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Krishnaswami Alladi and Paul Erdős

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K. ALLADI AND P. ERDÖS

We discuss in this paper arithmetic properties of the function $A(n) = \sum_{p^{\alpha} \| n} \alpha p$. Asymptotic estimates of $A(n)$ reveal the connection between $A(n)$ and large prime factors of n. The distribution modulo 2 of $A(n)$ turns out to be an interesting study and congruences involving $A(n)$ are considered. Moreover the very intimate connection between $A(n)$ and the partition of integers into primes provides a natural motivation for its study.

0. Introduction. Let a positive integer n be expressed as a product of distinct primes in the canonical fashion $n = \prod_{i=1}^{r} p_i^{\alpha_i}$. Define a function $A(n) = \sum_{i=1}^{r} \alpha_i p_i$.

(i) The function $A(n)$ is not injective. In fact for a fixed integer m, the number of solutions in n to $A(n) = m$. is the number of partitions of m into primes.

(ii) $A(n)$ fluctuates in size appreciably. It is easily seen that $A(n) = n$ when n is a prime, while $A(n) = O(\log n)$ when n is a power of a small prime. Actually the "average order" of $A(n)$ turns out

It is surprising that the function $A(n)$ with such nice arithmetic

properties has not been studied in detail. Besides the work of

[10]. Of course the contents of this paper are different.

for all $\delta \geq 2$. We have

$$1 \quad \int^{\infty} d\pi(y) \quad \int^{\infty} dy \quad \int^{\infty} d\{\pi(y) - li(y)\}$$

(1.2) $$= \int^{\infty} dy + O(1)$$

$$+ \int^{\infty} \{\pi(y) - li(y)\} O(1) dy$$

$$\frac{1}{\ldots} = \frac{1}{\ldots} + O\left(\frac{\log\log x}{\ldots}\right)$$

because log (z/x) and log x are of the same order of magnitude. Now

Note that each p_i can range from $P_1(k)$ up to the minimum of p_{i+1}

and $x/kp_m \cdots p_{i+1}$. So we shall break up the range of p_{i+1}, and discuss several cases, and in each of them we shall be able to decide without ambiguity which of p_{i+1} and $x/kp_m \cdots p_{i+1}$ is smaller, thereby

determining the range of p_i.

and p_{i+1} for $i = 1, 2, \cdots, m-1$.

Case 2. Now let $\sqrt[m]{x/k} < p_m \leq x/k$. We have now several choices.

[redacted]

$p_i \leq \sqrt[i]{x/kp_m \cdots p_{i+1}}$ $p_{i-1} \leq p_i$ $p_2 \leq p_3$ $p_1 \leq p_2$

$\cdots$

$\cdots$

Last term.

$\sum \quad \sum \quad \sum \quad \sum \quad \cdots$

[redacted]

Note that the term in (1.7) is just the term in (1.8) with i replaced by $i+1$. Thus making the first m summations of (S_i) gives

(1.8) $O\Big(\; x^{1+(1/m)} \; \Big)$

Obviously (1.8) summed over k gives $O(x^{1+(1/m)}/\log^m x)$.

Now we proceed to the asymptotic estimate. We shall see that the leading terms we get are exactly those mentioned above. But the error terms can be estimated just as we got upper bound estimates

$\sum$ $x^{1+(1/i)}$

$ix^{1+(1/i+1)}$

$(i+1)!\,(kp \cdots p)^{1+(1/i+1)} L(i+1, i+1, x/kp \cdots p)$

(1.12). What we are summing in (1.13) is the term in (1.11). In the process of going from (1.11) to (1.12) note that what has happened is that i has been replaced by $i + 1$ for the variables and there is an extra factor of i. So making the first j_1 summations we get a term which is

(1.14) $$i(i + 1) \cdots (i + j_1 - 1)x^{1+(1/i+j_1)}$$

we find that the exponent of x which had remained constant for

[illegible]

$$(1.18) \qquad (\text{constant}) \cdot x^{1+1/(i+j_1+i_1+1)}$$

[illegible]

as shown by our investigation of error terms. Our theorem will be and

$$\sum p \frac{x}{\ } = x \sum \frac{1}{\ } = O\Big(\sum p \Big) .$$

So

$$\sum_{i \geqq 3} \sum p \frac{x}{p^i} = O(x)$$

which proves Theorem 1.5.

THEOREM 1.6. *We have*

$$\sum \{A^*(n) - P(n) - P(n) - \cdots - P(n)\} \sim \sum P(n) \sim k_m x^{1+1/m}$$

Proof. The theorem follows by combining Theorems 1.1 and 1.5.

