_v$ is odd. If $a \in \mathcal{O}_k$ generates a prime ideal, let v_a be the associated valuation, and let $\mathbb{F}_a = \mathbb{F}_{v_a}$. For $a \in k$, let $a \gg 0$ mean that a is totally positive, i.e., positive for every real embedding of k. For any field L, let $L^\times$ be the unit group $L - \{0\}$.

3. Conic bundles

A *conic* over k is the zero locus in $\mathbb{P}^2 = \operatorname{Proj} k[x_0, x_1, x_2]$ of a nonzero degree-2 homogeneous polynomial s in $k[x_0, x_1, x_2]$. If E is the k-vector space with basis x_0, x_1, x_2, then we may view $\mathbb{P}^2$ as $\mathbb{P}E := \operatorname{Proj} \operatorname{Sym} E$, and s as a nonzero element of $\operatorname{Sym}^2 E$. By analogy, a *conic bundle* C over a k-scheme B is the zero locus in $\mathbb{P}\mathcal{E}$ of a nowhere-vanishing global section s of $\operatorname{Sym}^2 \mathcal{E}$, where $\mathcal{E}$ is some rank-3 vector sheaf on B. We will consider only the special case where $\mathcal{E} = \mathcal{L}_0 \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$ for some line sheaves $\mathcal{L}_i$ and $s = s_0 + s_1 + s_2$ where $s_i \in \Gamma(B, \mathcal{L}_i^{\otimes 2})$; we then call $C \to B$ a *diagonal conic bundle*.

Lemma 3.1. *Let B be a smooth curve over k. Let $\overline{k}$ be an algebraic closure of k. Let $C \to B$ be a diagonal conic bundle, with notation as above, such that $\sum_{i=0}^{2} \operatorname{ord}_P(s_i) \leq 1$ for every $P \in B(\overline{k})$. Then the total space C is smooth over k.*

Proof. We may assume that k is algebraically closed. Let π be the morphism $C \to B$. Since B is smooth, C is smooth over k at any point where π is smooth. Thus we need only check the singular points of the fibers of π.

Given $P \in B$, choose a neighborhood U of P in B such that $\mathcal{L}_i|_U \simeq \mathcal{O}_U$; then $\pi^{-1}(U)$ is isomorphic to a conic bundle $a_0 x_0^2 + a_1 x_1^2 + a_2 x_2^2 = 0$ in $\mathbb{P}^2_U$ where the $a_i \in \mathcal{O}_B(U)$ satisfy $\sum_{i=0}^{2} \operatorname{ord}_P(a_i) \leq 1$. If all the a_i are nonvanishing at P, then the fiber $\pi^{-1}(P)$ is a smooth conic. Otherwise, at most one of the a_i vanishes at P, say $\operatorname{ord}_P(a_2) = 1$. Then $\pi^{-1}(P)$ has a unique singularity Q, the point $((0,0), P)$ in the affine patch $a_0 X_0^2 + a_1 X_1^2 + a_2 = 0$ in the smooth k-variety $\mathbb{A}^2 \times U$, where $X_i := x_i/x_2$. Let $\mathfrak{m}_Q$ be the maximal ideal of Q in $\mathbb{A}^2 \times U$. Then $a_0 X_0^2 + a_1 X_1^2 \in \mathfrak{m}_Q^2$ but $\operatorname{ord}_P(a_2) = 1$, so $a_0 X_0^2 + a_1 X_1^2 + a_2 \notin \mathfrak{m}_Q^2$. Thus C is regular at Q, and hence C is smooth over k even at Q.

4. Hilbert symbol

For $v \in \Omega_k$ and $t, u \in k_v^\times$, let $(t, u)_v \in \{\pm 1\}$ be the v-adic *Hilbert symbol*: by definition, $(t, u)_v = 1$ if and only if $x^2 - ty^2 - uz^2 = 0$ has a solution $(x, y, z) \neq (0, 0, 0)$ in k_v^3. We recall some basic properties of the Hilbert symbol:

Lemma 4.1.
For all $t, t', u, u' \in k_v^\times$, we have

(a) $(t, u)_v = 1$ *if and only if t belongs to the image of the norm map $k_v(\sqrt{u})^\times \to k_v^\times$.*
(b) $(t, u)_v = (u, t)_v$.
(c) $(tt', u)_v = (t, u)_v (t', u)_v$ *and* $(t, uu')_v = (t, u)_v (t, u')_v$; *in particular,* $(t, u^2)_v = 1$.
(d) $(t, -t)_v = 1$.
(e) $(t, 1 - t)_v = 1$, *assuming in addition that* $t \neq 1$.
(f) *Suppose that v is odd and $v(t) = 0$. Then $(t, u)_v = -1$ if and only if $v(u)$ is odd and the image of t in $\mathbb{F}_v$ is a non-square.*

If $t, u \in k^\times$, we have the product formula

(g) $\prod_{v \in \Omega_k} (t, u)_v = 1$.

Proof. See [Ser73, Chapter III] for the case $k = \mathbb{Q}$. See [Ser79, Chapter XIV] for the general case: in particular, Proposition 7 there yields (a), (b), (c), (d), (e); Proposition 8 implies (f); and p. 222 contains (g).

