

THE VALUATION OF THE DISCRIMINANT OF A HYPERSURFACE

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ABSTRACT. Let R be a discrete valuation ring, with valuation $v: R \rightarrow \mathbb{Z} \cup \{\infty\}$ and residue field k . Let H be a hypersurface $\text{Proj } R[x_0, \dots, x_n]/(f)$. Let H_k be the special fiber, and let $(H_k)_{\text{sing}}$ be its singular subscheme. Let $\Delta(f)$ be the discriminant of f . We use Zariski's main theorem and degeneration arguments to prove that $v(\Delta(f)) = 1$ if and only if H is regular and $(H_k)_{\text{sing}}$ consists of a nondegenerate double point over k . We also give lower bounds on $v(\Delta(f))$ when H_k has multiple singularities or a positive-dimensional singularity.

1. INTRODUCTION

Throughout the paper, R denotes a discrete valuation ring, with valuation $v: R \rightarrow \mathbb{Z} \cup \{\infty\}$, maximal ideal $\mathfrak{m} = (\pi)$, and residue field k (except in a few places where k is an arbitrary field).

Let $E \subset \mathbb{P}_R^2$ be defined by a Weierstrass equation, with generic fiber an elliptic curve. If the discriminant of the equation has valuation 1, then E is regular and the singular locus of its special fiber consists of a node; this follows from Tate's algorithm [Tat75], for example; see also [Sil94, Lemma IV.9.5(a)]. Our main theorem (Theorem 1.1) generalizes this to hypersurfaces of arbitrary degree and dimension (terminology will be explained later).

Theorem 1.1. *Let $f \in R[x_0, \dots, x_n]$ be a homogeneous polynomial. Let $\Delta(f)$ be its discriminant. Let $H = \text{Proj } R[x_0, \dots, x_n]/(f)$. Then the following are equivalent:*

- (i) $v(\Delta(f)) = 1$;
- (ii) H is regular, and $(H_k)_{\text{sing}}$ consists of a nondegenerate double point in $H(k)$.

We also prove that if $(H_k)_{\text{sing}}$ consists of r isolated closed points, then $v(\Delta(f)) \geq r$ (Theorem 6.2). If $\dim (H_k)_{\text{sing}} \geq 1$, we show that H_k is a limit of hypersurfaces whose singular subscheme is finite but with many points, and we combine this and an argument using the Greenberg functor to deduce that $v(\Delta(f)) \geq \max(\lfloor (\deg f - 1)/2 \rfloor, 2)$ (Theorem 8.4).

2. DISCRIMINANT

Fix $n \geq 1$ and $d \geq 2$. Let $x^{\mathbf{i}}$ range over the degree d monomials in $\mathbb{Z}[x_0, \dots, x_n]$, and let $a_{\mathbf{i}}$ be independent indeterminates, so that $F := \sum_{\mathbf{i}} a_{\mathbf{i}} x^{\mathbf{i}}$ is the generic degree d homogeneous polynomial in $x_0, \dots, x_n$. Then the affine space $\mathbb{A}^N := \text{Spec } \mathbb{Z}[\{a_{\mathbf{i}}\}]$ may be viewed as a moduli space for hypersurfaces (one could also remove the origin, or projectivize as in [Sai12, §2.4]). Let $\mathcal{H} \subset \mathbb{P}^n \times \mathbb{A}^N$ be the closed subscheme defined by $F = 0$, so the projection $\phi: \mathcal{H} \rightarrow \mathbb{A}^N$ is the **universal hypersurface**. Let $\mathcal{H}_{\text{sing}}$ be the **relative singular subscheme**, the closed subscheme

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defined by $F = \partial F/\partial x_0 = \cdots = \partial F/\partial x_n = 0$. More precisely, $\mathcal{H}_{\text{sing}}$ is the locus of points where ϕ is not smooth of relative dimension $n - 1$.

