

EXTENDING SELF-MAPS TO PROJECTIVE SPACE OVER FINITE FIELDS

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ABSTRACT. Using the closed point sieve, we extend to finite fields the following theorem proved by A. Bhatnagar and L. Szpiro over infinite fields: if X is a closed subscheme of $\mathbb{P}^n$ over a field, and $\phi: X \rightarrow X$ satisfies $\phi^*\mathcal{O}_X(1) \simeq \mathcal{O}_X(d)$ for some $d \geq 2$, then there exists $r \geq 1$ such that ϕ^r extends to a morphism $\mathbb{P}^n \rightarrow \mathbb{P}^n$.

1. INTRODUCTION

Let k be a field. Given a closed subscheme $X \subseteq \mathbb{P}^n$ over k , and given a self-map (i.e., k -scheme endomorphism) $\phi: X \rightarrow X$, does ϕ extend to a self-map $\psi: \mathbb{P}^n \rightarrow \mathbb{P}^n$? Such questions have applications in arithmetic dynamics: for instance, [Fak03, Corollary 2.4] uses a positive answer to a variant of this to show that the Morton–Silverman uniform boundedness conjecture for preperiodic points of a self-map of projective space over a number field [MS94, p. 100] implies the uniform boundedness conjecture for torsion points on abelian varieties over a number field.

If the extension ψ exists, then $\psi^*\mathcal{O}(1) \simeq \mathcal{O}(d)$ for some integer d , and then $\phi^*\mathcal{O}_X(1) \simeq \mathcal{O}_X(d)$. But A. Bhatnagar and L. Szpiro [BS12, Proposition 2.3] gave an example showing that the existence of d such that $\phi^*\mathcal{O}_X(1) \simeq \mathcal{O}_X(d)$ is not sufficient for the extension ψ to exist.

To obtain an extension theorem, one can relax the requirements. Two ways of doing this lead to the following questions:

Question 1.1 (Changing the embedding). Let X be a projective k -scheme. Let $\mathcal{L}$ be an ample line bundle on X . Let $\phi: X \rightarrow X$ be a morphism such that $\phi^*\mathcal{L} \simeq \mathcal{L}^{\otimes d}$ for some $d \geq 1$. Does there exist a closed immersion $X \hookrightarrow \mathbb{P}^n$ such that ϕ extends to a morphism $\mathbb{P}^n \rightarrow \mathbb{P}^n$?

Question 1.2 (Replacing the self-map by a power). Let X be a closed subscheme of $\mathbb{P}^n$ over k . Let $\phi: X \rightarrow X$ be a morphism such that $\phi^*\mathcal{O}_X(1) \simeq \mathcal{O}_X(d)$ for some $d \geq 2$. Then there exists $r \geq 1$ such that ϕ^r extends to a morphism $\mathbb{P}^n \rightarrow \mathbb{P}^n$.

Remark 1.3. Section 4 explains why we cannot allow $d = 1$ in Question 1.2.

Suppose that k is infinite. Then the answer to both questions is yes: see [Fak03, Corollary 2.3] and [BS12, Theorem 2.1], respectively (in the proof of the latter, one should replace the prime avoidance lemma there by the lemma used in [Fak03], that a finite union of proper

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subspaces in a vector space over an infinite field cannot cover the whole space). A positive answer to Question 1.2 is also an immediate consequence of [Fak03, Proposition 2.1] if one notices that the statement and proof there remain valid if hypothesis (1) is imposed only for $n = d$ instead of all $n \geq 0$. (The word “variety” in [Fak03] and [BS12] may be read as “scheme of finite type”, so there is no difference between “projective variety” and “projective scheme”.)

Our main result is the following:

Theorem 1.4. *Question 1.2 has a positive answer over any field k .*

In the case where k is finite, the general position arguments in [Fak03] and [BS12] fail, so a new idea is needed. To prove Theorem 1.4, we use the closed point sieve introduced in [Poo04] to show that a *random* choice leads to an extension of ϕ , even though we cannot exhibit one explicitly. As far as we know, this is the first time that sieve techniques have been applied to a problem in dynamics.

