

# JORDAN–HÖLDER THEOREM

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The following exposition is based on [Ser16, §1.3] and a discussion with Christopher Xu. A finite filtration of a group  $G$  is a descending sequence of subgroups

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = 1,$$

where  $G_i \triangleright G_{i+1}$  denotes that  $G_{i+1}$  is a normal subgroup of  $G_i$ ; then define  $\text{gr}_i(G) := G_i/G_{i+1}$ . The filtration is a **Jordan–Hölder filtration** of length  $n$  if each  $\text{gr}_i(G)$  is simple; then the multiset  $\{\text{gr}_i(G) : 0 \leq i < n\}$  records the multiplicity of each isomorphism type among the  $\text{gr}_i(G)$ .

**Jordan–Hölder theorem.** *Let  $G$  be a group. Every Jordan–Hölder filtration of  $G$  has the same multiset  $\{\text{gr}_i(G)\}$  of composition factors.*

*Proof.* We may assume  $G \neq 1$ , and that  $G$  has a Jordan–Hölder filtration; let  $\ell(G)$  denote the length of a shortest one. We use induction on  $\ell(G)$ .

*Base case:*  $\ell(G) = 1$ . Then the only Jordan–Hölder filtration is  $1 \triangleleft G$ .

*Inductive step:* Suppose that  $\ell(G) \geq 2$ . Let  $N$  be the group just below  $G$  in a shortest Jordan–Hölder filtration. Thus  $N \triangleleft G$ , and  $\ell(N)$  and  $\ell(G/N)$  are strictly less than  $\ell(G)$ .

Let  $(G_i)$  be *any* Jordan–Hölder filtration of  $G$ . It induces filtrations on  $N$  and  $G/N$ : namely,  $N_i := G_i \cap N$  and  $(G/N)_i := \text{im}(G_i \rightarrow G/N)$ . Applying the snake lemma to

$$\begin{array}{ccccccc} 1 & \longrightarrow & N_{i+1} & \longrightarrow & G_{i+1} & \longrightarrow & (G/N)_{i+1} \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & N_i & \longrightarrow & G_i & \longrightarrow & (G/N)_i \longrightarrow 1 \end{array}$$

yields an exact sequence

$$1 \longrightarrow \text{gr}_i(N) \longrightarrow \text{gr}_i(G) \longrightarrow \text{gr}_i(G/N) \longrightarrow 1.$$

Since  $\text{gr}_i(G)$  is simple, one of  $\text{gr}_i(N)$  and  $\text{gr}_i(G/N)$  is 1 and the other is isomorphic to  $\text{gr}_i(G)$ . In particular,  $(N_i)$  with duplicates removed is a Jordan–Hölder filtration of  $N$ , and likewise for  $((G/N)_i)$  and  $G/N$ , and

$$\{\text{gr}_i(G)\} = \{\text{gr}_i(N)\} \amalg \{\text{gr}_i(G/N)\}$$

as multisets, where it is understood that on the right we discard all quotients isomorphic to 1. By the inductive hypothesis,  $\{\text{gr}_i(N)\}$  and  $\{\text{gr}_i(G/N)\}$  are each independent of the choice of filtration  $(G_i)$ , so  $\{\text{gr}_i(G)\}$  is too.  $\square$

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*Date:* November 29, 2020.

The writing was supported in part by National Science Foundation grant DMS-1601946 and Simons Foundation grants #402472 (to Bjorn Poonen) and #550033.

## REFERENCES

- [Ser16] Jean-Pierre Serre, *Finite groups: an introduction*, Surveys of Modern Mathematics, vol. 10, International Press, Somerville, MA; Higher Education Press, Beijing, 2016. With assistance in translation provided by Garving K. Luli and Pin Yu. MR3469786 ↑1

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