The other projection $\mathcal{H}_{\text{sing}} \rightarrow \mathbb{P}^n$ is a rank $N - n - 1$ vector bundle since the equations $F = \partial F/\partial x_0 = \cdots = \partial F/\partial x_n = 0$ are linear in the a_i and independent above each point of $\mathbb{P}^n$ except for the Euler relation $d \cdot F = \sum x_i (\partial F/\partial x_i)$. Thus $\mathcal{H}_{\text{sing}}$ is integral and smooth of relative dimension $N - 1$ over $\mathbb{Z}$. Its scheme-theoretic image under the proper morphism ϕ is a closed subscheme $D \subset \mathbb{A}^N$, the locus parametrizing singular hypersurfaces. In fact, $D \subset \mathbb{A}^N$ is a divisor and the restriction $\mathcal{H}_{\text{sing}} \rightarrow D$ of ϕ is birational (cf. [Sai12, §2.9]); this is a Bertini-type statement saying essentially that among hypersurfaces singular at a point, most have singular subscheme consisting of just that point. Thus $D \subset \mathbb{A}^N$ is the zero locus of some polynomial $\Delta \in \mathbb{Z}[\{a_i\}]$ determined up to a unit, i.e., up to sign; Δ is called the **discriminant**. (See [GKZ08, Dem12, Sai12] for other descriptions of Δ .) By definition, if the a_i are specialized to elements of a field k , the resulting hypersurface in $\mathbb{P}_k^n$ is singular (not smooth of dimension $n - 1$) if and only if Δ specializes to 0 in k .

3. QUADRATIC FORMS

Proposition 3.1. *Suppose that $d = 2$. Let $\text{Det} = \det(\partial^2 F/\partial x_i \partial x_j) \in \mathbb{Z}[\{a_i\}]$. If n is odd, then $\Delta = \pm \text{Det}$. If n is even, then $\Delta = \pm \text{Det}/2$.*

Proof. This is well known, except perhaps the power of 2, which can be determined by evaluating Det for a quadratic form defining a smooth quadric over $\mathbb{Z}$, since $\Delta = \pm 1$ for such a form. Use $x_0 x_1 + \cdots + x_{n-1} x_n$ if n is odd, and $x_0 x_1 + \cdots + x_{n-2} x_{n-1} + x_n^2$ if n is even. $\square$

A **symmetric bilinear space** over R is a pair (M, β) where M is a finite-rank projective module R (hence free since R is a discrete valuation ring) and $\beta: M \times M \rightarrow R$ is a symmetric R -bilinear pairing.

Proposition 3.2. *Let R be a discrete valuation ring.*

- (a) *Each symmetric bilinear space over R is an orthogonal direct sum of spaces of rank 1 and 2.*
- (b) *Every quadratic form $f(x_0, \dots, x_n)$ over R is equivalent to one of the form*

$$\sum_{i=1}^I (a_i x_i^2 + b_i x_i y_i + c_i y_i^2) + \sum_{j=1}^J d_j z_j^2$$

with $2I + J = n + 1$ and $a_i, b_i, c_i, d_j \in R$.

- (c) *Let f be as in (b). Let $H = \text{Proj } R[x_0, \dots, x_n]/(f)$. Then $v(\Delta(f)) \geq \dim(H_k)_{\text{sing}} + 1$.*

Proof.

- (a) (We paraphrase an argument of Jean-Pierre Tignol adapted from the proof of [Ver19, Proposition 4.10].) Let (M, β) be a nonzero symmetric bilinear space. We may assume that $\beta \neq 0$. By dividing β by a nonzero element of R , we may assume that $\beta(M, M) \not\subset \mathfrak{m}$. We claim that there exists a free R -module N of rank 1 or 2 with a homomorphism $N \rightarrow M$ such that β induces a **regular** pairing on N (i.e., the composition $N \rightarrow M \xrightarrow{\beta} M^\vee \rightarrow N^\vee$ is an isomorphism); then $N \rightarrow M$ is injective, and M is the orthogonal direct sum of N and $N^\perp := \ker(M \rightarrow N^\vee)$, so we are done by induction on $\text{rank}(M)$.

If there exists $e \in M$ with $\beta(e, e) \in R^\times$ a unit, then let $N = Re$. Otherwise, choose $c, d \in M$ with $\beta(c, d) \in R^\times$ and let $N = Rc \oplus Rd$; the induced pairing is regular since its matrix is invertible, being congruent mod π to $\begin{pmatrix} 0 & \beta(c, d) \\ \beta(c, d) & 0 \end{pmatrix}$.

- (b) Decomposing a quadratic space is equivalent to decomposing the associated symmetric bilinear space, even if $\text{char } k = 2$.
- (c) First suppose $\text{char } k \neq 2$. Then f is equivalent to $\sum a_i x_i^2$ for some $a_i \in R$, and

$$\dim (H_k)_{\text{sing}} = \#\{i : v(a_i) \geq 1\} - 1 \leq v(\text{Det}(f)) - 1 = v(\Delta(f)) - 1,$$

by Proposition 3.1.

