THE LOCAL-GLOBAL PRINCIPLE FOR INTEGRAL POINTS ON STACKY CURVES

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Abstract. We construct a stacky curve of genus 1/2 (i.e., Euler characteristic 1) over $\mathbb{Z}$ that has an $\mathbb{R}$-point and a $\mathbb{Z}_p$-point for every prime p but no $\mathbb{Z}$-point. This is best possible: we also prove that any stacky curve of genus less than 1/2 over a ring of S-integers of a global field *satisfies* the local-global principle for integral points.

1. Introduction

Let k be a global field, i.e., a finite extension of either $\mathbb{Q}$ or $\mathbb{F}_p(t)$. For each nontrivial place v of k, let k_v be the completion of k at v. Let X be a smooth projective geometrically integral curve of genus g over k. If X has a k-point, then of course X has a k_v-point for every v. The converse holds if $g = 0$ (by the Hasse–Minkowski theorem), but there are well-known counterexamples of higher genus; in fact, counterexamples exist over every global field [Poo10]. This motivates the question: What is the smallest g such that there exists a counterexample of genus g over some global field? The answer is 1. Indeed, the first counterexample discovered was a genus 1 curve, the smooth projective model of $2y^2 = 1 - 17x^4$ over $\mathbb{Q}$ [Lin40, Rei42]. In fact, a positive proportion of genus 1 curves in the weighted projective space $\mathbb{P}(1, 1, 2)$ given by $z^2 = f(x, y)$, where $f(x, y)$ is an integral binary quartic form, violate the local-global principle over $\mathbb{Q}$ [Bha13].

Let us now generalize to allow X to be a *stacky* curve over k. (See Sections 2 and 3 for our conventions.) Then the genus g of X — defined by the formula $\chi = 2 - 2g$, where χ is the topological Euler characteristic of X — is no longer constrained to be a natural number; certain *fractional* values are also possible. Therefore we may now ask: What is the smallest g such that there exists a stacky curve of genus g over some global field k violating the local-global principle? It turns out that if we formulate the local-global principle using *rational* points over k and its completions, then the answer is not interesting, because rational points are almost the same as rational points on the coarse moduli space of X: see Section 4.

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Therefore we will answer our question in the context of a local-global principle for *integral* points on a stacky curve.

Our first theorem gives a proper stacky curve of genus $1/2$ over $\mathbb{Z}$ that violates the local-global principle.

Theorem 1. *Let p, q, r be primes congruent to* $7 \pmod 8$ *such that p is a square* $\pmod q$ *and* $\pmod r$, *and q is a square* $\pmod r$. *Let $f(x,y) = ax^2 + bxy + cy^2$ be a positive definite integral binary quadratic form of discriminant $-pqr$ such that a is a nonzero square* $\pmod q$ *but a nonsquare* $\pmod p$ *and* $\pmod r$. *Let $\mathcal{Y} := \operatorname{Proj} \mathbb{Z}[x,y,z]/(z^2 - f(x,y))$. Define a μ_2-action on $\mathcal{Y}$ by letting $\lambda \in \mu_2$ act as $(x : y : z) \mapsto (x : y : \lambda z)$. Let $\mathcal{X}$ be the quotient stack $[\mathcal{Y}/\mu_2]$. Then*
(a) *the genus of $\mathcal{X}$ is $1/2$ (i.e., $\chi(\mathcal{X}) = 1$);*
(b) *$\mathcal{X}(\mathbb{Z}_\ell) \neq \emptyset$ for every rational prime ℓ and $\mathcal{X}(\mathbb{R}) \neq \emptyset$;*
(c) *$\mathcal{X}(\mathbb{Z}) = \emptyset$, and even $\mathcal{X}(\mathbb{Z}[1/(2pqr)]) = \emptyset$.*
The same conclusions hold if instead we define $\mathcal{X}$ as $[\mathcal{Y}/(\mathbb{Z}/2\mathbb{Z})]$, where $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathcal{Y}$ through the nontrivial homomorphism $\mathbb{Z}/2\mathbb{Z} \to \mu_2$; this $\mathcal{X}$ is a Deligne–Mumford stack even over $\mathbb{Z}$.