THEOREM 1.7. *For any fixed integer M, the set of solutions to $A(n) - A^*(n) = M$ has a natural density* > 0. (Note: A sequence $\{a_n\}_{n=1}^{\infty}$ has a natural density $\delta(A)$ if $\lim_{n \to \infty} n/a_n = \delta(A)$.)

$$\delta\left(\bigcup_{j=1}^{p(M)} S_{m_j}\right) = \sum_{j=1}^{p(M)} \delta(S_{m_j}) \tag{1.24}$$

as $S_{m_i} \cap S_{m_j} = \varnothing$ if $i \neq j$. In fact because of (1.23) and (1.24) the

density is a rational multiple of $1/\zeta(2) = 6/\pi^2$.

2. Congruences involving $A(n)$.

We now recall some results in [1]. For any integer m, the number of solutions to $A(n) = m$ is the number of partitions of m into primes. Note that $A(n) = n$ if and

$$\psi(x, y) < \frac{c_1 x \log x \log\log x}{[illegible]}$$

[illegible]

(2.4) $$= O\left(\frac{x}{[illegible]}\right)$$

[illegible]

So we will restrict our attention to $P_1(n) > e^{\sqrt{\log x \log\log x}}$ for the number of n not satisfying this is given by (2.4). We also assume that if $\tau(n)$ is the number of divisors of n then

(2.5) $$\tau(n) < e^{1/2\sqrt{\log x \log\log x}}$$

[illegible]

For the number of integers not satisfying (2.5) is easily seen to be

(2.6) $$O\left(\frac{x \log x}{[illegible]}\right)$$

[illegible]

because $\sum \tau(n) = O(x \log x)$. So we confine ourselves to $n \leq x$ satis-

[illegible]

[redacted]

Thus for fixed t, the number of solutions to (2.7) in special numbers

[redacted]

is at most

$$\log x + e^{1/2\sqrt{\log x \log\log x}} = O(e^{1/2\sqrt{\log x \log\log x}}) .$$

But by (2.8) we have an upper bound on the number of choices of t. Thus the $\{l_i\}$ among the n_i do not exceed

[redacted]

(2.11) $$O\left(\frac{x}{\quad} e^{1/2\sqrt{\log x \log\log x}}\right) = O\left(\frac{x}{\quad}\right)$$

[redacted]

But the number of integers not included among the $\{n_i\}$ is by (2.6)

[redacted]

and (2.4)

(2.12) $$O\left(\frac{x \log x}{e^{1/2\sqrt{\log x \log\log x}}}\right) .$$

So (2.12) and (2.11) prove the theorem with any $c < 1/2$.

Now for a lower bound.

[redacted]

Let us put [illegible] > 0 arbitrary. So we have at least for $A(2 \cdot p \cdot p_1 \cdot p_2 \cdot \cdots \cdot p_r) = 2p$.

One can show that for sufficiently large composite numbers n, there exists m with $m \equiv 0 (\bmod A(m))$, $A(m) = n$ and m/n square free and prime to n. This follows from Vinogradov's theorem, and here we partition $n - A(n)$ into primes. It might be of interest to de-

[redacted]

[redacted]

true.

3. Distribution modulo 2. First we shall show that $A(n)$ is uniformly distributed modulo 2, and the error is of the order of

the sum of the Möbius function $M(x)$. Here we shall concentrate

[redacted]

Now if μ is the Möbius function then

Now by (3.5), (3.10) is rewritten as

$$x\sum \frac{\alpha(d)}{[illegible]} = c_1 x \sum \frac{\mu(m)}{[illegible]} - \sum \mu(m)\frac{x}{[illegible]} \sum \frac{a(n)}{[illegible]}$$

[illegible]

We can deduce Theorem 3.2 from (3.11), if we appeal to Axer's Theorem 267 in [5] stated below.

[illegible]

interest to consider the relative sizes of $f(n)$ and $F(n)$.

In this context we mention the following curious problem. Replace

$$G(n) = \min \quad \prod a_i^2; \; g(n) = \prod b_i^2 \quad \sum a_i^2 = n$$

where b_i^2 is the largest square $\leq n$. and so on. It might be true that both $G(n)$ and $g(n)$ are both $< c \cdot n^2$ where c is a constant. In $G(n)$ above, we require that not more than three of the $a_i = 1$. for $3 = 1 + 1 + 1$ is the only decomposition of 3.

For more results on $A(n)$. see a forthcoming paper of Erdös and Pomerance where it is proved that the set of solutions to $A(n) = A(n+1)$ is of density zero. One could also consider equations involv-

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[redacted]

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