Now suppose $\text{char } k = 2$. Let $I_0 = \#\{i : v(b_i) = 0\}$ and $I_1 = \#\{i : v(b_i) \geq 1\}$. Let $J_0 = \#\{j : v(d_j) = 0\}$ and $J_1 = \#\{j : v(d_j) \geq 1\}$. If n is odd, let $J' := J$. If n is even, then J is odd, so let $J' := J - 1$. In both cases $J' \geq 0$. The common zero locus in $\mathbb{P}_k^n$ of the polynomials $\partial f / \partial x_i$ and $\partial f / \partial y_i$ for $i \in I_0$ is of dimension $n - 2I_0$, and including the condition $f = 0$ drops the dimension by 1 more if $J_0 \geq 1$. Thus $\dim (H_k)_{\text{sing}} \leq n - 2I_0$, with strict inequality if $J_0 \geq 1$. On the other hand, $v(4a_i c_i - b_i^2) \geq 2$ whenever $v(b_i) \geq 1$, and $v(2d_j) \geq v(2) + v(d_j)$ for all j , so Proposition 3.1 implies

$$\begin{aligned} v(\Delta(f)) &\geq 2I_1 + J'v(2) + J_1 \\ &= (n - 2I_0) + J'v(2) - J_0 + 1 \\ &\geq \dim (H_k)_{\text{sing}} + J'v(2) - J_0 + 1. \end{aligned}$$

If $J_0 \geq 1$, then the inequality above is strict and $J'v(2) \geq (J_0 - 1)v(2) \geq J_0 - 1$, so $v(\Delta(f)) \geq \dim (H_k)_{\text{sing}} + 1$. If $J_0 = 0$, then instead use $J'v(2) \geq 0$ to again get $v(\Delta(f)) \geq \dim (H_k)_{\text{sing}} + 1$. $\square$

4. NONDEGENERATE DOUBLE POINTS

Definition 4.1 ([SGA 7_I, VI.6]). Let k be a field. Let X be a finite-type k -scheme. A k -point $Q \in X$ is called a **nondegenerate double point** (or **nondegenerate quadratic point**) if there exist $n \geq 1$ and $f \in k[[x_1, \dots, x_n]]$ such that there is an isomorphism of complete k -algebras $\widehat{\mathcal{O}}_{X,Q} \simeq k[[x_1, \dots, x_n]]/(f)$ and an equality of ideals $(\partial f / \partial x_1, \dots, \partial f / \partial x_n) = (x_1, \dots, x_n)$.

Remark 4.2. The ideal equality is equivalent to saying that Q is an isolated reduced point of the singular subscheme X_{sing} .

Remark 4.3. Suppose that n and f exist. Then f can be taken to be a quadratic form [SGA 7_I, VI.6.1]. If, moreover, k is algebraically closed, then

- if $\text{char } k \neq 2$, then one can take $f := x_1^2 + \dots + x_n^2$;
- if $\text{char } k = 2$, then n must be even and one can take $f := x_1 x_2 + x_3 x_4 + \dots + x_{n-1} x_n$.

Remark 4.4 ([SGA 7_I, Definition VI.6.6]). There is also notion of **ordinary double point**, which is the same except that when $\text{char } k = 2$ and n is odd, since nondegeneracy is impossible one allows singularities analytically equivalent over an algebraic closure to the singularity defined by the “least degenerate” quadratic form $f := x_1 x_2 + \dots + x_{n-2} x_{n-1} + x_n^2$.

5. COMMUTATIVE ALGEBRA

A ring extension $R' \supset R$ is called a **weakly unramified extension** if R' too is a discrete valuation ring and π is also a uniformizer of R' .