Remark 1.5. See [MZMS13, Theorem 3] for an analogous statement on self-maps of equicharacteristic complete local rings.

Remark 1.6. We still do not know if Question 1.1 has a positive answer when k is finite.

2. EXTENDING MORPHISMS TO PROJECTIVE SPACE

The finite field case of Theorem 1.4 will be proved with the aid of the following quantitative theorem, involving a zeta function $\zeta_U(s)$ defined as in [Poo04]:

Theorem 2.1. *Let k be a finite field $\mathbb{F}_q$. Fix a closed subscheme X of $\mathbb{P}^n = \text{Proj } S$ over k . Let $U := \mathbb{P}^n - X$. Let $I = \bigoplus_{d \geq 0} I_d \subseteq S = \bigoplus_{d \geq 0} S_d$ be the homogeneous ideal of $X \subseteq \mathbb{P}^n$. Let $N \geq n$. Fix $f_0, \dots, f_N \in S_d$. Then if $g_0, \dots, g_N$ are chosen independently and uniformly at random from the finite set I_d ,*

$$\text{Prob}(f_0 + g_0, \dots, f_N + g_N \text{ have no common zeros on } U) = \zeta_U(N+1)^{-1} + o(1),$$

where the $o(1)$ is bounded by a function of k , X , n , N , and d that tends to 0 as $d \rightarrow \infty$ while k , X , n , and N are fixed.

Theorem 2.1 will be proved in Section 3. For now, we show how it implies Theorem 1.4, through the following:

Theorem 2.2. *Fix a closed subscheme X of $\mathbb{P}^n$ over a field k . If d is sufficiently large and $N \geq n$, then any morphism $\phi: X \rightarrow \mathbb{P}^N$ such that $\phi^* \mathcal{O}(1) \simeq \mathcal{O}_X(d)$ extends to a morphism $\mathbb{P}^n \rightarrow \mathbb{P}^N$.*

Proof. Let $z_0, \dots, z_N$ be the homogeneous coordinates on $\mathbb{P}^N$. For sufficiently large d , the restriction map $S_d = \Gamma(\mathbb{P}^n, \mathcal{O}(d)) \rightarrow \Gamma(X, \mathcal{O}_X(d))$ is surjective. So each $\phi^*(z_i)$ is the restriction of some $f_i \in S_d$.

If k is infinite, the proof of [Fak03, Proposition 2.1] applies for any d that is moreover large enough that X is cut out in $\mathbb{P}^n$ by homogeneous polynomials of degree at most d .

If k is finite, Theorem 2.1 implies that for sufficiently large d , there exist $g_0, \dots, g_N \in I_d$ such that $f_0 + g_0, \dots, f_N + g_N$ have no common zeros in $\mathbb{P}^n - X$. On the other hand, restricted to X , they define the same map ϕ as $f_0, \dots, f_N$ do, so they have no common zeros on X either. Thus $f_0 + g_0, \dots, f_N + g_N$ define a morphism $\mathbb{P}^n \rightarrow \mathbb{P}^N$ extending ϕ . $\square$

Proof of Theorem 1.4. Apply Theorem 2.2 with $N = n$ and with ϕ equal to a sufficiently large power of the ϕ given in Theorem 1.4. $\square$

3. PROOF OF THEOREM 2.1

The idea of the proof of Theorem 2.1, borrowed from [Poo04], is to sieve out, for each closed point $P \in U$, the $(g_0, \dots, g_N)$ for which $f_0 + g_0, \dots, f_N + g_N$ have a common zero at P . Heuristically, the probability that a given $f_i + g_i$ vanishes at P is $q^{-\deg P}$, so, assuming independence, the probability that $f_0 + g_0, \dots, f_N + g_N$ have no common zeros on U should be

$$\prod_{\text{closed } P \in U} (1 - q^{-(N+1)\deg P}) = \zeta_U(N+1)^{-1}.$$

But independence holds only for finitely many P , so to make this rigorous, we impose the conditions only for P of degree up to some bound ρ , and then prove that the number of $(g_0, \dots, g_N)$ sieved out by higher-degree closed points is negligible.