Remark 2. The hypotheses in Theorem 1 can be satisfied. For example, let $p = 7$, $q = 47$, $r = 31$, and $f(x,y) = 3x^2 + xy + 850y^2$.

Remark 3. The reason for considering $\mathbb{Z}[1/(2pqr)]$ in (c) is that $\mathcal{X}$ is smooth over that base.

Remark 4. Section 8 of [DG95] implicitly contains a similar counterexample, but of genus $2/3$. Let

$$\mathcal{Y} := \operatorname{Spec} \frac{\mathbb{Z}[x,y,z]}{(x^2 + 29y^2 - 3z^3)} - \{x = y = z = 0\},$$

so $\mathcal{Y}(\mathbb{Z})$ consists of *primitive* integer solutions to $x^2 + 29y^2 = 3z^3$, those such that no prime divides all of x, y, z. Let each $\lambda \in \mathbb{G}_m$ act on $\mathcal{Y}$ as $(x,y,z) \mapsto (\lambda^3 x, \lambda^3 y, \lambda^2 z)$. The quotient stack $\mathcal{X} := [\mathcal{Y}/\mathbb{G}_m]$ is a proper stacky curve. Since every $\mathbb{G}_m$-torsor over $\operatorname{Spec} \mathbb{Z}$ is trivial, the map $\mathcal{Y}(\mathbb{Z}) \to \mathcal{X}(\mathbb{Z})$ is surjective, and likewise with $\mathbb{Z}$ replaced by $\mathbb{R}$ or $\mathbb{Z}_p$ for any prime p. Thus Section 8 of [DG95] says that $\mathcal{X}$ is a counterexample to the local-global principle.

Our second theorem shows that any stacky curve of genus less than $1/2$ over a ring of S-integers of a global field *satisfies* the local-global principle. Let k be a global field, and let k_v denote the completion of k at v. Let S be a finite nonempty set of places of k containing all the archimedean places. Let $\mathcal{O}$ be the ring of S-integers in k; that is, $\mathcal{O} := \{x \in k : v(x) \geq 0 \text{ for all } v \notin S\}$. For each $v \notin S$, let $\mathcal{O}_v$ be the completion of $\mathcal{O}$ at v. For each $v \in S$, let $\mathcal{O}_v = k_v$.

Theorem 5. *Let $\mathcal{X}$ be a stacky curve over $\mathcal{O}$ of genus less than $1/2$ (i.e., $\chi(\mathcal{X}) > 1$). If $\mathcal{X}(\mathcal{O}_v) \neq \emptyset$ for all places v of k, then $\mathcal{X}(\mathcal{O}) \neq \emptyset$.*

2. Stacks

By a stack, we mean an algebraic (Artin) stack $\mathcal{X}$ over a scheme S [SP, Tag 026O]. For any object $T \in (Sch/S)_{\mathrm{fppf}}$, we write $\mathcal{X}(T)$ for the set of isomorphism classes of S-morphisms $T \to \mathcal{X}$, or equivalently (by the 2-Yoneda lemma [SP, Tag 04SS]), the set of isomorphism classes of the fiber category $\mathcal{X}_T$. If $T = \operatorname{Spec} A$, we write $\mathcal{X}(A)$ for $\mathcal{X}(T)$.

3. Stacky curves

Let k be an algebraically closed field. Let X be a stacky curve over k, i.e., a smooth separated irreducible 1-dimensional Deligne–Mumford stack over k containing a nonempty open substack isomorphic to a scheme. (This definition is slightly more general than [VZB19, Definition 5.2.1] in that we require only separatedness instead of properness, to allow punctures.)