5. Châtelet surfaces

Fix $\alpha \in k^\times$ and $P(x) \in k[x]$ of degree at most 4. Let V_0 be the affine surface in $\mathbb{A}^3$ given by $y^2 - \alpha z^2 = P(x)$. We want a smooth projective model V of V_0. Define $\tilde{P}(w, x) := w^4 P(x/w)$; view $\tilde{P}$ as a section of $\mathcal{O}(4)$ on $\mathbb{P}^1 := \operatorname{Proj} k[w, x]$. The construction of Section 3 with $B = \mathbb{P}^1$, $\mathcal{L}_0 = \mathcal{L}_1 = \mathcal{O}$, $\mathcal{L}_2 = \mathcal{O}(2)$, $s_0 := 1$, $s_1 := -\alpha$, and $s_2 := -\tilde{P}$ gives a diagonal conic bundle $V \to \mathbb{P}^1$ containing V_0 as an affine open subvariety. Since $V \to \mathbb{P}^1$ is projective, V is projective over k too. If $P(x)$ is not identically 0, then V is geometrically integral. If $P(x)$ is separable and of degree 3 or 4, then $\tilde{P}(w, x)$ is separable and V is smooth over k by Lemma 3.1; in this case V is called the Châtelet surface given by $y^2 - \alpha z^2 = P(x)$.

Iskovskikh [Isk71] showed that the Châtelet surface over $\mathbb{Q}$ given by

$$y^2 + z^2 = (x^2 - 2)(3 - x^2)$$

violated the Hasse principle. Several years later it was shown that this violation could be explained by the Brauer-Manin obstruction, and that more generally, any Châtelet surface over a number field given by $y^2 - az^2 = f(x)g(x)$ with f and g distinct irreducible quadratic polynomials satisfies the Hasse principle if and only if there is no Brauer-Manin obstruction [CTCS80, Theorem B]. Finally, the two-part paper [CTSSD87a, CTSSD87b] generalized this to all Châtelet surfaces over number fields. For an introduction to the Brauer-Manin obstruction, see [Sko01, §5.2].

Proposition 5.1. *There exists a Châtelet surface V over k that violates the Hasse principle.*

The rest of this section is devoted to the proof of Proposition 5.1, so a reader interested in only the case $k = \mathbb{Q}$ may accept the Iskovskikh example and proceed to Section 6. We generalize the argument presented in [Sko01, p. 145].

By the Chebotarev density theorem and global class field theory applied to a ray class field, we can find $b \in \mathcal{O}_k$ generating a prime ideal such that $b \gg 0$ and $b \equiv 1 \pmod{8\mathcal{O}_k}$. Similarly we find $a \in \mathcal{O}_k$ generating a prime ideal such that $a \gg 0$ and $a \equiv 1 \pmod{8\mathcal{O}_k}$ and a is a not a square modulo b. We may assume that $\#\mathbb{F}_a, \#\mathbb{F}_b > 5$. Fix $c \in \mathcal{O}_k$ such that $b | (ac + 1)$.

We use the abbreviation $(t, u)_b := (t, u)_{v_b}$. We will need the following Hilbert symbol calculations later:

Lemma 5.2. *We have*

(i) $(-1, a)_v = 1$ *for all* $v \in \Omega_k$.
(ii) $(-1, b)_v = 1$ *for all* $v \in \Omega_k$.
(iii) $(ab, a)_b = -1$.
(iv) $(ab, c)_b = -1$.

Proof.

(i) For v archimedean or 2-adic, we have $a \in k_v^{\times 2}$, so Lemma 4.1(c) implies $(-1, a)_v = 1$. For all other v except v_a, we have $v(-1) = v(a) = 0$, so Lemma 4.1(f) implies $(-1, a)_v = 1$. For $v = v_a$, it follows from Lemma 4.1(g).
(ii) The proof is the same as that of (i).
(iii) By (i) and Lemma 4.1(c,d,f), we have $(ab, a)_b = (-1, a)_b (ab, a)_b = (-a, a)_b (b, a)_b = 1 \cdot (b, a)_b = -1$.

(iv) Since $b \mid (ac+1)$, we have $ac \in (-1)k_{v_b}^{\times 2}$, so Lemma 4.1(c) implies $(ab, ac)_b = (ab, -1)_b = (a, -1)_b (b, -1)_b = 1$, where we used (i) and (ii) in the last step. Divide by (iii) to get $(ab, c)_b = -1$.

Let V be the Châtelet surface given by

$$y^2 - abz^2 = (x^2 + c)(ax^2 + ac + 1). \tag{1}$$

(The quadratic factors on the right are separable and generate the unit ideal of $k[x]$, so V is smooth over k.)

Lemma 5.3. *The variety V has a k_v-point for every place v of k.*

Proof. Suppose that v is archimedean or 2-adic. Then $ab \in k_v^{\times 2}$, so the left hand side of (1) factors as $(y + dz)(y - dz)$ for some $d \in k_v^{\times}$; now, choose $x \in k_v$, write the value of the right hand side of (1) as $x_1 x_2$ for some $x_1, x_2 \in k_v$, and solve the system $y + dz = x_1$, $y - dz = x_2$ for $y, z \in k_v$ to obtain a k_v-point of V.

Suppose that v is odd and $v \notin \{v_a, v_b\}$. Choose $x \in k$ with $v(x) < 0$. Then the right hand side of (1) has even valuation and is hence a norm for the unramified extension $k_v(\sqrt{ab})/k_v$. So V has a k_v-point.