3.1. Points of low degree. Let $f = (f_0, \dots, f_N)$ and $g = (g_0, \dots, g_N)$. Let $V(f + g)$ be the common zero locus of the $f_i + g_i$. Given $\rho \in \mathbb{Z}_{>0}$ and a k -scheme Z , let $Z_{<\rho}$ be the set of closed points of Z of degree less than ρ , and define $Z_{>\rho}$ similarly.

Lemma 3.1 (Points of low degree). *For fixed ρ , if d is sufficiently large, then*

$$\text{Prob}(V(f + g) \cap U_{<\rho} = \emptyset) = \prod_{P \in U_{<\rho}} (1 - q^{-(N+1)\deg P}).$$

Proof. Let $\mathcal{I}$ be the ideal sheaf of $X \subseteq \mathbb{P}^n$. View $U_{<\rho}$ as a 0-dimensional closed subscheme of $\mathbb{P}^n$. By [Poo08, Lemma 2.1], if d is sufficiently large, then the restriction map $I_d \rightarrow \Gamma(U_{<\rho}, \mathcal{I} \cdot \mathcal{O}_{U_{<\rho}}(d))$ is surjective. In particular, for each i , the tuple of “values” $((f_i + g_i)(P))_{P \in U_{<\rho}}$ is equidistributed. The residue field at P has size $q^{\deg P}$, so the probability that $f + g$ vanishes at P is $q^{-(N+1)\deg P}$, and the probability that $f + g$ is nonvanishing at all $P \in U_{<\rho}$ is

$$\prod_{P \in U_{<\rho}} (1 - q^{-(N+1)\deg P}). \quad \square$$

3.2. Points of medium degree. Let $U_{a \leq ? \leq b}$ be the set of closed points of U of degree between a and b . As in [Poo08, Section 2], fix c so that $S_1 I_m = I_{m+1}$ for all $m \geq c$.

Lemma 3.2 (Points of medium degree). *If d is sufficiently large, then*

$$\text{Prob}(V(f + g) \cap U_{\rho \leq ? \leq d-c} = \emptyset) = O(q^{-\rho}).$$

Proof. By [Poo08, Lemma 2.2], the fraction of $h \in I_d$ vanishing at a closed point P of degree $e \in [\rho, d - c]$ is at most $q^{-\min(d-c, e)} = q^{-e}$. The set of $g_i \in I_d$ such that $f_i + g_i$ vanishes at P is either empty or a coset of this set of polynomials h , so $\text{Prob}(f_i + g_i \text{ vanishes at } P) \leq q^{-e}$. Hence $\text{Prob}(f + g \text{ vanishes at } P) \leq q^{-(N+1)e}$. Summing over all $P \in U_{\rho \leq ? \leq d-c}$ and using the trivial bound that U contains $O(q^{Ne})$ closed points of degree e yields

$$\sum_{e=\rho}^{d-c} O(q^{Ne}) q^{-(N+1)e} = O(q^{-\rho}). \quad \square$$

3.3. Points of high degree.

Lemma 3.3. *Given a closed subvariety $Z \subset \mathbb{P}^n$ such that $\dim Z \cap U > 0$, the probability that a random $h \in I_d$ vanishes identically on Z is at most $q^{-(d-c)}$.*

Proof. Choose $P \in (Z \cap U)_{>d-c}$. If h vanishes on Z , it vanishes at P . By [Poo08, Lemma 4.1], $\text{Prob}(h(P) = 0) \leq q^{-(d-c)}$. $\square$

Lemma 3.4 (Points of high degree). *We have*

$$\text{Prob}(V(f+g) \cap U_{>d-c} = \emptyset) = 1 - o(1)$$

as $d \rightarrow \infty$.