By the Keel–Mori theorem [KM97] in the form given in [Con05] and [Ols16, Theorem 11.1.2], X has a morphism to a coarse moduli space X_{coarse} that is a smooth integral curve over k. We have $X_{\mathrm{coarse}} = \widetilde{X}_{\mathrm{coarse}} - Z$ for some smooth projective integral curve $\widetilde{X}_{\mathrm{coarse}}$ and some finite set of closed points Z. Moreover, by [Ols16, Theorem 11.3.1], each $P \in X_{\mathrm{coarse}}(k)$ has an étale neighborhood U above which $X \to X_{\mathrm{coarse}}$ has the form $[V/G] \to U$ for some possibly ramified finite G-Galois cover $V \to U$ (by a scheme), where G is the stabilizer of X above P. The stacky curve X is called tame above P if $\operatorname{char} k \nmid |G|$, and tame if it is tame above every P. Let $\mathcal{P} \subset X_{\mathrm{coarse}}(k)$ be the (finite) set above which the stabilizer is nontrivial; then the morphism $X \to X_{\mathrm{coarse}}$ is an isomorphism above $X_{\mathrm{coarse}} - \mathcal{P}$.

Let $\tilde{g}_{\mathrm{coarse}}$ be the genus of $\widetilde{X}_{\mathrm{coarse}}$; then the Euler characteristic $\chi(X_{\mathrm{coarse}})$ is $(2 - 2\tilde{g}_{\mathrm{coarse}}) - \#Z$. We now follow [Kob20] to define $\chi(X)$ and $g(X)$. For P, U, V, G as above, let $G_i \le G$ be the ramification subgroups for $V \to U$ above P, and define

$$\delta_P := \sum_{i \ge 0} \frac{|G_i| - 1}{|G|}$$

(which simplifies to only the first term $(|G| - 1)/|G|$ if X is tame above P). Then define the Euler characteristic by

$$\chi(X) := \chi(X_{\mathrm{coarse}}) - \sum_{P \in \mathcal{P}} \delta_P.$$

(This is motivated by the Riemann–Hurwitz formula. See [VZB19, Kob20] for other motivation.) Finally, define the genus $g = g(X)$ by $\chi(X) = 2 - 2g$.

Lemma 6. *Let X be a stacky curve over an algebraically closed field k with $g < 1/2$. Then $X_{\mathrm{coarse}} \simeq \mathbb{P}^1$ and $\#\mathcal{P} \le 1$ and X is tame.*

Proof. Since $g < 1/2$, we have $\chi(X) > 1$. For each $P \in \mathcal{P}$, note that $\delta_P \ge (|G| - 1)/|G| \ge 1/2$. Now

$$\chi(X) = 2 - 2\tilde{g}_{\mathrm{coarse}} - \#Z - \sum_{P \in \mathcal{P}} \delta_P,$$

which is ≤ 1 if $\widetilde{g}_{\mathrm{coarse}} \geq 1$ or $\#Z \geq 1$ or $\#\mathcal{P} \geq 2$. Thus $\widetilde{g}_{\mathrm{coarse}} = 0$, $\#Z = 0$, and $\#\mathcal{P} \leq 1$. Furthermore, if X is not tame, then there exists $P \in \mathcal{P}$ with $\delta_P \geq (|G|-1)/|G| + 1/|G| \geq 1$, which again forces $\chi(X) \leq 1$, a contradiction. $\square$

Now let k be any field. Let $\overline{k}$ be an algebraic closure of k, and let k_{s} be the separable closure of k in $\overline{k}$. By a stacky curve over k, we mean an algebraic stack X over k such that the base extension $X_{\overline{k}}$ is a stacky curve over $\overline{k}$. Define $\chi(X) := \chi(X_{\overline{k}})$ and $g(X) := g(X_{\overline{k}})$.

Lemma 7. *If X is a tame stacky curve over k, then the set $\mathcal{P} \subset X_{\mathrm{coarse}}(\overline{k})$ for $X_{\overline{k}}$ consists of points whose residue fields are separable over k.*

Proof. Let $\bar{P} \in \mathcal{P}$. Let P be the closed point of X_{coarse} associated to $\bar{P}$. By working étale locally on X_{coarse}, we may assume that $X = [V/G]$ for a smooth curve V over k that is a G-Galois cover of X_{coarse} totally tamely ramified above P. Analytically locally above P, the tame cover is given by the equation $y^n = \pi$ for some uniformizer π at $P \in X_{\mathrm{coarse}}$. After base change to $\overline{k}$, however, $\pi = u\bar{\pi}^i$, where u is a unit, $\bar{\pi}$ is a uniformizer at $\bar{P}$, and i is the inseparable degree of $k(P)/k$. Thus $V_{\overline{k}}$ is analytically locally given by $y^n = u\bar{\pi}^i$. Since $V_{\overline{k}}$ is smooth, $i = 1$. Thus $k(P)/k$ is separable. $\square$

