

The projective line minus three fractional points

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(grew out of discussions with many people at the Spring 2006 MSRI program on Rational and Integral Points on Higher-Dimensional Varieties, especially Frédéric Campana, Jordan Ellenberg, and Aaron Levin)

1 3 kinds of integral points

- Darmon's M -curves
- Campana's orbifolds
- Almost integral points

2 Counting points of bounded height

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Motivation: a generalized Fermat equation

- Let

$$S(\mathbb{Z}) := \left\{ (x, y, z) \in \mathbb{Z}^3 : \begin{array}{l} x^2 + y^3 = z^7 \\ \gcd(x, y, z) = 1 \end{array} \right\}.$$

- Then

$$\begin{aligned} S(\mathbb{Z}) &\rightarrow \mathbb{P}^1(\mathbb{Q}) := \mathbb{Q} \cup \left\{ \frac{1}{0} \right\} \\ (x, y, z) &\mapsto \frac{x^2}{z^7} \quad \left(= 1 - \frac{y^3}{z^7} \right). \end{aligned}$$

induces a bijection

$$\frac{S(\mathbb{Z})}{\text{sign}} \leftrightarrow \left\{ q \in \mathbb{P}^1(\mathbb{Q}) : \begin{array}{l} \text{num}(q) \text{ is a square} \\ \text{num}(q-1) \text{ is a cube} \\ \text{den}(q) \text{ is a } 7^{\text{th}} \text{ power} \end{array} \right\}.$$

- Darmon and Granville applied Faltings' theorem to covers of $\mathbb{P}^1$ ramified only over $\{0, 1, \infty\}$ to prove that the right hand side is finite, and hence deduce that $S(\mathbb{Z})$ is finite.

Geometric interpretation

- Define a $\mathbb{Z}$ -scheme

$$S := (x^2 + y^3 = z^7 \text{ in } \mathbb{A}^3) - \{(0, 0, 0)\}.$$

- Then the morphism

$$\begin{array}{ccc} S & & (x, y, z) \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & & x^2/y^7 \end{array}$$

has multiple fibers above 0, 1, ∞ , having multiplicities 2, 3, 7, respectively.

- So $S \rightarrow \mathbb{P}^1$ factors through a stack $\tilde{\mathbb{P}}^1 := [S/\mathbb{G}_m]$ that looks like $\mathbb{P}^1$ except that the points 0, 1, ∞ have been replaced by a $1/2$ -point, a $1/3$ -point, and a $1/7$ -point, respectively. Points in $S(\mathbb{Z})$ map to $\tilde{\mathbb{P}}^1(\mathbb{Z}) \subset \mathbb{P}^1(\mathbb{Z}) = \mathbb{P}^1(\mathbb{Q})$.
- **Moral: Multiple fibers impose conditions on images of integral points.**

Numerator with respect to a point

- We saw that a fiber of multiplicity 2 above $0 \in \mathbb{P}^1(\mathbb{Q})$ imposes the condition that $\text{num}(q)$ be a square.
- What condition is imposed, say, by a fiber of multiplicity 2 above the point $3/5 \in \mathbb{P}^1(\mathbb{Q})$?
- Answer: The value of $\text{num}_{3/5}(a/b) := |5a - 3b|$ should be a square.

In general:

Definition (Numerator with respect to the point c/d)

For $c/d \in \mathbb{P}^1(\mathbb{Q})$, define $\text{num}_{c/d}(a/b) := |ad - bc|$.

Examples

- If $c \in \mathbb{Z}$, then $\text{num}_c(a/b) = \text{num}(a/b - c)$.
- $\text{num}_\infty(q) = \text{den}(q)$.

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Darmon's M-curves

- **M-curve data:**
points $P_1, \dots, P_N \in \mathbb{P}^1(\mathbb{Q})$, with
multiplicities $m_1, \dots, m_N \in \{2, 3, \dots\} \cup \{\infty\}$.
- An **M-curve** may be denoted formally by $\mathbb{P}^1 - \Delta$, where

$$\Delta := \sum_{i=1}^N \left(1 - \frac{1}{m_i}\right) [P_i].$$

(It is really a kind of stack.)

- Define the **Euler characteristic**

$$\begin{aligned}\chi(\mathbb{P}^1 - \Delta) &:= \chi(\mathbb{P}^1) - \deg \Delta \\ &= 2 - \sum_{i=1}^{\infty} \left(1 - \frac{1}{m_i}\right).\end{aligned}$$

Definition (Integral points in Darmon's sense)

$(\mathbb{P}^1 - \Delta)(\mathbb{Z}) := \{q \in \mathbb{P}^1(\mathbb{Q}) : \text{num}_{P_i}(q) \text{ is an } m_i\text{-th power } \forall i\}$

Note: “ ∞ -th power” means unit (i.e., ± 1).