Suppose that $v = v_b$. Because a is not a square modulo b, all $\mathbb{F}_b$-points on the projective closure of the affine curve $y^2 = a(x^2 + c)$ over $\mathbb{F}_b$ lie on the affine part, so there are $\#\mathbb{F}_b + 1$ solutions $(x, y) \in \mathbb{F}_b^2$. Then the number of solutions with $x^2 + c \neq 0$ and $x \neq 0$ is at least $(\#\mathbb{F}_b + 1) - 2 - 2 - 2 > 0$. Choose $x \in \mathcal{O}_k$ reducing to the x-coordinate of such a solution. The right hand side of (1) is congruent modulo b to $(x^2 + c)(ax^2)$, so by Hensel's lemma it is in $k_v^{\times 2}$. Thus V has a k_v-point.

Suppose that $v = v_a$. The same argument as in the previous paragraph shows that we may choose $x \in \mathcal{O}_k$ such that $x^2 + c \in k_v^{\times 2}$. The other factor $ax^2 + ac + 1$ is 1 mod a, hence in $k_v^{\times 2}$. Therefore the right hand side of (1) is in $k_v^{\times 2}$, so V has a k_v-point.

Let $\mathbf{k}(V)$ be the function field of V. Let $A \in \operatorname{Br} \mathbf{k}(V)$ be the class of the quaternion algebra $(ab, x^2 + c)$. Since for any $g \in \mathbf{k}(V)^{\times}$ the class of (ab, g) is unaffected by multiplying g by a square or by a norm from $\mathbf{k}(V)(\sqrt{ab})$, the class A equals the class of $(ab, 1 + c/x^2)$ and of $(ab, ax^2 + ac + 1)$.

Lemma 5.4. *The element A belongs to the subgroup* $\operatorname{Br} V$ *of* $\operatorname{Br} \mathbf{k}(V)$.

Proof. First of all, V is a regular integral scheme, so $\operatorname{Br} V$ is a subgroup of $\operatorname{Br} \mathbf{k}(V)$, and it consists of the elements whose residue at every codimension-1 point P of V vanishes. To check that A satisfies this residue condition at P, it is sufficient to show that A can be represented by a quaternion algebra (f, g) where $f, g \in \mathbf{k}(V)^{\times}$ are regular and nonvanishing at P. In fact, at every $P \in V$, one of the three representations of A given in the paragraph preceding Lemma 5.4 is of this form.

We will show that A gives a Brauer-Manin obstruction to the Hasse principle. For $P_v \in V(k_v)$, let $A(P_v) \in \operatorname{Br} k_v$ be the evaluation of A at P_v. Let $\operatorname{inv}_v \colon \operatorname{Br} k_v \hookrightarrow \mathbb{Q}/\mathbb{Z}$ be the usual invariant map. Given $P_v \in V(k_v)$, if A is represented by (f, g) with $f, g \in \mathbf{k}(V)^{\times}$ regular and nonvanishing at P_v, then $\operatorname{inv}_v(A(P_v))$ is 0 or $1/2$ according to whether the Hilbert symbol $(f(P_v), g(P_v))_v$ is 1 or -1.

Lemma 5.5. *For any $P_v \in V(k_v)$,*

$$\operatorname{inv}_v(A(P_v)) = \begin{cases} 0, & \text{if } v \neq v_b, \\ 1/2 & \text{if } v = v_b. \end{cases}$$

Proof. Since V is smooth, the implicit function theorem shows that $V_0(k_v)$ is v-adically dense in $V(k_v)$. Since $\operatorname{inv}_v(A(P_v))$ is a continuous function on $V(k_v)$ with the v-adic topology, it suffices to prove the result for $P_v \in V_0(k_v)$.

Suppose that v is archimedean or 2-adic. Then $ab \in k_v^{\times 2}$, so for any $t \in k_v^{\times}$ the Hilbert symbol $(ab, t)_v$ is 1. Hence $\operatorname{inv}_v(A(P_v)) = 0$.

Suppose that v is odd and $v \notin \{v_a, v_b\}$. If $v(x) < 0$ at P_v, then $v(x^2 + c)$ is even, so Lemma 4.1(f) implies $\operatorname{inv}_v(A(P_v)) = 0$. If $v(x) \geq 0$, then either $x^2 + c$ or $ax^2 + ac + 1$ is a v-adic unit, so using an appropriate representation of A and applying Lemma 4.1(f) shows that $\operatorname{inv}_v(A(P_v)) = 0$.

Suppose that $v = v_a$. If $v(x) < 0$ at P_v, then $x^2 + c \in k_v^{\times 2}$, so Lemma 4.1(c) implies $\operatorname{inv}_v(A(P_v)) = 0$. If $v(x) \geq 0$, then $ax^2 + ac + 1$ is $1 \bmod a$ so it is in $k_v^{\times 2}$, and again $\operatorname{inv}_v(A(P_v)) = 0$.

Finally, suppose that $v = v_b$. Each of the following two sentences will use the following observation: if elements $t, u \in k_v^{\times}$ and $\epsilon \in k_v$ satisfy $v(u) \leq 0 < v(\epsilon)$, then $(u + \epsilon)/u \in k_v^{\times 2}$, so Lemma 4.1(c) implies $(t, u + \epsilon)_b = (t, u)_b$. If $v(x) \leq 0$, then taking $\epsilon = ac + 1$ yields $(ab, ax^2 + ac + 1)_b = (ab, ax^2)_b = (ab, a)_b = -1$, by Lemma 5.2(iii). If $v(x) > 0$, then taking $\epsilon = x^2$ yields $(ab, x^2 + c)_b = (ab, c)_b = -1$, by Lemma 5.2(iv). In either case, $\operatorname{inv}_v(A(P_v)) = 1/2$.