Next, let $\mathcal{O}$ be a ring of S-integers in a global field k. By a stacky curve $\mathcal{X}$ over $\mathcal{O}$, we mean a separated finite-type algebraic stack over $\operatorname{Spec} \mathcal{O}$ such that $\mathcal{X}_k$ is a stacky curve. (To be as general as possible, we do not impose Deligne–Mumford, tameness, smoothness, or properness conditions on the fibers above closed points of $\operatorname{Spec} \mathcal{O}$.) Define $\chi(\mathcal{X}) := \chi(\mathcal{X}_{\overline{k}})$ and $g(\mathcal{X}) := g(\mathcal{X}_{\overline{k}})$.

4. Local-global principle for rational points

We now explain why the local-global principle for *rational* points is not so interesting.

Proposition 8. *Let k be a global field. Let X be a stacky curve over k with $g < 1$. If $X(k_v) \neq \emptyset$ for all nontrivial places v of k, then $X(k) \neq \emptyset$.*

Proof. We have $0 < \chi(X) \leq 2 - 2\widetilde{g}_{\mathrm{coarse}}$, so $\widetilde{g}_{\mathrm{coarse}} = 0$. Thus X_{coarse} is a smooth geometrically integral curve of genus 0. Because of the morphism $X \to X_{\mathrm{coarse}}$, we have $X_{\mathrm{coarse}}(k_v) \neq \emptyset$ for every v. By the Hasse–Minkowski theorem, $X_{\mathrm{coarse}}(k) \neq \emptyset$, so X_{coarse} is a dense open subscheme of $\mathbb{P}^1_k$. In particular, $X_{\mathrm{coarse}}(k)$ is Zariski dense in X_{coarse}, and all but finitely many of these k-points correspond to k-points on X. $\square$

Because of Proposition 8, our main theorems are concerned with the local-global principle for *integral* points.

5. Proof of Theorem 1: counterexample to the local-global principle

(a) Since $(\mathcal{X}_{\mathbb{Q}})_{\text{coarse}}$ is dominated by the genus 0 curve $\mathcal{Y}_{\mathbb{Q}}$, we have $\widetilde{g}_{\text{coarse}} = 0$. The action of μ_2 on $\mathcal{Y}_{\overline{\mathbb{Q}}}$ fixes exactly two $\overline{\mathbb{Q}}$-points, namely those with $z = 0$; thus $\mathcal{P} = 2$, and $\delta_P = 1/2$ for each $P \in \mathcal{P}$. Hence $\chi(\mathcal{X}) = (2 - 2 \cdot 0) - (1/2 + 1/2) = 1$. (Alternatively, $\chi(\mathcal{X}) = \chi(\mathcal{Y})/2 = 2/2 = 1$.)

(b) Let R be a principal ideal domain. By definition of the quotient stack, a morphism $\operatorname{Spec} R \to \mathcal{X}$ is given by a μ_2-torsor T equipped with a μ_2-equivariant morphism $T \to \mathcal{Y}$. The torsors are classified by $\mathrm{H}^1_{\text{fppf}}(R, \mu_2)$, which is isomorphic to $R^\times/R^{\times 2}$, since $\mathrm{H}^1_{\text{fppf}}(R, \mathbb{G}_m) = \operatorname{Pic} R = 0$. Explicitly, if $t \in R^\times$, the corresponding μ_2-torsor is $T_t := \operatorname{Spec} R[u]/(u^2 - t)$. Define the twisted cover

$$\mathcal{Y}_t := \operatorname{Proj} R[x, y, z]/(tz^2 - f(x, y))$$

with its morphism $\pi_t \colon \mathcal{Y}_t \to \mathcal{X}$. To give a μ_2-equivariant morphism $T_t \to \mathcal{Y}$ is the same as giving a morphism $\operatorname{Spec} R \to \mathcal{Y}_t$. Thus we obtain

