Generalized Euler and Class Numbers

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1. Introduction. In [1] we discussed the Dirichlet series

$$(1) \qquad L_a(s) = \sum_{k=0}^{\infty} \left(\frac{-a}{2k+1}\right)(2k+1)^{-s}$$

The first row are the *Euler* numbers: [redacted]

$$c_{1,n} = E_n \,, \tag{5}$$

which are also called *secant* numbers since [redacted]

$$\sec w = \sum_{n=0}^{\infty} E_n \frac{w^{2n}}{(2n)!} \,. \tag{6}$$

The first column are the *class* numbers; that is, there are $c_{a,0}$ inequivalent classes of primitive binary quadratic forms

$$Cu^2 + 2Buv + Av^2$$

with

$$AC - B^2 = a \,,$$

the principal form of which is represented by [redacted]

This time the first … $u^2 + av^2$.

Our two-dimensional array $c_{a,n}$ therefore generalizes both the Euler numbers and the class numbers—thus our title. [redacted]

Similarly, a short table of $d_{a,n}$ is shown below. (The number $D_{a,n}$ in (9) actuall [redacted] and the known Fourier expansions:

$$(10)\qquad \begin{aligned} E_{2n}(x) &= \frac{(-1)^n 4(2n)!}{\pi^{2n+1}} S_{2n+1}\left(\frac{x}{2}\right), \\ E_{2n-1}(x) &= \frac{(-1)^n 4(2n-1)!}{\pi^{2n}} C_{2n}\left(\frac{x}{2}\right), \end{aligned}$$

where [1, Eq. (18)]

$$(11)\qquad \begin{aligned} S_s(x) &= \sum_{k=0}^{\infty} \frac{\sin 2\pi(2k+1)x}{(2k+1)^s}, \\ C_s(x) &= \sum_{k=0}^{\infty} \frac{\cos 2\pi(2k+1)x}{(2k+1)^s}. \end{aligned}$$

It follows, if we put

$$x = 2y \quad \text{and} \quad t = 2vi$$

in (9), that

$$(12)\qquad \begin{aligned} \frac{\pi}{4}\frac{\cos v(1-4y)}{\cos v} &= \sum_{n=0}^{\infty} \left(\frac{2v}{\pi}\right)^{2n} S_{2n+1}(y), \\ \frac{\pi}{4}\frac{\sin v(1-4y)}{\cos v} &= \sum_{n=1}^{\infty} \left(\frac{2v}{\pi}\right)^{2n-1} C_{2n}(y). \end{aligned}$$

Now, clearly,

$$(13)\qquad L_1(s) = S_s(\tfrac{1}{4}) \quad \text{and} \quad L_{-1}(s) = C_s(0),$$

so that from (12) and (4), together with (6) and (8), we find that $c_{1,n}$ and $d_{1,n}$ are indeed the secant and tangent numbers, respectively.

$$(14)\qquad a = bm^2$$

with b square-free, we have [1, Eq. (23)]

$$(15)\qquad L_a(s) = L_b(s) \prod_{p_i \mid m} \left[1 - \left(\frac{-b}{p_i}\right) p_i^{-s}\right],$$

the product being taken over all odd primes p_i (if any) that divide m.

if $b = 1$. In any case, the $c_{a,n}$ and $d_{a,n}$ are integral multiples of the $c_{b,n}$ and $d_{b,n}$, respectively.

It remains, then, to compute $c_{b,n}$ and $d_{b,n}$ for square-free $b > 1$. We showed, in [1], that for such b we have

$$(18)\qquad L_b(2n+1) = \frac{2}{\sqrt{b}} \sum_k \epsilon_k S_{2n+1}(y_k) , \qquad L_{-b}(2n) = \frac{2}{\sqrt{b}} \sum_k \epsilon_k C_{2n}(y_k) ,$$

where in the linear combinations on the right the ϵ_k are Jacobi symbols, and the y_k are rational numbers, both dependent upon b. In all such cases, we therefore have

$$
\begin{aligned}
\mathfrak{C}_{2,n} &= (-1)^n\,, \\
\mathfrak{C}_{3,n} &= (-1)^n\,, \\
\mathfrak{C}_{5,n} &= (-1)^n[4^{2n} + 2^{2n}]\,, \\
\mathfrak{C}_{6,n} &= (-1)^n[5^{2n} + 1^{2n}]\,, \\
\mathfrak{C}_{7,n} &= (-1)^n[3^{2n} + 1^{2n} - 5^{2n}]\,, \\
\mathfrak{C}_{10,n} &= (-1)^n[9^{2n} - 7^{2n} + 3^{2n} + 1^{2n}]\,, \\
\mathfrak{D}_{2,n} &= (-1)^{n-1}\,, \\
\mathfrak{D}_{3,n} &= (-1)^{n-1}2^{2n-1}\,, \\
\mathfrak{D}_{5,n} &= (-1)^{n-1}[1^{2n-1} + 3^{2n-1}]\,, \\
\mathfrak{D}_{6,n} &= (-1)^{n-1}[5^{2n-1} + 1^{2n-1}]\,, \\
\mathfrak{D}_{7,n} &= (-1)^{n-1}[6^{2n-1} + 4^{2n-1} - 2^{2n-1}]\,, \\
\mathfrak{D}_{10,n} &= (-1)^{n-1}[9^{2n-1} + 7^{2n-1} - 3^{2n-1} + 1^{2n-1}]\,.
\end{aligned}
\tag{23}
$$

[redacted] as a linear combina-

[redacted]

are all of these numbers integers.

[redacted]

2143, 3142, 3241, 4132, and 4231

are the five ways in which 1234 may be permuted in which successive differences

MTAC 22, 1968

CORRIGENDA

DANIEL SHANKS & JOHN W. WRENCH, JR., "The calculation of certain Dirichlet

[redacted]