COMMENTS ON SERRE, FINITE GROUPS: AN INTRODUCTION

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These are comments on the book Jean-Pierre Serre, *Finite groups: an introduction*, 2016, International Press, ISBN 978-1-57146-327-2 (revised hardcover edition). The blue comments have significant mathematical content. The following people have contributed to this list: Anlong Chua, Peter Mizes, Timothy Ngotiaoco, Bjorn Poonen, Ahaan Rungta, Jean Pierre-Serre, Adam Theriault-Shay, Chase Vogeli.

(1) Definition 1.5: faithful, free, G-torsor should be boldface (not just their first letters).
(2) p.2: It would be nice to replace the "We have" before (1.2) by "By (1.1),"
(3) Chapter 1, Exercise 5: At the end it should say $(H : H') = 2$ instead of $(H : H') = 3$.
(4) Chapter 1, Exercise 7(a): x/h should be h/x.
(5) Chapter 1, Exercise 11(i): Change "$gx_i = y_i$ for $i = 1, 2$" to "$gx_1 = x_2$ and $gy_1 = y_2$".
(6) Chapter 1, Exercise 21: In the final sentence, $\mathcal{S}_q$ should be C_q, a cyclic group of order q.
(7) Section 2.4: In the first sentence, H should be S.
(8) Chapter 2, proof of Proposition 2.11(2): $(G : H)$ should be $(G : S)$.
(9) Chapter 2, Exercise 3(b), middle of the Hint: ordre should be order.
(10) Chapter 2, Exercise 7(b): The last displayed line should be
$$x_m \mapsto x_m + a_m(x_1, \ldots, x_{m-1}),$$
with m instead of n each time it occurs.
(11) Chapter 2, Exercise 8(b): "The of G" should say "The conjugation action of G".
(12) Chapter 2, Exercise 9(a) is wrong: $|J| = 36$.
(13) Chapter 2, Exercise 15(b): There is a double comma after c in the set.
(14) Chapter 2: The use of c in footnote 3 in Exercise 14 conflicts with the use of c in Exercise 15(a).
(15) Chapter 2, Exercise 15(e): In the Hint, there is a condition that a_i is locally conjugate to a_{i+1} for $i = 0$, but a_0 has not been defined. Probably it would be best to require $a_0 = s_1$, as in Theorem 2.19, instead of starting with a_1.
(16) Chapter 2, Exercise 2.16(b): The remark says "This holds more generally when $G = \mathrm{SL}_2(\mathbf{F}_q)$ with $q \equiv 3 \pmod{16}$." The congruence is more restrictive than it needs to be. In fact, if q is a power of an odd prime p, then
    (i) $\mathrm{SL}_2(\mathbf{F}_q)$ contains a subgroup S isomorphic to the quaternion group;
    (ii) the order 4 subgroups of S are conjugate in $N_{\mathrm{SL}_2(\mathbf{F}_q)}(S)$ (total fusion); and
    (iii) if $q \equiv \pm 3 \pmod 8$, then S is a 2-Sylow subgroup of $\mathrm{SL}_2(\mathbf{F}_q)$.
    One way to prove (i) and (ii) is to let $\mathcal{O}$ be the Hurwitz ring of integral quaternions spanned by $1, i, j, k, \frac{1+i+j+k}{2}$, and to consider the injective homomorphism $\mathcal{O}^\times \to (\mathcal{O} \times \mathbf{F}_p)^\times \simeq \mathrm{GL}_2(\mathbf{F}_p)$ for any odd prime p.

Date: October 16, 2018.

(17) Chapter 2, Exercise 16(c1), Hint: Even more direct would be to use Corollary 2.15 instead of Proposition 2.14.

(18) Chapter 2, Exercise 16(c2): The last sentence should say "Show that S acts freely on Z, ...".

(19) End of Section 3.1, Example 1: We have $\mathrm{GL}_2(\mathbf{F}_3)^{\mathrm{ab}} = \mathbf{F}_3^\times$, so the only exceptional case is $n = 2$ and $K \simeq \mathbf{F}_2$.

(20) Corollary 3.3, Proof of (i): The proof for $d\ell(G) > 1$ works also for $d\ell(G) = 1$, so there is no need to break into cases. Instead start with "Choose an abelian subgroup A...".