Lemma 5.1. *For any field extension $k' \supset k$, there exists a weakly unramified extension $R' \supset R$ with residue field k' (i.e., isomorphic to k' as k -algebra).*

Proof. If k'/k is generated by one algebraic element, say a zero of a monic irreducible polynomial $\bar{f} \in k[x]$, then we may take $R' := R[x]/(f)$ for any monic $f \in R[x]$ reducing to $\bar{f}$ [Ser79, I.6, Proposition 15]. If k'/k is generated by one transcendental element t , then we may take the localization $R' := R[t]_{(\pi)}$ of the (regular) polynomial ring $R[t]$ at the codimension 1 prime (π) ; the residue field of R' is $\text{Frac}(R[t]/(\pi)) = k(t)$. The general case follows from Zorn's lemma, using direct limits. $\square$

Lemma 5.2. *Let A be a noetherian local domain. Let $\hat{A}$ be its completion. Let B be the integral closure of A . Then*

$$\#\{\text{minimal primes of } \hat{A}\} \geq \#\{\text{maximal ideals of } B\}.$$

Proof. Combine [SP, Tag 0C24] and [SP, Tag 0C28(1)]. $\square$

6. HYPERSURFACES WITH SEVERAL SINGULARITIES

Let notation be as in Theorem 1.1. We use subscripts to denote base change: e.g., $D_A := D \times_{\text{Spec } \mathbb{Z}} \text{Spec } A$ for any ring A . Restricting ϕ_R yields a proper morphism $\varphi: (\mathcal{H}_R)_{\text{sing}} \rightarrow D_R$.

Proposition 6.1. *The proper morphism $\varphi: (\mathcal{H}_R)_{\text{sing}} \rightarrow D_R$ is birational.*

Proof. This follows from [Sai12, Proposition 2.12] applied over $\text{Frac}(R)$. $\square$

Theorem 6.2. *If the space $(H_k)_{\text{sing}}$ consists of r closed points, then $v(\Delta(f)) \geq r$.*

Proof. Using Lemma 5.1, we may reduce to the case in which k is algebraically closed.

Let $P \in D_R(k)$ correspond to H_k , so $\varphi^{-1}(P) = (H_k)_{\text{sing}}$. Since R is regular, the local ring $\mathcal{O}_{\mathbb{A}_R^N, P}$ is regular, and hence factorial [AB59, Theorem 5].

Let $D' := \{d \in D_R : \dim \varphi^{-1}(d) = 0\}$, so $P \in D'$. By [EGA IV₃, Corollaire 13.1.5], D' is open in D_R . By Proposition 6.1, $\varphi^{-1}(D') \rightarrow D'$ is birational. It is also quasi-finite and proper, hence finite by Zariski's main theorem [EGA III₁, Corollaire 4.4.11]. Moreover, $(\mathcal{H}_R)_{\text{sing}}$ is smooth over a discrete valuation ring, hence normal. The previous three sentences imply that $\varphi^{-1}(D') \rightarrow D'$ is the normalization of D' .

Take $A := \mathcal{O}_{D', P} = \mathcal{O}_{D, P} = \mathcal{O}_{\mathbb{A}_R^N, P}/(\Delta)$, and define $\hat{A}$ and B as in Lemma 5.2. Then the maximal ideals of B correspond to the points of $\varphi^{-1}(D')$ above P , which are the r points of $(H_k)_{\text{sing}}$. Lemma 5.2 implies that $\hat{A}$ has at least r minimal primes. Their inverse images in $\mathcal{O}_{\mathbb{A}_R^N, P}$, correspond to prime factors of Δ in this factorial ring, so $\Delta = p_1 \cdots p_r q$, for some $p_1, \dots, p_r, q \in \mathcal{O}_{\mathbb{A}_R^N, P}$ with each p_i vanishing at P . Evaluating both sides at (the coefficient tuple of) f shows that $v(\Delta(f)) \geq 1 + \cdots + 1 + 0 = r$. $\square$

7. VALUATIONS OF POLYNOMIAL VALUES

Lemma 7.1. *Suppose that k is infinite, and $\ell \geq n$. Let $\rho: \mathbb{A}_k^\ell \rightarrow \mathbb{A}_k^n$ be a projection. Let $V \subset \mathbb{A}_k^\ell$ be a closed subscheme. Then $\{a \in k^n : \rho^{-1}(a)(k) \subseteq V(k)\}$ is the set of k -points of a closed subscheme $Z \subseteq \mathbb{A}_k^n$.*

Proof. Since k is infinite, $\rho^{-1}(a)(k) \subset V(k)$ is equivalent to $\rho^{-1}(a) \subset V$, which fails if and only if $a \in \rho(\mathbb{A}_k^n - V)$. Since ρ is flat, ρ is open, so $\rho(\mathbb{A}_k^n - V)$ is open; let Z be its complement. $\square$

For $b \in R$, let $\bar{b}$ be its image in k . Likewise, given $b \in R^n$, define $\bar{b} \in k^n$.