Lemma 5.5, together with the reciprocity law $\sum_{v \in \Omega_k} \operatorname{inv}_v(\alpha) = 0$ for $\alpha \in \operatorname{Br} k$ (or the special case for quaternion algebras given by Lemma 4.1(g)), implies that V has no k-point. This completes the proof of Proposition 5.1.

6. Châtelet surface bundles

By a *Châtelet surface bundle over* $\mathbb{P}^1$ we mean a flat proper morphism $\mathcal{V} \to \mathbb{P}^1$ such that the generic fiber is a Châtelet surface; then for $t \in \mathbb{P}^1(k)$, we let $\mathcal{V}_t$ be the fiber above t.

We retain the notation of Section 5. Let $\tilde{P}_0(w, x) \in k[w, x]$ be the homogeneous form of degree-4 obtained by homogenizing the right hand side of (1). Let $\tilde{P}_\infty(w, x)$ be any irreducible degree-4 form in $k[w, x]$. Thus $\tilde{P}_0$ and $\tilde{P}_\infty$ are linearly independent.

Let $\mathcal{V}$ be the diagonal conic bundle over $\mathbb{P}^1 \times \mathbb{P}^1 := \operatorname{Proj} k[u, v] \times \operatorname{Proj} k[w, x]$ obtained by taking $\mathcal{L}_0 = \mathcal{L}_1 := \mathcal{O}$, $\mathcal{L}_2 := \mathcal{O}(1, 2)$, $s_0 := 1$, $s_1 := -ab$, and $s_2 := -(u^2\tilde{P}_\infty + v^2\tilde{P}_0)$. Composing $\mathcal{V} \to \mathbb{P}^1 \times \mathbb{P}^1$ with the first projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ lets us view $\mathcal{V}$ as a Châtelet surface bundle over $\mathbb{P}^1 = \operatorname{Proj} k[u, v]$ with projective geometrically integral fibers. If $u, v \in k$ are not both 0, the fiber above $(u : v) \in \mathbb{P}^1(k)$ is the Châtelet surface given by

$$y^2 - abz^2 = u^2\tilde{P}_\infty(1, x) + v^2\tilde{P}_0(1, x),$$

if smooth over k. In particular, the fiber $\mathcal{V}_{(0:1)}$ is isomorphic to V.

Call a subset T of $\mathbb{P}^1(k)$ *thin* if and only if there exist finitely many regular projective geometrically integral curves C_i and morphisms $\pi_i\colon C_i \to \mathbb{P}^1$ of degree greater than 1 such that $T \subseteq \bigcup \pi_i(C_i(k))$; cf. [Ser97, §9.1]. Such sets arise in the context of the Hilbert irreducibility theorem.

Lemma 6.1. *The set of specializations* $(u : v) \in \mathbb{P}^1(k)$ *such that* $u^2\tilde{P}_\infty + v^2\tilde{P}_0 \in k[w, x]$ *is reducible (for any or all choices of* $(u, v) \in k^2 - \{(0, 0)\}$ *representing* $(u : v)$*) is a thin set.*

Proof. We may assume $u = 1$. The degree-4 form $\tilde{P}_\infty + v^2\tilde{P}_0$ over $k(v)$ is irreducible since it has an irreducible specialization, namely $\tilde{P}_\infty$. Apply [Ser97, §9.2, Proposition 1].

Lemma 6.2. *There exists a finite set S of non-complex places of k and a neighborhood N_v of* $(0 : 1)$ *in* $\mathbb{P}^1(k_v)$ *for each* $v \in S$ *such that for* $t \in \mathbb{P}^1(k)$ *belonging to* N_v *for each* $v \in S$*, the fiber* $\mathcal{V}_t$ *has a* k_v*-point for* every $v \in \Omega_k$.

Proof. This is an application of the "fibration method", which has been used previously in various places (e.g., [CTSSD87a], [CT98, 2.1], [CTP00, Lemma 3.1]). Since all geometric fibers of the k-morphism $\mathcal{V} \to \mathbb{P}^1$ are integral, the same is true for a model over some ring $\mathcal{O}_{k,S}$ of S-integers. By adding finitely many v to S, we can arrange that for nonarchimedean $v \notin S$ the residue field $\mathbb{F}_v$ is large enough that every $\mathbb{F}_v$-fiber has a smooth $\mathbb{F}_v$-point by the Weil conjectures; then by Hensel's lemma any k_v-fiber has a k_v-point. Include the real places in S, and exclude the complex places since for complex v the existence of k_v-points on fibers is automatic. For $v \in S$, since the fiber above $(0 : 1)$ has a k_v-point, and since $\mathcal{V} \to \mathbb{P}^1$ is smooth above $(0 : 1)$, the implicit function theorem implies that the image of $\mathcal{V}(k_v) \to \mathbb{P}^1(k_v)$ contains a v-adic neighborhood N_v of $(0 : 1)$ in $\mathbb{P}^1(k_v)$.

7. Base change

The following lemma combines the idea of [CTP00, Lemma 3.3] with some new ideas.

Lemma 7.1. *Let* $P \in \mathbb{P}^1(k)$*. Let S be a finite set of non-complex places of k. For each* $v \in S$*, let N_v be a neighborhood of P in* $\mathbb{P}^1(k_v)$*. Let T be a thin subset of* $\mathbb{P}^1(k)$ *containing P. Then there exists a k-morphism* $\pi\colon \mathbb{P}^1 \to \mathbb{P}^1$ *such that both of the following hold:*

(1) $\pi(\mathbb{P}^1(k_v)) \subseteq N_v$ *for each* $v \in S$.
(2) $\pi^{-1}(T) \cap \mathbb{P}^1(k)$ *consists of a single point Q with* $\pi(Q) = P$.