$$\mathcal{X}(R) = \coprod_{t \in R^\times} \pi_t(\mathcal{Y}_t(R)).$$

For any $\ell \notin \{p, q, r\}$, the rank 3 form $z^2 - f(x, y)$ has good reduction at ℓ, so $\mathcal{Y}(\mathbb{F}_\ell) \neq \emptyset$, and Hensel's lemma yields $\mathcal{Y}(\mathbb{Z}_\ell) \neq \emptyset$. Since the discriminant of $f(x, y)$ is divisible only by p and not p^2, the form is not identically 0 modulo p, so there exist $\bar{a}, \bar{b} \in \mathbb{F}_p$ with $f(\bar{a}, \bar{b}) \in \mathbb{F}_p^\times$. Lift $\bar{a}, \bar{b}$ to $a, b \in \mathbb{Z}_p$, so $f(a, b) \in \mathbb{Z}_p^\times$. Then $\mathcal{Y}_{f(a,b)}(\mathbb{Z}_p) \neq \emptyset$. The same argument applies at q and r. Since f is positive definite, $\mathcal{Y}(\mathbb{R}) \neq \emptyset$. Thus $\mathcal{X}(\mathbb{Z}_\ell) \neq \emptyset$ for all primes ℓ, and $\mathcal{X}(\mathbb{R}) \neq \emptyset$.

(c) We now show that $\mathcal{X}(\mathbb{Z}[1/(2pqr)]) = \emptyset$, i.e., that $\mathcal{Y}_t(\mathbb{Z}[1/(2pqr)]) = \emptyset$ for all $t \in \mathbb{Z}[1/(2pqr)]^\times$, or equivalently, that the quadratic form $f(x, y)$ does not represent any element of $\mathbb{Z}[1/(2pqr)]^\times$ times a square in $\mathbb{Z}[1/(2pqr)]$.

Completing the square shows that f is equivalent over $\mathbb{Q}$ to the diagonal form $[a, apqr]$. If we use $u = u_v$ to denote a unit nonresidue in $\mathbb{Z}_v$, then

- over $\mathbb{Q}_p$, the form f is equivalent to $[u, up]$ and represents the squareclasses u, up;
- over $\mathbb{Q}_q$, the form f is equivalent to $[1, uq]$ and represents the squareclasses $1, uq$;
- over $\mathbb{Q}_r$, the form f is equivalent to $[u, ur]$ and represents the squareclasses u, ur.

Therefore,

- f takes square values in $\mathbb{R}$ and $\mathbb{Q}_q$, but not in $\mathbb{Q}_p$ and $\mathbb{Q}_r$.
- $-f$ takes square values in $\mathbb{Q}_p$ and $\mathbb{Q}_r$, but not in $\mathbb{R}$ and $\mathbb{Q}_q$.

It follows that f and $-f$ together represent squares locally at all places, but do not globally represent squares.

We now further check that sf, for *every* factor s of pqr, fails to globally represent a square (by quadratic reciprocity, r is not a square $\pmod p$ and $\pmod q$, and q is not a square $\pmod p$):

- pf takes square values in $\mathbb{R}$ and $\mathbb{Q}_q$, but not in $\mathbb{Q}_p$ and $\mathbb{Q}_r$.
- qf takes square values in $\mathbb{R}$ and $\mathbb{Q}_p$, but not in $\mathbb{Q}_q$ and $\mathbb{Q}_r$.
- rf takes square values in $\mathbb{R}$ and $\mathbb{Q}_p$, but not in $\mathbb{Q}_q$ and $\mathbb{Q}_r$.
- pqf takes square values in $\mathbb{R}$ and $\mathbb{Q}_p$, but not in $\mathbb{Q}_q$ and $\mathbb{Q}_r$.
- prf takes square values in $\mathbb{R}$ and $\mathbb{Q}_p$, but not in $\mathbb{Q}_q$ and $\mathbb{Q}_r$.
- qrf takes square values in $\mathbb{R}$ and $\mathbb{Q}_q$, but not in $\mathbb{Q}_p$ and $\mathbb{Q}_r$.
- $pqrf$ takes square values in $\mathbb{R}$ and $\mathbb{Q}_q$, but not in $\mathbb{Q}_p$ and $\mathbb{Q}_r$.