Proposition 7.2. *Let $\delta \in R[x_1, \dots, x_n]$ and $m \in \mathbb{Z}_{\geq 0}$. If k is infinite and perfect, then*

$$\{a \in k^n : v(\delta(b)) \geq m \text{ for all } b \in R^n \text{ with } \bar{b} = a\}$$

is the set of k -points of a closed subscheme of $\mathbb{A}_k^n$.

Proof. The m th Greenberg functor Gr^m satisfies $\text{Gr}^m(X)(k) = X(R/\mathfrak{m}^m)$ for any R -scheme X ; see [Gre61; Gre63; NS08, §2.2; BGA18]. Applying Gr_m to $\delta: \mathbb{A}_R^n \rightarrow \mathbb{A}_R^1$ yields a morphism

$$\text{Gr}^m(\mathbb{A}_R^n) \longrightarrow \text{Gr}^m(\mathbb{A}_R^1);$$

let V be the fiber above 0. On the other hand, the reduction map $R/\mathfrak{m}^m \rightarrow k$ induces a morphism $\rho: \text{Gr}^m(\mathbb{A}_R^n) \rightarrow \text{Gr}^1(\mathbb{A}_R^n)$ that is a projection $\mathbb{A}_k^{mn} \rightarrow \mathbb{A}_k^n$ as in Lemma 7.1. For $a \in k^n$,

$$v(\delta(b)) \geq m \text{ for all } b \in R^n \text{ with } \bar{b} = a \iff \rho^{-1}(a)(k) \subset V(k),$$

so the result follows from Lemma 7.1. $\square$

8. HYPERSURFACES WITH A POSITIVE-DIMENSIONAL SINGULARITY

In Lemma 8.1, Corollary 8.2, and Lemma 8.3, we assume that $n \geq 2$, $r \geq 1$, and $P_1, \dots, P_r$ are distinct points in $\mathbb{P}^n(k)$. Let $\mathcal{O} = \mathcal{O}_{\mathbb{P}^n}$. For each $P \in \mathbb{P}^n(k)$, let $\mathfrak{m}_P \subset \mathcal{O}$ be the ideal sheaf of P .

Lemma 8.1. *If $d \geq 2r - 1$, then $\mathcal{O}(d) \rightarrow \prod_i (\mathcal{O}/\mathfrak{m}_{P_i}^2)(d)$ induces a surjection on global sections.*

Proof. Let ℓ_i be a linear form vanishing at P_i but not P_j for any $j \neq i$. Let h be a homogeneous polynomial of degree $d - (2r - 1)$ not vanishing at any P_i . For each s , as g ranges over linear forms, the image of g in $(\mathcal{O}/\mathfrak{m}_{P_s}^2)(1)$ ranges over all its sections, so the images of $gh \prod_{j \neq s} \ell_j^2$ in $\prod_i (\mathcal{O}/\mathfrak{m}_{P_i}^2)(d)$ exhaust the s th factor of $\prod_i (\mathcal{O}/\mathfrak{m}_{P_i}^2)(d)$. $\square$

Corollary 8.2. *Let $N = \dim_k \Gamma(\mathbb{P}^n, \mathcal{O}(d))$. For $f \in \Gamma(\mathbb{P}^n, \mathcal{O}(d))$, let $H_f := \text{Proj } k[x_0, \dots, x_n]/(f)$. Then the f for which $(H_f)_{\text{sing}} \supset \{P_1, \dots, P_r\}$ form a vector space of dimension $N - r(n + 1)$.*

Lemma 8.3. *If $d \geq 3$ and $1 \leq r \leq \max((d - 1)/2, 2)$, then in the locus $\mathbb{A}$ of f for which $(H_f)_{\text{sing}} \supset \{P_1, \dots, P_r\}$, the open sublocus U for which $(H_f)_{\text{sing}}$ is finite is dense.*

Proof. Since $\mathbb{A}$ is defined by the vanishing of values of f and its partial derivatives at the P_i , it is cut out by linear forms in the coefficients of f , so $\mathbb{A}$ is an affine space. Applying [EGA IV₃, Corollaire 13.1.5] the relative singular subscheme over $\mathbb{A}$ shows that U is open in $\mathbb{A}$, so it remains to show that $U \neq \emptyset$.