Proof. We will construct π as a composition. But we present the argument as a series of reductions, each step of which involves taking the inverse image of all the data under some $\beta\colon \mathbb{P}^1 \to \mathbb{P}^1$ and replacing P by some $P' \in \beta^{-1}(P) \cap \mathbb{P}^1(k)$.

Choose finitely many regular projective geometrically integral curves C_i and morphisms $\nu_i\colon C_i \to \mathbb{P}^1$ of degree greater than 1 such that $T \subseteq \bigcup \nu_i(C_i(k))$. By choosing a suitable coordinate on $\mathbb{P}^1$, we may assume that $0, \infty \in \mathbb{P}^1(k)$ are disjoint from the branch points of every ν_i, and that $1 \in \mathbb{P}^1(k)$ is the point P. Choose $n \in \mathbb{Z}_{>0}$ such that $n > 2 \deg \nu_i$ for every i, and let $\beta\colon \mathbb{P}^1 \to \mathbb{P}^1$ be the morphism corresponding to the rational function $t \mapsto t^n$; note that there exists $P' \in \mathbb{P}^1(k)$ with $\beta(P') = P$. Define the fiber product C_i' with morphisms $\nu_i'\colon C_i' \to \mathbb{P}^1$ and $\beta_i\colon C_i' \to C_i$ making a cartesian diagram

$$\begin{array}{ccc} C_i' & \xrightarrow{\beta_i} & C_i \\ {\scriptstyle \nu_i'}\downarrow & & \downarrow{\scriptstyle \nu_i} \\ \mathbb{P}^1 & \xrightarrow{\beta} & \mathbb{P}^1. \end{array}$$

Since β is totally ramified above 0, the morphism β_i is totally ramified above $\nu_i^{-1}(0)$, so C_i' is geometrically integral. Since the branch loci of β and ν_i are disjoint, C_i' is regular and the rami cation divisors R'_i and R_i of $'_i$ and $_i$, respectively, satisfy $R'_i = {}^*_i R_i$. Since $\mathbb{P}^1$ has no everywhere unrami ed cover, $\deg R_i > 0$. By the Hurwitz formula, the genus g'_i of C'_i satis es

$$2g'_i - 2 = (\deg \ '_i)(-2) + \deg R'_i = (\deg \ _i)(-2) + n \deg R_i \geq -2 \deg \ _i + n > 0;$$

so $g'_i > 1$. By Faltings' theorem [Fal83], $C'_i(k)$ is nite. We have $^{-1}(T) \cap \mathbb{P}^1(k) \subseteq \bigcup \ '_i(C'_i(k))$, so $^{-1}(T) \cap \mathbb{P}^1(k)$ is nite. By pulling all the data back under and replacing P by P', we reduce to the case where T is nite.

Choose a new coordinate on $\mathbb{P}^1$ for which P is $0 \in \mathbb{P}^1(k)$. Then the rational function $t \mapsto 1/(t^2 + m)$ maps ∞ to 0 and maps $\mathbb{P}^1(\mathbb{R})$ into N_v for each real v if $m \in \mathbb{Z}_{>0}$ is chosen large enough. Pulling all the data back under the corresponding endomorphism of $\mathbb{P}^1$, we reduce to the case where S contains no archimedean places. Now suppose that S contains a nonarchimedean place v. Let $q = \#\mathbb{F}_v$. Choose a large positive integer r, and let $m := q^r(q-1)$. Then the rational function $t \mapsto t^m$ maps all $t \in k_v$ with $v(t) > 0$ into a small v-adic neighborhood of 0, all $t \in k_v$ with $v(t) < 0$ (including ∞) into a small neighborhood of ∞, and all $t \in k_v$ with $v(t) = 0$ into a small neighborhood of 1 (raising to the power $q-1$ already maps the t with $v(t) = 0$ into the 1-units, and successively raising to the power q brings these closer and closer to 1). Choose a rational function g mapping $\{0, 1, \infty\}$ to P; then choosing r large enough arranges that the rational function $g(t^m)$ maps $\mathbb{P}^1(k_v)$ into N_v. Pulling back everything under the corresponding endomorphism of $\mathbb{P}^1$ lets us replace S by $S - \{v\}$. Eventually we reduce to the case in which $S = \emptyset$.

For a suitable choice of coordinate, P is the point $0 \in \mathbb{P}^1(k)$, and $\infty \notin T$. Choose $c \in k^\times$ such that the images of c and $T - \{0\}$ in $k^\times/k^{\times 2}$ do not meet. Let $: \mathbb{P}^1 \to \mathbb{P}^1$ be given by the rational function ct^2. Then $^{-1}(T) \cap \mathbb{P}^1(k)$ consists of the single point 0.

Proposition 7.2. *There exists a Châtelet surface bundle* $: \mathcal{W} \to \mathbb{P}^1$ *over k such that*

(i) *is smooth over* $\mathbb{P}^1(k)$*, and*
(ii) $(\mathcal{W}(k)) = \mathbb{A}^1(k)$.