(21) Section 3.2, Examples, 3., last sentence: It is easier to check that the sequence satisfies condition (2′) of Proposition 3.2 than to check condition (2).

(22) Section 3.3, Proposition 3.9: The translation is not good here. It would be better to put "For each $n > 0$," at the beginning of the sentence, so that this quantifier is clearly outside the "if and only if" statement.

(23) Section 3.4.2: The independence of [,] on the choice of representatives follows from (1.3) (for independence in the second argument) and from (1.3) combined with the formula $(y, x) = (x, y)^{-1}$ (for independence in the first argument). Formula (1.4) does not seem to help.

(24) Section 3.4.2: The same formulas, (1.3) and $(y, x) = (x, y)^{-1}$, are used for bilinearity.

(25) Section 3.6, Theorem 3.18(3): One could remove "proper" to get a cleaner statement. (If $H = G$, it is a sequence with 0 steps: namely, $m = 1$ and $H = H_1 = G$.)

(26) Section 3.6, Theorem 3.18: It is not immediate that (1) implies (3), though the French arXiv edition shows how it can be done using induction and part (5). An easier alternative might be to deduce (3) from Propositions 3.8 and 3.5.

(27) Section 3.6: Corollary 3.19 is a corollary of Proposition 3.17, not of Theorem 3.18. Perhaps include a sentence "Another consequence of Proposition 3.17 is the following:" before Corollary 3.19?

(28) Section 3.6, Theorem 3.20(5): Clearer wording would be "Any two elements...".

(29) Section 3.6, proof of Corollary 3.21: Replace "part (4) of th.3.18" by "part (3) of th.3.18 with $H = 1$". (This is more direct.)

(30) Section 3.7.1, first paragraph: The argument refers to a nonexistent "cor.3.18". Also there is no need to assume that G is nontrivial. Perhaps simplify the paragraph to

> Let K be a field and let L be a finite Galois extension of K whose Galois group G is a 2-group. By part (3) of th.3.18, there is a tower of subgroups from 1 up to G such that each group has index 2 in the next. The corresponding subfields form a tower of quadratic extensions from K up to L.

(31) Section 3.8, proof of Theorem 3.23: $i(g)$ should be $i(s)$ each time it appears.

(32) Section 3.9, proof of Theorem 3.27: To be completely unambiguous in the first sentence, it might be better to write "Theorem 3.26" instead of "the theorem above".

(33) Section 3.9, proof of Proposition 3.29: The equation

$$t(g)e(g)t(g')e(g') = t(gg').e(g)e(g')$$

should be replaced by

$$g.e(g)\ g'.e(g') = gg'.e(gg')$$

(this is the statement that t is a homomorphism).

(34) Section 3.11, Exercise 1: This exact statement was proved in the text as Corollary 3.3(ii). Probably either Corollary 3.3(ii) or this exercise should be removed.

(35) Section 3.11, Exercise 2(iii): "exists" should be "exist".

(36) Section 3.11, Exercise 5: There are two parts both labeled b_1).

(37) Section 3.11, Exercise 5: In the hint to the first b_1), x is used in two different ways.

(38) Section 3.11, Exercise 7: In the first sentence of the first hint, $\lambda \in k$ should be $\lambda \in K$.

(39) Section 3.11, Exercise 7: Starting near the end of the second sentence of the first hint, change "U; this proves (a). As for (e)". to "$\widetilde{U}$; this proves (e). As for (a)".

(40) Section 3.11, Exercise 8(a): The hypothesis that the A_i/A_{i+1} are abelian is not needed.

(41) Section 3.11, Exercise 19: It would be good to clarify whether the reader is expected to prove that all the groups $\mathrm{PSL}_2(\mathbf{F}_q)$ for these q are minimal simple groups, or only to prove that these groups for other prime powers q are not minimal simple groups. (The former seems rather difficult for an exercise.)

(42) Section 4.3: The end of the sentence defining **semidirect product** refers to an action of G of A, but such an action has not been mentioned before. Perhaps one could add a sentence earlier saying "The conjugation action of C on A defines an action of G of A."