First suppose that $r \leq (d - 1)/2$. Let

$$I = \{(f, P_{r+1}) : f \in \mathbb{A}, P_{r+1} \in (H_f)_{\text{sing}} - \{P_1, \dots, P_r\}\}.$$

The fiber of $I \rightarrow \mathbb{P}_k^n - \{P_1, \dots, P_r\}$ above P_{r+1} consists of the f for which $(H_f)_{\text{sing}} \supset \{P_1, \dots, P_{r+1}\}$, so its dimension is $N - (r+1)(n+1)$ by Corollary 8.2; similarly, $\dim \mathbb{A} = N - r(n+1)$. Thus $\dim I = n + N - (r+1)(n+1) = \dim \mathbb{A} - 1$. Therefore $I \rightarrow \mathbb{A}$ is not dominant, and U contains the complement of its image.

Now suppose instead that $r \leq 2$. Choose a homogeneous degree d form $g(x_3, \dots, x_n)$ defining a smooth hypersurface in $\mathbb{P}^{n-3}$, let $c_1, \dots, c_{d-1} \in k$ be distinct (enlarge k if necessary), and let

$$f = x_0 \prod_{i=1}^{d-1} (x_1 - c_i x_2) + g.$$

At a point P where f and its partial derivatives vanish, $\prod_{i=1}^{d-1} (x_1 - c_i x_2) = 0$, so $g = 0$, so g and its derivatives vanish, so $x_3 = \dots = x_n = 0$; thus P is a singular point of the plane curve $x_0 \prod_{i=1}^{d-1} (x_1 - c_i x_2) = 0$, i.e., an intersection point of two components. By a linear change of variable, we may assume that the P_i (of which there are at most two!) are among these singular points. Then f gives a k -point of U . $\square$

Theorem 8.4. *Let $H = \text{Proj } R[x_0, \dots, x_n]/(f)$ for some homogeneous f of degree d . If $\dim (H_k)_{\text{sing}} \geq 1$, then $v(\Delta(f)) \geq \max(\lfloor (d-1)/2 \rfloor, 2)$.*

Proof. We may assume that $n, d \geq 2$. Using Lemma 5.1, we may reduce to the case in which k is algebraically closed. If $d = 2$, then Proposition 3.2(c) implies that $v(\Delta(f)) \geq \dim (H_k)_{\text{sing}} + 1 \geq 2$.

So assume $d \geq 3$. Let Z be the closed subscheme of Proposition 7.2 for $\delta := \Delta \in R[\{a_i\}]$ and $r := \max(\lfloor (d-1)/2 \rfloor, 2)$. Choose distinct $P_1, \dots, P_r \in (H_k)_{\text{sing}}(k)$. If $j \in R[x_0, \dots, x_n]$ is a degree d homogeneous polynomial, and $J = \text{Proj } R[x_0, \dots, x_n]/(j)$ is such that $(J_k)_{\text{sing}} = \{P_1, \dots, P_r\}$, then $v(\Delta(j)) \geq r$ by Theorem 6.2, so the corresponding coefficient tuple mod $\mathfrak{m}$ belongs to $Z(k)$. By Lemma 8.3, any coefficient tuple mod $\mathfrak{m}$ corresponding to a hypersurface whose singular locus contains $\{P_1, \dots, P_r\}$ also belongs to $Z(k)$. This applies in particular to the coefficient tuple of f mod $\mathfrak{m}$, so $v(\Delta(m)) \geq r$ by definition of Z . $\square$

9. WHEN THE DISCRIMINANT HAS VALUATION 1

Proof of Theorem 1.1. Case 1: $\text{char } k = 2$ and n is odd. By [Sai12, Theorem 4.2], if the sign of Δ is chosen appropriately, then $\Delta = A^2 + 4B$ for some polynomials A, B , so $v(\Delta(f)) \neq 1$. On the other hand, by Remark 4.3, H_k cannot have a nondegenerate double point. Thus (i) and (ii) both fail.

Case 2: $\text{char } k \neq 2$ or n is even. The hypersurface $H \rightarrow \text{Spec } R$ is the pullback of $\mathcal{H}_R \rightarrow \mathbb{A}_R^N$ by some R -morphism $\iota: \text{Spec } R \rightarrow \mathbb{A}_R^N$. Let $P = \iota(\text{Spec } k) \in \mathbb{A}^N(k)$.