Proof. Obtain $: \mathbb{P}^1 \to \mathbb{P}^1$ from Lemma 7.1 with $P = (0:1)$, with S and N_v as in Lemma 6.2, with T the thin set of Lemma 6.1; note that T contains the nitely many $t \in \mathbb{P}^1(k)$ above which $\mathcal{V} \to \mathbb{P}^1$ is not smooth. We may assume that the Q in Lemma 7.1 is ∞. De ne $\mathcal{W}$ as the ber product

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{V} \\ {\scriptstyle\mu}\downarrow & & \downarrow \\ \mathbb{P}^1 & \xrightarrow{\gamma} & \mathbb{P}^1 \end{array}$$

and let be the projection $\mathcal{W} \to \mathbb{P}^1$ shown. Then is smooth above $\mathbb{P}^1(k)$, and for every $t \in \mathbb{P}^1(k)$ the ber $\mathcal{W}_t$ has a k_v-point for every v. If $t \in \mathbb{A}^1(k)$, then $(t) \notin T$, so $\mathcal{W}_t$ is a Châtelet surface de ned by an irreducible degree-4 polynomial, so by [CTSSD87a, Theorem B(i)(b)] $\mathcal{W}_t$ satis es the Hasse principle; thus $\mathcal{W}_t$ has a k-point. But if $t = \infty$, then $\mathcal{W}_t$ is isomorphic to $\mathcal{V}_{(0:1)} \simeq V$, which has no k-point.

The following proposition will not be needed elsewhere. Its role is only to illustrate that Theorem 1.3 and Proposition 7.2 depend subtly upon properties of k: for instance, they are not true over all elds of cohomological dimension 2.

Proposition 7.3. *Let k_0 be an uncountable algebraically closed field, and let k be a field extension of k_0 generated by a set S of cardinality less than $\#k_0$. Then there is no morphism $: \mathcal{W} \to \mathbb{P}^1$ of projective k-varieties such that $(\mathcal{W}(k)) = \mathbb{A}^1(k)$.*

Proof. Suppose that $(\mathcal{W}(k)) = \mathbb{A}^1(k)$. Fix a projective embedding $\mathcal{W} \hookrightarrow \mathbb{P}^n$. Let $\subseteq \mathcal{W} \times \mathbb{P}^1 \subseteq \mathbb{P}^n \times \mathbb{P}^1$ be the graph of . Since $\mathcal{W}$ is projective, is the zero locus of a finite set of bihomogeneous polynomials $_i$ with coefficients in k.

Let L be a finite-dimensional k_0-subspace of k. Let

$$\mathbb{P}^n[L] = \{ (a_0 : \cdots : a_n) \in \mathbb{P}^n(k) \mid (a_0, \ldots, a_n) \in L^n - \{0\} \}.$$

Let $\mathcal{W}[L] = \mathbb{P}^n[L] \cap \mathcal{W}(k) \subseteq \mathbb{P}^n(k)$.

We claim that the subset $I_L := (\mathcal{W}[L]) \cap \mathbb{P}^1(k_0)$ of $\mathbb{P}^1(k)$ is finite. Choose a k_0-basis of L to identify $(L^{n+1} - \{0\})/k_0^\times$ with $\mathbb{P}^N(k_0)$, where $N + 1 = (n+1)\dim_{k_0} L$. For each i, the coefficients obtained when the value of $_i$ at

$$(v_0, \ldots, v_N, w_0, w_1) \in k_0^{N+1} \times k_0^2 \simeq L^{n+1} \times k_0^2$$

is expressed as a linear combination of elements in a fixed k_0-basis of k are bihomogeneous polynomials in $k_0[v_0, \ldots, v_N, w_0, w_1]$. These bihomogeneous polynomials, taken for all i, define a Zariski closed subset $C_L \subseteq \mathbb{P}^N(k_0) \times \mathbb{P}^1(k_0)$. By definition of and the $_i$, the projection of C_L onto the second factor equals I_L. Thus I_L is Zariski closed in $\mathbb{P}^1(k_0)$. On the other hand, $I_L \subseteq (\mathcal{W}[L]) \subseteq (\mathcal{W}(k)) = \mathbb{A}^1(k)$, so $\infty \notin I_L$. Thus I_L is finite, as claimed.

Let $\mathcal{L}$ be the collection of finite-dimensional k_0-subspaces L of k spanned by a finite set of monomials in the elements of S. Then $\bigcup_{L \in \mathcal{L}} L$ is the k_0-subalgebra of k generated by S, and its fraction field is k. Therefore $\bigcup_{L\in\mathcal{L}} \mathbb{P}^n[L] = \mathbb{P}^n(k)$ and $\bigcup_{L\in\mathcal{L}} \mathcal{W}[L] = \mathcal{W}(k)$. Applying and intersecting with $\mathbb{P}^1(k_0)$ yields $\bigcup_{L\in\mathcal{L}} I_L = (\mathcal{W}(k)) \cap \mathbb{P}^1(k_0)$. Thus

$$\#\left((\mathcal{W}(k)) \cap \mathbb{P}^1(k_0) \right) = \#\bigcup_{L\in\mathcal{L}} I_L \le \#\mathcal{L} \cdot \aleph_0 = \max\{\#S, \aleph_0\} < \#k_0 = \#\mathbb{A}^1(k_0).$$

The strict inequality implies $(\mathcal{W}(k)) \neq \mathbb{A}^1(k)$.