Since 2 is a square in $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{Q}_q$, and $\mathbb{Q}_r$, multiplying each of the sf's in the above statements by 2 would not change the truth of any these statements. Meanwhile, since -1 and -2 are nonsquares in $\mathbb{R}$, $\mathbb{Q}_p$, $\mathbb{Q}_q$, and $\mathbb{Q}_r$, multiplying the sf's in the statements above by -1 or -2 would simply reverse all the conditions (in particular, all would fail to represent squares in $\mathbb{R}$).

We conclude that $\mathcal{Y}_t(\mathbb{Z}[1/(2pqr)]) = \emptyset$ for all $t \in \mathbb{Z}[1/(2pqr)]^\times$, i.e., $\mathcal{X}(\mathbb{Z}[1/(2pqr)]) = \emptyset$, as claimed.

The same arguments apply to $\mathcal{X}' := [\mathcal{Y}/(\mathbb{Z}/2\mathbb{Z})]$; in particular,

$$\mathcal{X}'(\mathbb{Z}[1/(2pqr)]) = \mathcal{X}(\mathbb{Z}[1/(2pqr)]) = \emptyset,$$

because the homomorphism $\mathbb{Z}/2\mathbb{Z} \to \mu_2$ is an isomorphism over $\mathbb{Z}[1/2]$ and hence over $\mathbb{Z}[1/(2pqr)]$.

6. Stacks over local rings

This section contains some results to be used in the proof of Theorem 5.

Proposition 9. *Let A be a noetherian local ring. Let X be an algebraic stack of finite type over A. Let $x \in X(A)$. Then there exists a finite-type algebraic space U over A, a smooth surjective morphism $f\colon U \to X$, and an element $u \in U(A)$ such that $f(u) = x$.*

Proof. By definition, there exists a finite-type A-scheme V and a smooth surjective morphism $V \to X$. Taking the 2-fiber product with $\operatorname{Spec} A \xrightarrow{x} X$ yields an algebraic space $V_x \to \operatorname{Spec} A$. Then $V_x \to \operatorname{Spec} A$ is smooth, so it admits étale local sections. Thus we can find a Galois étale extension A' of A, say with group G, such that x lifts to a morphism $\operatorname{Spec} A' \xrightarrow{v} V$ equipped with a compatible system of isomorphisms between the conjugates of v.

Let $n = \#G$. Let V_X^n be the 2-fiber product over X of n copies of V, indexed by G. The left translation action of G on G induces a right G-action on V_X^n respecting the morphism $V_X^n \to X$, and there is also a right G-action on $\operatorname{Spec} A'$. Therefore we may twist V_X^n to obtain a new algebraic space U lying over X (a quotient of $V_X^n \times_A A'$ by a twisted action of G) such that the element of $V_X^n(A')$ given by the conjugates of v and the isomorphisms between them descends to an element of $U(A)$. □

Remark 10. Atticus Christensen, combining a variant of our proof with other arguments, has extended Proposition 9 to other rings A, such as arbitrary products of complete noetherian local rings, and adèle rings of global fields [Chr20, Theorem 7.0.7 and Propositions 12.0.5 and 12.0.8].

For any valued field K, let $\widehat{K}$ denote its completion.

Proposition 11. *Let A be an excellent henselian discrete valuation ring. Let $K = \operatorname{Frac} A$. Let U be a separated finite-type algebraic space over K.*

(a) *The set $U(K)$ has a topology inherited from the topology on K.*
(b) *If U is smooth and irreducible, then any nonempty open subset of $U(K)$ is Zariski dense in U.*

Proof.