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Campana's orbifolds: motivation

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- Suppose $\pi: S \rightarrow \mathbb{P}^1$ is such that the fiber above 0 consists of two irreducible components, one of multiplicity 2 and one of multiplicity 5.
- If $s \in S(\mathbb{Z})$, then $\pi(s)$ is again restricted: its numerator is of the form u^2v^5 .
- Equivalently, in the prime factorization of $\text{num}(\pi(s))$, every exponent is a nonnegative integer combination of 2 and 5.
- In particular (but not equivalently), $\text{num}(\pi(s))$ is a **squareful** integer, i.e., $p_1^{e_1} \cdots p_r^{e_r}$ with all $e_i \geq 2$.

More generally:

Definition

An integer a is called **m -powerful** if in its prime factorization all (nonzero) exponents are $\geq m$.

An integer a is called **∞ -powerful** if $a = \pm 1$.

Definition (Integral points in Campana's sense)

For an M-curve $\mathbb{P}^1 - \Delta$, define

$$(\mathbb{P}^1 - \Delta)_C(\mathbb{Z}) := \{q \in \mathbb{P}^1(\mathbb{Q}) : \text{num}_{P_i}(q) \text{ is } m_i\text{-powerful } \forall i\}$$

Example

Let $\Delta = \frac{1}{2}[0] + \frac{1}{2}[3] + [\infty]$. Then

$$\begin{aligned} (\mathbb{P}^1 - \Delta)_C(\mathbb{Z}) &= \left\{ \frac{a}{b} \in \mathbb{P}^1(\mathbb{Q}) : \begin{array}{l} a \text{ is squareful,} \\ a - 3b \text{ is squareful, and} \\ b = 1 \end{array} \right\} \\ &= \{a \in \mathbb{Z} : a, a - 3 \text{ are both squareful}\} \end{aligned}$$

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Definition (Height and penalty)

For an M -curve $\mathbb{P}^1 - \Delta$ and $q = a/b \in \mathbb{P}^1(\mathbb{Q})$, define

$$H(q) := \max(|a|, |b|)$$

$$\text{penalty}_{\mathbb{P}^1 - \Delta}(q) := \prod_{i=1}^N \prod_{\substack{p \text{ such that} \\ m_i \nmid v_p(\text{num}_{P_i}(q))}} p^{1 - \frac{1}{m_i}}.$$

Remark: If Δ consists of whole points, then $\log(\text{penalty})$ is the “truncated counting function” in Vojta’s “more general *abc* conjecture”.

Fix a real number $r \in [0, \deg \Delta]$ (“tolerance level”).

Definition (Almost integral points)

$$(\mathbb{P}^1 - \Delta + r)(\mathbb{Z}) := \{q \in \mathbb{P}^1(\mathbb{Q}) : \text{penalty}_{\mathbb{P}^1 - \Delta}(q) \leq H(q)^r\}$$

Also define $\chi(\mathbb{P}^1 - \Delta + r) := \chi(\mathbb{P}^1 - \Delta) + r$.

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- We will study when the set of integral points (in each of the three senses) is finite.
- When it is infinite, we will measure it by counting points of bounded height.

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Definition (Counting functions)

$$(\mathbb{P}^1 - \Delta)(\mathbb{Z})_{\leq B} := \{q \in (\mathbb{P}^1 - \Delta)(\mathbb{Z}) : H(q) \leq B\}.$$

$$(\mathbb{P}^1 - \Delta)_C(\mathbb{Z})_{\leq B} := \{q \in (\mathbb{P}^1 - \Delta)_C(\mathbb{Z}) : H(q) \leq B\}.$$

$$(\mathbb{P}^1 - \Delta + r)(\mathbb{Z})_{\leq B} := \{q \in (\mathbb{P}^1 - \Delta + r)(\mathbb{Z}) : H(q) \leq B\}.$$

Heuristics for Darmon's M-curves

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Δ	$(\mathbb{P}^1 - \Delta)(\mathbb{Z})$	$\#(\mathbb{P}^1 - \Delta)(\mathbb{Z})_{\leq B}$
0	$\left\{ \frac{a}{b} : \gcd(a, b) = 1 \right\}$	$\sim B^2$
$(1 - \frac{1}{m}) [\infty]$	$\left\{ \frac{a}{b} : b \text{ is } m^{\text{th}} \text{ power} \right\}$	$\sim B \cdot B^{1/m}$
$\sum \left(1 - \frac{1}{m_i}\right) [P_i]$	$\left\{ \frac{a}{b} : \dots \right\}$	$\sim B^x?$

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Heuristic:

- In the case $\Delta = (1 - \frac{1}{m}) [\infty]$, the probability that a point satisfies the condition at ∞ is $\sim \frac{B \cdot B^{1/m}}{B^2} = \frac{1}{B^{1-1/m}}$.
- If conditions at different points are independent, the count for $\Delta = \sum \left(1 - \frac{1}{m_i}\right) [P_i]$ should be

$$\sim B^2 \left(\frac{1}{B^{1-1/m_1}} \right) \cdots \left(\frac{1}{B^{1-1/m_N}} \right) = B^x.$$

Heuristics for Darmon's M-curves

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$(1 - \frac{1}{m}) [\infty]$	$\left\{ \frac{a}{b} : b \text{ is } m^{\text{th}} \text{ power} \right\}$	$\sim B \cdot B^{1/m}$
$\sum \left(1 - \frac{1}{m_i}\right) [P_i]$	$\left\{ \frac{a}{b} : \dots \right\}$	$\sim B^\chi?$

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$$\sim B^2 \left(\frac{1}{B^{1-1/m_1}} \right) \cdots \left(\frac{1}{B^{1-1/m_N}} \right) = B^\chi.$$

Heuristics for Campana's orbifolds and for almost integral points

We use two facts.