8. Reductions

Lemma 8.1. *There exists a projective k-variety Z and a morphism $: Z \to \mathbb{P}^n$ such that $(Z(k)) = \mathbb{A}^n(k)$ and is smooth above $\mathbb{A}^n(k)$.*

Proof. Start with the birational map $(\mathbb{P}^1)^n \dashrightarrow \mathbb{P}^n$ given by the isomorphism $(\mathbb{A}^1)^n \to \mathbb{A}^n$. Resolve the indeterminacy; i.e., find a projective k-variety J and a birational morphism $J \to (\mathbb{P}^1)^n$ whose composition with $(\mathbb{P}^1)^n \dashrightarrow \mathbb{P}^n$ extends to a morphism $J \to \mathbb{P}^n$ that is an isomorphism above $\mathbb{A}^n$. Define Z to make a cartesian square

$$\begin{array}{ccccc} Z & \longrightarrow & \mathcal{W}^n & & \\ \downarrow & & \downarrow \mu^n & & \\ J & \longrightarrow & (\mathbb{P}^1)^n & \dashrightarrow & \mathbb{P}^n \end{array}$$

where $\mathcal{W} \xrightarrow{\mu} \mathbb{P}^1$ is as in Proposition 7.2. Let be the composition $Z \to J \to \mathbb{P}^n$.

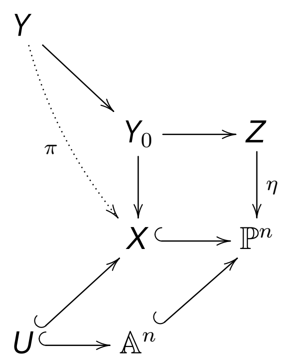
By construction of $\mathcal{W}$, we have $^n(\mathcal{W}^n(k)) = (\mathbb{A}^1)^n(k)$, so the image of $Z(k) \to J(k)$ is contained in the copy of $\mathbb{A}^n$ in J. Therefore $(Z(k)) \subseteq \mathbb{A}^n(k)$.

On the other hand, if $t \in \mathbb{A}^n(k)$, then $J \to (\mathbb{P}^1)^n$ is a local isomorphism above t, and $\mathcal{W}^n \to (\mathbb{P}^1)^n$ is smooth above t, so $Z \to J$ is smooth above t, and the fiber $\eta^{-1}(t)$ is isomorphic to the corresponding fiber of $\mathcal{W}^n \to (\mathbb{P}^1)^n$ so it has a k-point. Thus $\eta(Z(k)) = \mathbb{A}^n(k)$.

Proof of existence in Theorem 1.3. We use strong induction on $\dim X$. The case where X is empty is trivial. We may assume that X is integral; then X is generically smooth, and the non-smooth locus X_{sing} is of lower dimension. Let $U_{\text{sing}} = U \cap X_{\text{sing}}$. The inductive hypothesis gives $\pi_1 \colon Y_1 \to X_{\text{sing}}$ such that $\pi_1(Y_1(k)) = U_{\text{sing}}(k)$. If we prove the conclusion for the smooth open subvariety $U - U_{\text{sing}} \subseteq X$, i.e., if we find $\pi_2 \colon Y_2 \to X$ such that $\pi_2(Y_2(k)) = (U - U_{\text{sing}})(k)$, then the disjoint union $Y_1 \sqcup Y_2$ serves as a Y for $U \subseteq X$. Thus we reduce to the case where U is smooth over k.

If U is a finite union of open subvarieties U_i, then it suffices to prove the conclusion for each $U_i \subseteq X$ and take the disjoint union of the resulting Y's. In particular, by choosing a projective embedding of X and expressing $X - U$ as a finite intersection of hypersurface sections of X, we may reduce to the case where $U = X - D$ for some very ample effective divisor $D \subseteq X$. In other words, we may assume that $X \subseteq \mathbb{P}^n$ and $U = X \cap \mathbb{A}^n$.

Let $Z \to \mathbb{P}^n$ be as in Lemma 8.1. Define Y_0 to make a cartesian diagram



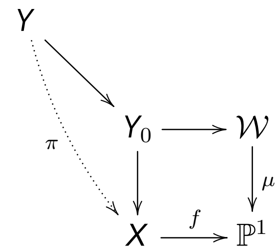
and let $Y \to Y_0$ be a resolution of singularities that is an isomorphism above the smooth locus of Y_0, so Y is a regular projective variety. Let π be the composition $Y \to Y_0 \to X$.

Suppose that $t \in U(k)$. Then $Z \to \mathbb{P}^n$ is smooth above t, by choice of Z. So $Y_0 \to X$ is smooth above t. Moreover, $U \to \operatorname{Spec} k$ is smooth, so $Y_0 \to \operatorname{Spec} k$ is smooth above t. Therefore $Y \to Y_0$ is a local isomorphism above t. Thus $\pi^{-1}(t) \simeq \eta^{-1}(t)$, and the latter has a k-point.

On the other hand, if $t \in X(k) - U(k)$, then $\pi^{-1}(t)$ cannot have a k-point, since such a k-point would map to a k-point of Z lying over $t \in \mathbb{P}^n(k) - \mathbb{A}^n(k)$, contradicting the choice of Z.

Thus $\pi(Y(k)) = U(k)$.

Remark 8.2. In the special case where X is a regular projective curve and U is an affine open subvariety of X, the reductions may be simplified greatly. Namely, using the Riemann-Roch theorem, construct a morphism $f \colon X \to \mathbb{P}^1$ such that $f^{-1}(\infty) = X - U$; now define Y_0 to make a cartesian diagram



and let Y be a resolution of singularities of Y_0.

9. Effectivity

The construction of Y in Theorem 1.3 as given is not e ective, because it used Faltings' theorem. More speci cally, in the proof of Lemma 7.1 we know that $C_i'(k)$ is nite but might not know what it is, so when we reach the last paragraph of the proof, we might not know what the nite set T is, and hence we have no algorithm for computing a good c, where good means that the images of c and $T - \{0\}$ in $k^\times/k^{\times 2}$ do not meet.