(a) In fact, much more is true: if $K = \widehat{K}$, then the analytification of U exists as a rigid analytic space [CT09, Theorem 1.2.1]. If $K \neq \widehat{K}$, equip $U(K)$ with the subspace topology inherited from $U(\widehat{K})$.
(b) If $K = \widehat{K}$, this follows from the fact that a nonzero power series in n variables over K cannot vanish on a nonempty open subset of K^n. If $K \neq \widehat{K}$, use Artin approximation: any point of $U(\widehat{K})$ can be approximated by a point of $U(K)$. □

Proposition 12. *Let A be an excellent henselian discrete valuation ring. Let $K = \operatorname{Frac} A$. Let U be a separated finite-type algebraic space over A. Then $U(A)$ is an open subset of $U(K)$.*

Proof. Since U is separated over A, the map $U(A) \to U(K)$ is injective. Let $u \in U(A)$. Choose a separated A-scheme V with an étale surjective morphism $f\colon V \to U$. Then u lifts to some $v \in V(A')$ for some finite étale A-algebra A'. Let $K' = \operatorname{Frac} A'$. Since V is a separated A-scheme, $V(A')$ is an open subset of $V(K')$. If A is complete, then the étale morphism $V \to U$ induces an étale morphism of analytifications [CT09, Theorem 2.3.1], so $V(K') \to U(K')$ is a local homeomorphism; in particular, it defines a homeomorphism from a neighborhood N_V of v in $V(K')$ to a neighborhood N_U of u in $U(K')$, and we may assume that $N_V \subseteq V(A')$. In the general case, a given point of $V(\widehat{K'})$ maps to some point of $U(K')$ if and only if it is in $V(K')$, so the homeomorphism for $\widehat{K'}$-points restricts to a homeomorphism for K'-points, which we again denote $N_V \xrightarrow{\sim} N_U$. If $u_1 \in N_U \cap U(K)$, then u_1 lies in the image of $N_V \subseteq V(A')$, so $u_1 \in U(A')$; now $u_1 \in U(A') \cap U(K)$, which is $U(A)$ since U is a sheaf on $(\operatorname{Spec} A)_{\mathrm{fppf}}$. Hence $U(A)$ is open in $U(K)$. □

7. Proof of Theorem 5

By Lemma 6, we have $(\mathcal{X}_{\overline{k}})_{\text{coarse}} \simeq \mathbb{P}^1_{\overline{k}}$, and hence $(\mathcal{X}_k)_{\text{coarse}}$ is a smooth proper curve of genus 0. Since $\mathcal{X}$ has an $\mathcal{O}_v$-point for every v, the stack $\mathcal{X}_k$ has a k_v-point for every v, so $(\mathcal{X}_k)_{\text{coarse}}$ has a k_v-point for every v. Thus $(\mathcal{X}_k)_{\text{coarse}} \simeq \mathbb{P}^1_k$.

If $\mathcal{X}_k \to (\mathcal{X}_k)_{\text{coarse}}$ is not an isomorphism, then by Lemma 6, there is a unique $\overline{k}$-point above which it fails to be an isomorphism, and by Lemma 7, it is a k_{s}-point, and that point must be $\operatorname{Gal}(k_{\text{s}}/k)$-stable, hence a k-point of $\mathbb{P}^1$, which we may assume is ∞. Thus $\mathcal{X}_k$ contains an open substack isomorphic to $\mathbb{A}^1_k$.

Since all the stacks are of finite presentation, the isomorphism just constructed extends above some affine open neighborhood of the generic point in $\operatorname{Spec} \mathcal{O}$. That is, there exists a finite set of places $S' \supseteq S$ such that if $\mathcal{O}'$ is the ring of S'-integers in k, then the stack $\mathcal{X}_{\mathcal{O}'}$ contains an open substack isomorphic to $\mathbb{A}^1_{\mathcal{O}'}$.