Proof. Let $W_{-1} = \mathbb{P}^n$. For $i = 0, \dots, N$, let W_i be the common zero locus of $f_0 + g_0, \dots, f_i + g_i$. We pick $g_0, \dots, g_N$ randomly one at a time.

Claim 1: For $i = -1, \dots, n-2$, conditioned on a choice of $g_0, \dots, g_i$ for which $\dim W_i \cap U = n - i - 1$, the probability that $\dim W_{i+1} \cap U = n - i - 2$ is $1 - o(1)$ as $d \rightarrow \infty$.

Proof of Claim 1: We have $\dim W_{i+1} \cap U = n - i - 2$ if $f_{i+1} + g_{i+1}$ does not vanish identically on any irreducible component of $W_i \cap U$. The number of such components is at most the number of components of W_i , which, by Bézout's theorem as in [Ful84, p. 10], is at most $O(d^{i+1})$. For each component Z meeting U , the set of g_{i+1} such that $f_{i+1} + g_{i+1}$ vanishes identically on Z is either empty or a coset of the subspace of $h \in I_d$ vanishing identically on Z , and the probability that h vanishes on Z is at most $q^{-(d-c)}$, by Lemma 3.3. Thus the desired probability is at least $1 - O(d^{i+1})q^{-(d-c)} = 1 - o(1)$.

Claim 2: Conditioned on a choice of $g_0, \dots, g_{n-1}$ for which $\dim W_{n-1} \cap U$ is finite, $\text{Prob}(W_n \cap U_{>d-c} = \emptyset) = 1 - o(1)$ as $d \rightarrow \infty$.

Proof of Claim 2: By Bézout's theorem again, $\#(W_{n-1} \cap U) = O(d^n)$. For each $P \in W_{n-1} \cap U$, the set of $g_n \in I_d$ such that $f_n + g_n$ vanishes at P is either empty or a coset of the subspace of $h \in I_d$ vanishing at P . If, moreover, $\deg P > d - c$, then $\text{Prob}(h(P) = 0) \leq q^{-(d-c)}$ by [Poo08, Lemma 4.1]. Thus the desired probability is at least $1 - O(d^n)q^{-(d-c)} = 1 - o(1)$ as $d \rightarrow \infty$.

Applying Claim 1 inductively and finally Claim 2 shows that with probability $1 - o(1)$, we have $W_n \cap U_{>d-c} = \emptyset$ and hence also $V(f+g) \cap U_{>d-c} = \emptyset$ since $V(f+g) \subseteq W_n$. $\square$

3.4. End of proof. Combining Lemmas 3.1, 3.2, and 3.4 shows that for any $\rho \in \mathbb{Z}_{>0}$,

$$\text{Prob}(V(f+g) \cap U = \emptyset) = \prod_{P \in U_{<\rho}} (1 - q^{-(N+1)\deg P}) - O(q^{-\rho}) - o(1)$$

as $d \rightarrow \infty$. Applying this to larger and larger ρ completes the proof of Theorem 2.1.

4. A COUNTEREXAMPLE

Here we show that Question 1.2 has a negative answer if we allow $d = 1$, even for projective integral varieties over $k = \mathbb{C}$. Our counterexample is inspired by [BS12, Proposition 2.3].

Let $k = \mathbb{C}$. Let X be the image of the morphism $\mathbb{P}^1 \rightarrow \mathbb{P}^3$ given by $(x : y) \mapsto (x^4 : x^3y : xy^3 : y^4)$. Let $\phi : X \rightarrow X$ correspond under $X \simeq \mathbb{P}^1$ to the automorphism of $\mathbb{P}^1$ given by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. For $r \geq 1$, the self-map ϕ^r corresponds to $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$. But this does not preserve the span

of $\{x^4, x^3y, xy^3, y^4\}$, since the coefficient of x^2y^2 in $(x + ry)^4$ is nonzero. Thus ϕ^r cannot be the restriction of an automorphism of $\mathbb{P}^3$.

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