Existence of an algorithm for Theorem 1.3. Let F be the (nite) set of $t \in \mathbb{P}^1(k)$ such that $\mathcal{V}_t$ is not smooth. Suppose that instead of requiring that c be good, we require only the e ectively checkable condition that the images of c and F in $k^\times/k^{\times 2}$ do not meet. Then the proof of existence in Theorem 1.3 still yields a regular projective variety Y_c and a morphism $_c$: $Y_c \to X$, but it might not have the desired property $_c(Y_c(k)) = U(k)$. Indeed, in the proof of Proposition 7.2, some of the Châtelet surfaces $\mathcal{W}_t$ other than $\mathcal{W}_\infty$ may be de ned by a reducible degree-4 polynomial and hence may violate the Hasse principle; thus the conclusion $(\mathcal{W}(k)) = \mathbb{A}^1(k)$ in Proposition 7.2 must be weakened to $(\mathcal{W}(k)) \subseteq \mathbb{A}^1(k)$, and this eventually implies $_c(Y_c(k)) \subseteq U(k)$.

On the other hand, an argument ofy ay

exists an open neighborhood U of y in Y that is smooth, or equivalently, geometrically regular [EGA IV_4, IV.17.15.2], which implies geometrically reduced. For an integral variety, being geometrically reduced depends only on the function field [EGA IV_2, IV.4.6.1], so Y is geometrically reduced too.

Combining the two previous paragraphs shows that Y is geometrically integral [EGA IV_2, IV.4.6.2].

Proof of Theorem 1.1(i). Suppose that we want to know whether the k-variety U has a k-point. By passing to a finite open cover, we may assume that U is affine. Let X be a projective closure of U. Construct $Y \to X$ be as in Theorem 1.3. Then U has a k-point if and only if Y has a k-point, so we reduce to the problem of deciding whether a regular projective variety Y has a k-point. Connected components are computable, so we may assume that Y is also connected. Check whether Y is geometrically integral; if so, by assumption we can decide whether Y has a k-point; if not, Lemma 10.1 implies that Y has no k-point.

Proof of Theorem 1.1(ii). We want to compute $\#X(k)$. Apply the algorithm of Theorem 1.1(i) to X. If it says that X has no k-point, we are done. Otherwise, search until a k-point P on X is found, and start over with the variety $X - \{P\}$. If $X(k)$ is finite, this algorithm will eventually terminate. (This kind of argument was used also in [Kim03].)

11. Global function fields

In this section, we investigate whether the proofs of the previous sections carry over to the case where k is a global function field of characteristic $p > 2$.

The main issues are

(1) The two-part paper [CTSSD87a, CTSSD87b], which is key to all our main results, works only over number fields. But it seems likely that the same proofs work, with at most minor modifications, over any global field of characteristic not 2.

(2) The proof of Theorem 1.3 uses resolution of singularities, which is not proved in positive characteristic. Moreover, the proof of Theorem 1.1 uses Theorem 1.3 so it also is in question. Without assuming resolution of singularities, one would obtain the weaker versions of Theorem 1.1 and 1.3 in which the word "regular" is removed from both.

There are a few other issues, but these can be circumvented, as we now discuss.

The proof of Proposition 5.1 works for any global function field k of characteristic not 2: fix a place ∞ of k, let $\mathcal{O}_k$ be the ring of functions that are regular outside ∞, and replace the archimedean and 2-adic conditions on a and b by the condition that a and b be squares in the completion k_∞; then the proof proceeds as before.

The second paragraph of the proof of Lemma 7.1 encounters two problems in positive characteristic: first, it needs $\;_i$ and to be separable, and second, to apply the function field analogue [Sam66] of Faltings' theorem it needs C_i' to be non-isotrivial. As for the first problem, if in Section 6 we choose $\tilde{P}_\infty(w; x)$ to be separable, then the same will be true of $\tilde{P}_\infty + v^2\tilde{P}_0$ over $k(v)$, and the same will be true of the $\;_i$ in the application of Lemma 7.1, since the $\;_i$ correspond to field extensions of $k(v)$ contained in the splitting field of $\tilde{P}_\infty + v^2\tilde{P}_0$ over $k(v)$; moreover, can be made separable simply by choosing n not divisible by p. As for the second problem, the flexibility in the choice of coordinate used to define in the proof of

Lemma 7.1 lets us arrange for C_i' to be non-isotrivial. Moreover, in this case, one can bound not only the number of k-points on each C_i', but also their height [Szp81, §8, Corollaire 2].

There is another thing that is better over global function elds k than over number elds. Namely, by a proved extension of Hilbert's tenth problem to such k [Phe91, Shl92, Vid94, Eis03], it is already known that there is no algorithm for deciding whether a k-variety has a k-point. Therefore, if k is a global function eld of characteristic not 2, and we assume that [CTSSD87a, CTSSD87b] works over k, then there is no algorithm for deciding whether a projective geometrically integral k-variety has a k-point (and if we moreover assume resolution of singularities, we can add the adjective \regular" in this nal statement).

Remark 11.1. Bianca Viray [Vir09] has proved an analogue of Proposition 5.1 for every global function eld of characteristic 2.

12. Open questions

(i) Can one generalize Remark 1.2(f) to show that to have algorithms as in (i) and (ii) of Theorem 1.1 for n-folds, it would su ce to be able to decide the existence of rational points on regular projective geometrically integral $(n + 2)$-folds?
(ii) Is there a proof of Proposition 5.1 that does not require such explicit calculations?
(iii) Is the problem of deciding whether a smooth projective geometrically integral *hypersurface* over k has a k-point also equivalent to the problem for arbitrary k-varieties?

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