Let $v \in S' - S$. Let $\mathcal{O}_{(v)}$ be the localization of $\mathcal{O}$ at v, and let $\mathcal{O}_{v,h}$ be its henselization in $\mathcal{O}_v$, so $\mathcal{O}_{v,h}$ is the set of elements of $\mathcal{O}_v$ that are algebraic over k. Let $k_{v,h} = \operatorname{Frac} \mathcal{O}_{v,h}$. We are given $x \in \mathcal{X}(\mathcal{O}_v)$. Let U, f, and u be as in Proposition 9 with $A = \mathcal{O}_v$. By Proposition 12, $U(\mathcal{O}_v)$ is open in $U(k_v)$. Let U_0 be the connected component of U_{k_v} containing u, so $U_0(k_v)$ is open in $U(k_v)$. The morphisms $U_0 \to U_{k_v} \to \mathcal{X}_{k_v} \to \operatorname{Spec} k_v$ are smooth, so U_0 is smooth and irreducible. Therefore, by Proposition 11(b), the set $U(\mathcal{O}_v) \cap U_0(k_v)$ is Zariski dense in U_0. On the other hand, U_0 dominates $\mathcal{X}_{k_v}$ since $U_0 \to \mathcal{X}_{k_v}$ is smooth and $\mathcal{X}_{k_v}$ is irreducible. By the previous two sentences, there exists $u_0 \in U(\mathcal{O}_v) \cap U_0(k_v)$ mapping into the subset $\mathbb{A}^1(k_v)$ of $\mathcal{X}(k_v)$. By Artin approximation, we may replace u_0 by a nearby point to assume also that $u_0 \in U(\mathcal{O}_{v,h})$.

Let U_1 be the inverse image of $\mathbb{A}^1_{k_{v,h}}$ under $U_{k_{v,h}} \to \mathcal{X}_{k_{v,h}}$. By Proposition 12, $U(\mathcal{O}_{v,h})$ is open in $U(k_{v,h})$, so $U(\mathcal{O}_{v,h}) \cap U_1(k_{v,h})$ is an open neighborhood of u_0 in $U_1(k_{v,h})$. Since $U_1 \to \mathbb{A}^1_{k_{v,h}}$ is smooth, the image of this neighborhood is a nonempty open subset B_v of $\mathbb{A}^1(k_{v,h})$. By construction, B_v is contained in the image of $U(\mathcal{O}_{v,h}) \to \mathcal{X}(\mathcal{O}_{v,h}) \subseteq \mathcal{X}(k_{v,h})$, so $B_v \subseteq \mathcal{X}(\mathcal{O}_{v,h})$.

By strong approximation, there exists $x \in \mathbb{A}^1(\mathcal{O}')$ such that $x \in B_v$ for all $v \in S' - S$. For each $v \in S' - S$, since $B_v \subseteq \mathcal{X}(\mathcal{O}_{v,h})$, there exists $x_v \in \mathcal{X}(\mathcal{O}_{v,h})$ such that x and x_v become equal in $\mathcal{X}(k_{v,h})$. Finally, the following lemma shows that x comes from an element of $\mathcal{X}(\mathcal{O})$.

Lemma 13. *If $x \in \mathcal{X}(\mathcal{O}')$ and $x_v \in \mathcal{X}(\mathcal{O}_{v,h})$ for each $v \in S' - S$ are such that the images of x and x_v in $\mathcal{X}(k_{v,h})$ are equal for every $v \in S' - S$, then there exists an element of $\mathcal{X}(\mathcal{O})$ mapping to x in $\mathcal{X}(\mathcal{O}')$ and to x_v in $\mathcal{X}(\mathcal{O}_{v,h})$ for each $v \in S' - S$.*

Proof. Since $\mathcal{X}$ is of finite presentation over $\mathcal{O}$, the element x_v comes from an element $\widetilde{x}_v$ of some finitely generated $\mathcal{O}$-subalgebra A_v of $\mathcal{O}_{v,h}$. The schemes $\operatorname{Spec} A_v$ together with $\operatorname{Spec} \mathcal{O}'$ form an fppf covering of $\operatorname{Spec} \mathcal{O}$, so the stack property of $\mathcal{X}$ shows that x and the $\widetilde{x}_v$ come from an element of $\mathcal{X}(\mathcal{O})$. $\square$

Remark 14. Inspired by an earlier draft of our article, Christensen has found a natural way to define a topology on the set of adelic points of a finite-type algebraic stack, and has proved a strong approximation theorem for a stacky curve with $\chi > 1$ [Chr20, Theorem 13.0.6]. His argument can substitute for the three paragraphs before Lemma 13 and hence give a partially independent proof of Theorem 5.

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