Fact (Erdős-Szekeres 1935)

The number of m -powerful integers in $[1, B]$ is $\sim B^{1/m}$ as $B \rightarrow \infty$.

(In fact, they proved a more precise asymptotic formula.)

Since the number of m -powerful integers up to B is (up to a constant factor) the same as the number of m^{th} powers up to B , the asymptotic behavior of $\#(\mathbb{P}^1 - \Delta)_C(\mathbb{Z})_{\leq B}$ should match that of $\#(\mathbb{P}^1 - \Delta)(\mathbb{Z})_{\leq B}$.

Fact

For $r \in [0, 1]$, the number of integers in $[1, B]$ whose radical is $< B^r$ is $B^{r+o(1)}$ as $B \rightarrow \infty$.

This gives an analogous prediction for $\#(\mathbb{P}^1 - \Delta + r)(\mathbb{Z})_{\leq B}$.

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	Darmon $\mathbb{P}^1 - \Delta$	Campana $(\mathbb{P}^1 - \Delta)_C$	Almost integral $\mathbb{P}^1 - \Delta + r$
$\chi > 0$	$\sim B^\chi$ (Beukers)	$\sim B^\chi?$	$B^{\chi+o(1)}?$
$\chi = 0$	$(\log B)^{O(1)}$ (Mordell-Weil)	$(\log B)^{O(1)}?$	$B^{o(1)}?$
$\chi < 0$	finite (Siegel, Faltings, Darmon- Granville)	finite? (Campana)	finite?

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All are true if $N \leq 2$.

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$\chi = 0$	$(\log B)^{O(1)}$ (Mordell-Weil)	$(\log B)^{O(1)}?$	$B^{o(1)}?$
$\chi < 0$	finite (Siegel, Faltings, Darmon- Granville)	finite? (Campana) ($\Leftarrow abc$)	finite? ($\iff abc$)

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$\chi > 0$	$\sim B^\chi$ (Beukers)	$\sim B^\chi?$	$B^{\chi+o(1)}?$
$\chi = 0$	$(\log B)^{O(1)}$ (Mordell-Weil)	$(\log B)^{O(1)}?$ ($\Rightarrow *$)	$B^{o(1)}?$
$\chi < 0$	finite (Siegel, Faltings, Darmon- Granville)	finite? (Campana) ($\Leftarrow abc$)	finite? ($\Longleftrightarrow abc$)

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All are true if $N \leq 2$.

* Given an elliptic curve over a number field, the ranks of its twists are uniformly bounded.

Consequences of the Campana column

Example

Consider $(\mathbb{P}^1 - \Delta)_\mathbb{C}$ with $\Delta := \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[\infty]$.

So $\chi = 1/2$. Then the number of solutions to

$$\begin{cases} x + y = z, \\ x, y, z \in \mathbb{Z} \cap [1, B] \text{ squareful,} \\ \gcd(x, y, z) = 1 \end{cases}$$

is $\sim B^{1/2}$?

Is the following related?

Theorem (Blomer 2005)

The number of integers in $[1, B]$ expressible as the sum of two squareful integers is

$$\frac{B}{(\log B)^{1-2^{-1/3}+o(1)}}$$

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Example

Take $\Delta := [\infty] + \frac{1}{2}[0] + \frac{1}{2}[1]$. So $\chi = 0$. Then
 $\{a \in \mathbb{Z} \cap [1, B] : a, a+1 \text{ are both squareful}\} = (\log B)^{O(1)}?$

Is it $O(\log B)$?

Well known: the Pell equation $x^2 - 8y^2 = 1$ proves $\gtrsim \log B$.

Example

Take $\Delta := [\infty] + \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{2}[2]$. So $\chi = -1/2$. Then

$$\{a \in \mathbb{Z}_{\geq 1} : a, a+1, a+2 \text{ are all squareful}\}$$

is finite?

Conjecture (Erdős 1975)

The set in the previous example is empty.

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Example

Take $\Delta := [0] + [\infty] + \frac{1}{2}[1]$ over $\mathbb{Z}[1/5]$. So $\chi = -1/2$. Then

$$\{n \geq 1 : 5^n - 1 \text{ is squareful}\}$$

is finite?

Can linear forms in logarithms prove this?

It seems not.