(43) Section 4.3: Although there is nothing wrong here, perhaps it is worth emphasizing that a group E having subgroups G and A such that every element of E can written uniquely as ax with $a \in A$ and $g \in G$ is not necessarily a semidirect product of the two groups (for instance, if $n \geq 4$, then $\mathcal{S}_n$ is not a semidirect product of $\mathcal{S}_{n-1}$ and the group generated by an n-cycle); the hypothesis that E is an extension of G by A is part of the definition. Perhaps one could say explicitly that a semidirect product of G by A is the same thing as a split extension of G by A.

(44) Section 4.4, Theorem 4.13: It is not clear what is meant by "or, equivalently, of G".

(45) Section 4.5.1, sentence after the display with ψ: composion should be composition.

(46) Section 4.5.2, third sentence: $\psi\colon G \to \mathrm{Out}(G)$ should be $\psi\colon G \to \mathrm{Out}(A)$, and $G \to \mathrm{Aut}(G)$ should be $G \to \mathrm{Aut}(A)$.

(47) Theorem 4.16: "is and only if" should be "if and only if".

(48) Section 4.5.3, paragraph containing (4.10): The distinctive symbol $\bullet$ is used both (with a subscript) for multiplication in E_f and for the action of $H^2(G, Z(A))$ on $\mathrm{Ext}(G, A, \psi)$. Perhaps consider using a different symbol, such as $*$, for one or the other?

(49) Proof of Theorem 4.20, part II: It might be nice to make the logic more explicit by saying, after the sentence with the Frattini argument,

> "Therefore the homomorphism $E' \to G$ is surjective. Its kernel is $E' \cap A$, which we call A'. Thus we have an exact sequence…".

(50) Proof of Theorem 4.20, part III: The conditions on I are not sufficient for what follows. The sentence defining I should be replaced by "We may assume that $G \neq 1$. Then, by Corollary 3.3(i), G has a nontrivial abelian normal p-subgroup I for some prime p."

(51) Proof of Theorem 4.20, part III: "After replacing I_1 by aI_1a^{-1}" should really say "After replacing G_1 by aG_1a^{-1}".

(52) Proof of Theorem 4.20, part III: After "the induction hypothesis applied to N shows that G_1 and G_2 are conjugate" add "by an element of $A \cap N$".
(53) Proof of Corollary 4.22: It might be nice to make the logic more explicit in the sentence "Since S is contained in...", by replacing it by

"The Frattini argument (prop.2.11(4)) shows that $S.N = G$. On the other hand, p does not divide $|G'|$, so the homomorphism $\varphi\colon G \to G'$ maps the p-group S to 1. Thus $\varphi(N) = \varphi(S.N) = \varphi(G) = G'$."

(54) Chapter 4, Exercise 2(b): The identity should be "$\partial_i \partial_j F = \partial_{j+1} \partial_i F$ if $i \leq j$", or equivalently "$\partial_i \partial_j F = \partial_j \partial_{i-1} F$ if $i > j$".
(55) Chapter 4, Exercise 5, case (ii): "agument" should be "argument", "same image" should be "same images", and "than" should be "as".
(56) Chapter 4, Exercise 7(b), Hint: H should be A.
(57) Chapter 4, Exercise 8: $(1, z)$ should be $(1, nz)$.
(58) Chapter 4, Exercise 8(a): (g, z) should be $(g, 0)$.
(59) Chapter 4, Exercise 16(c), Hint: prop.4.22 should be cor.4.22.
(60) Chapter 5, just before Theorem 5.4: th.8.21 should be th.8.62.
(61) Chapter 5, Proof of Lemma 5.5: In both halves of the proof, one could insert $B^{-1}A^{-1}$ as a step between $(AB)^{-1}$ and BA (though probably most readers can figure this out).
(62) Chapter 5, Proof of Theorem 5.9, definition of H': It might be worth mentioning that despite the notation, this is not necessarily a direct product of groups, but just the image of the cartesian product of the sets.
(63) Chapter 5, Proof of Theorem 5.9: Going from $H' = \prod_{p \neq p_0} H'_p$ to $|H'| = \prod_{p \neq p_0} |H'_p|$ is not as trivial as the notation makes it seem, so it might be worth saying more. One possible argument is this:

Since H' is the image of the cartesian product of the H'_p for $p \neq p_0$, we have $|H'| \leq \prod_{p \neq p_0} |H'_p|$. On the other hand, for each $p \neq p_0$, the group H'_p is a subgroup of H', so $|H'_p|$ divides $|H'|$. The previous two sentences imply that $|H'| = \prod_{p \neq p_0} |H'_p|$.