(i) $\Rightarrow$ (ii): Suppose that $v(\Delta(f)) = 1$. By Theorem 8.4, $(H_k)_{\text{sing}}$ is finite. The surjection $R[\{a_i\}] \twoheadrightarrow R$ sending the a_i to the corresponding coefficients α_i of f maps Δ to $\Delta(f)$, so the $a_i - \alpha_i$ and Δ are local parameters for $\mathbb{A}_R^N$ at P . Thus $D_R = \text{Spec } R[\{a_i\}]/(\Delta)$ is regular at P , so D_R is normal at P . Let U be the largest normal open subscheme of D_R such that $\varphi^{-1}U \rightarrow U$ has finite fibers. The fiber above P is $(H_k)_{\text{sing}}$, so $P \in U$. By Proposition 6.1, φ is a proper birational morphism, so $\varphi^{-1}U \rightarrow U$ has finite fibers by Zariski's main theorem [EGA III₁, Corollaire 4.4.9]. In particular, the fiber $(H_k)_{\text{sing}}$ consists of a single reduced k -point Q . By Remark 4.2, Q is a nondegenerate double point of H_k .

Choose an $\mathbb{A}_R^n \subset \mathbb{P}_R^n$ containing Q ; let f_0 be the corresponding dehomogenization of f . The point $(H_k)_{\text{sing}}$ is cut out in $\mathbb{A}_R^n$ by f_0 and its partial derivatives; these $n+1$ functions are

therefore local parameters for $\mathbb{P}_R^n$ at Q , so the local ring $\mathcal{O}_{H,Q} = \mathcal{O}_{\mathbb{P}_R^n,Q}/(f_0)$ is regular too. On the other hand, $H - \{Q\}$ is smooth over $\text{Spec } R$. Thus H is regular everywhere.

(ii) $\Rightarrow$ (i): Now suppose that H is regular and $(H_k)_{\text{sing}}$ consists of a nondegenerate double point $Q \in H(k)$. Hence the underlying space of H_{sing} is $\{Q\}$.

Since the tangent space of $(H_k)_{\text{sing}}$ at Q is 0, the projection $(\mathcal{H}_k)_{\text{sing}} \rightarrow \mathbb{A}_k^N$ induces an injection between the tangent spaces at Q and P . Since Q is the only point in $(\mathcal{H}_k)_{\text{sing}}$ above P , this implies that $(\mathcal{H}_R)_{\text{sing}} \rightarrow D_R$ is étale at Q . Pulling back $(\mathcal{H}_R)_{\text{sing}} \rightarrow D_R \hookrightarrow \mathbb{A}_R^N$ by ι shows that $H_{\text{sing}} \rightarrow \text{Spec}(R/(\Delta(f)))$ is étale. These are connected 0-dimensional schemes with the same residue field, so $H_{\text{sing}} \simeq \text{Spec}(R/(\Delta(f)))$.

Let f_0 be as above, so f_0 and its partial derivatives lie in the maximal ideal $\mathfrak{m}_{\mathbb{P}_R^n,Q} \subset \mathcal{O}_{\mathbb{P}_R^n,Q}/(f_0)$. The partial derivatives are independent in $\mathfrak{m}_{\mathbb{P}_R^n,Q}/\mathfrak{m}_{\mathbb{P}_R^n,Q}^2$ since they form a basis for $\mathfrak{m}_{\mathbb{P}_k^n,Q}/\mathfrak{m}_{\mathbb{P}_k^n,Q}^2$, since Q is a nondegenerate double point. On the other hand, the image of f_0 in $\mathfrak{m}_{\mathbb{P}_R^n,Q}/\mathfrak{m}_{\mathbb{P}_R^n,Q}^2$ is nonzero (since $\mathcal{O}_{H,Q} = \mathcal{O}_{\mathbb{P}_R^n,Q}/(f_0)$ is regular) and in fact *independent* of the partial derivatives (since it maps to 0 in $\mathfrak{m}_{\mathbb{P}_k^n,Q}/\mathfrak{m}_{\mathbb{P}_k^n,Q}^2$). Thus f_0 and its partial derivatives form a basis of $\mathfrak{m}_{\mathbb{P}_R^n,Q}/\mathfrak{m}_{\mathbb{P}_R^n,Q}^2$, so by Nakayama's lemma, they generate $\mathfrak{m}_{\mathbb{P}_R^n,Q}$, so $H_{\text{sing}} \simeq \text{Spec } k$.

The conclusions of the two previous paragraphs imply $v(\Delta(f)) = 1$. $\square$

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