(64) Chapter 5, Proof of Theorem 5.9: In the sentence after the display, th.4.6 should be th.4.20(1).
(65) Chapter 5, proof of Proposition 5.11(1): p-Sylow should be π-Sylow.
(66) Chapter 5, proof of Theorem 5.13: lemma 5.8 should be cor.5.8.
(67) Section 6.2, Theorem 6.6: In $(1'')$, after "such that", it should say "p divides the residue field degree $[O_K/\mathfrak{Q} : O_k/\mathfrak{q}]$ for every prime ideal $\mathfrak{Q}$ of O_K lying above $\mathfrak{q}$.
(68) Section 6.2: In the "Note" at the end of the section, "Theorem" should be "theorem".
(69) Section 6.5, proof of Theorem 6.13(1)⇒(2): "If p a prime factor..." should be "If p is a prime factor...".
(70) Section 6.5, comment after the proof of Corollary 6.14: "property 6.14" should be "property $\mathcal{F}$".
(71) Chapter 6, Exercise 3: There are two Frobenius groups of order 18, namely the dihedral group $(\mathbf{Z}/9\mathbf{Z}) \rtimes \{\pm 1\}$ and the group $(\mathbf{Z}/3\mathbf{Z})^2 \rtimes \{\pm 1\}$.
(72) Chapter 6, Exercise 4: "as" should be "has".
(73) Section 7.3: In the second sentence, after the semicolon, one could insert "thus by cor.7.3" (and remove the "thus" later in the sentence).

(74) Section 7.6, proof of Corollary 7.19: Insert "minimal" before "odd order $N < 2000$". Otherwise, G could have a cyclic quotient of prime order, so one could not apply th.7.15.
(75) Section 7.6, Proof of Corollary 7.19: In the sentence starting with "Note that", $p_i^{m_i}$ should be $p_1^{m_1}$, and $p = p_i$ should be $p = p_1$, and number should be numbers.
(76) Section 7.7, proof of Theorem 7.23: In two places, P should be $\mathbf{P}$.
(77) Section 7.7, penultimate paragraph of proof of Theorem 7.23: Change "$w(2\lambda) = 4\lambda$, $w(4\lambda) = 2\lambda$" to "$w(2\lambda) = 4$, $w(4\lambda) = 2$".
(78) Chapter 7, Exercise 9(b), Hint: "transfert" should be "transfer".
(79) Chapter 7, Exercise 17(b): It is not clear what is meant by "Hence the action of B/U on $\mathbf{P} - \{\infty\}$ is faithful." since U does not act trivially on $\mathbf{P} - \{\infty\}$. Perhaps remove the sentence, or change it to "Hence the conjugation action of B/U on U is faithful."
(80) Chapter 7, Exercise 17(d): The suggestion to "Use th.7.5 to prove that $\mathrm{Ver}(t) = t.wtw^{-1}$ for some $w \in N_G(T)$" is misleading. A direct application of th.7.5 produces $w \in G$ such that $\mathrm{Ver}(t) = t.w^{-1}tw \bmod D(B)$, so $w^{-1}tw \equiv t^{-1} \bmod D(B)$ (with $D(B) = U$); then one can replace w by wu for some $u \in U$ to obtain a new $w \in G$ such that $w^{-1}tw = t^{-1}$. It seems that only at that point does it become clear that $w \in N_G(T)$.
(81) Chapter 8, Exercise 6(c): This should be replaced by "Construct an example of b) with $|G| = 4$ such that G has 3 orbits on X of sizes $(4, 1, 1)$, and 3 orbits on Y of sizes $(2, 2, 2)$."
(82) Section 9.1.1: Remove the stray symbols +.. after Theorem 9.1.
(83) Section 9.1.3, second sentence: In the text "if x in a non-zero integer", change "in" to "is".
(84) Section 9.2.7, complement 2: "the Lefschetz's principle" should be "the Lefschetz principle".
(85) Section 9.2.7, end of complement 2: It might be nice to mention here also the article
Larsen, Michael J.; Pink, Richard; Finite subgroups of algebraic groups, *J. Amer. Math. Soc.* **24** (2011), no. 4, 1105–1158.
(86) Section 10.1.3: "wigth" should be "with".

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