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## Generalized Euler and Class Numbers

By Daniel Shanks

1. Introduction. In [1] we discussed the Dirichlet series

$$(1) \quad L_a(s) = \sum^{\infty} \left( \frac{-a}{2k+1} \right) (2k+1)^{-s}$$

364 ✓

→ 3 ✓

233 ✓

362 ✓

508 ✓

281 ✓

1586 ✓

1587 ✓

435 ✓

190 ✓

187 ✓

192 ✓

608 ✓

191 ✓

173 ✓

191 ✓

61 ✓

115 ✓

119 ✓

Euler nos

2769 → 281 ✓

26528 → 36 ✓ 1586 ✓

109329 → 187 ✓

3437246 → 192 ✓

354600 → 608 ✓

305210 → 191 ✓

484 → 173 ✓

305 → 191 ✓

116 → 61 ✓

305 → 191 ✓

116 → 115 ✓

305 → 191 ✓

116 → 119 ✓

305 → 191 ✓

116 → 187 ✓

305 → 191 ✓

116 → 192 ✓

305 → 191 ✓

The first row are the *Euler numbers*.

$$(5) \quad c_{1,n} = E_n,$$

which are also called secant numbers since

$$(6) \quad \sec w = \sum_{n=0}^{\infty} E_n \frac{w^{2n}}{(2n)!}.$$

The first column are the *class numbers*; that is, there are  $c_{a,0}$  inequivalent classes of primitive binary quadratic forms

where [1, Eq. (18)]

$$Cu^2 + 2Buw + Av^2 \quad (11)$$

with

$$AC - B^2 = a,$$

the principal form of which is represented by

Our two-dimensional array  $c_{a,n}$  therefore generalizes both the Euler numbers and the class numbers—thus our title.

product being taken follows, from (3)

in (9) that

Similarly, a short table of  $d_{a,n}$  is shown below. (The number  $n$  is omitted.)

and the known Fourier expansions:

$$(10) \quad \begin{aligned} E_{2n}(x) &= \frac{(-1)^n 4(2n)!}{\pi^{2n+1}} S_{2n+1}\left(\frac{x}{2}\right), \\ E_{2n-1}(x) &= \frac{(-1)^n 4(2n-1)!}{\pi^{2n}} C_{2n}\left(\frac{x}{2}\right), \end{aligned}$$

where [1, Eq. (18)]

$$(11) \quad \begin{aligned} S_s(x) &= \sum_{k=0}^{\infty} \frac{\sin 2\pi(2k+1)x}{(2k+1)^s}, \\ C_s(x) &= \sum_{k=0}^{\infty} \frac{\cos 2\pi(2k+1)x}{(2k+1)^s}. \end{aligned}$$

It follows, if we put

$$x = 2y \quad \text{and} \quad t = 2vi$$

in (9), that

$$(12) \quad \begin{aligned} \frac{\pi}{4} \frac{\cos v(1-4y)}{\cos v} &= \sum_{n=0}^{\infty} \left(\frac{2v}{\pi}\right)^{2n} S_{2n+1}(y), \\ \frac{\pi}{4} \frac{\sin v(1-4y)}{\cos v} &= \sum_{n=1}^{\infty} \left(\frac{2v}{\pi}\right)^{2n-1} C_{2n}(y). \end{aligned}$$

Now, clearly,

$$(13) \quad L_1(s) = S_s\left(\frac{1}{4}\right) \quad \text{and} \quad L_{-1}(s) = C_s(0),$$

so that from (12) and (4), together with (6) and (8), we find that  $c_{1,n}$  and  $d_{1,n}$  are indeed the secant and tangent numbers, respectively.

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$$(14) \quad a = bm^2$$

with  $b$  square-free, we have [1, Eq. (23)]

$$(15) \quad L_a(s) = L_b(s) \prod_{p_i|m} \left[ 1 - \left(\frac{-b}{p_i}\right) p_i^{-s} \right],$$

the product being taken over all odd primes  $p_i$  (if any) that divide  $m$ .

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if  $b = 1$ . In any case, the  $c_{a,n}$  and  $d_{a,n}$  are integral multiples of the  $c_{b,n}$  and  $d_{b,n}$ , respectively.

It remains, then, to compute  $c_{b,n}$  and  $d_{b,n}$  for square-free  $b > 1$ . We showed, in [1], that for such  $b$  we have

$$(18) \quad L_b(2n+1) = \frac{2}{\sqrt{b}} \sum_k \epsilon_k S_{2n+1}(y_k), \quad (23)$$

$$L_{-b}(2n) = \frac{2}{\sqrt{b}} \sum_k \epsilon_k C_{2n}(y_k),$$

where in the linear combinations on the right the  $\epsilon_k$  are Jacobi symbols, and the  $y_k$  are rational numbers, both dependent upon  $b$ . In all such cases, we therefore have

multiples of the  $c_{b,n}$  and  $d_{b,n}$ ,  
-free  $b > 1$ . We showed, in

$y_k$  , (23)

) ,

Jacobi symbols, and the  $y_k$   
ch cases, we therefore have

$$\begin{aligned} C_{2,n} &= (-1)^n, \\ C_{3,n} &= (-1)^n, \\ C_{5,n} &= (-1)^n[4^{2n} + 2^{2n}], \\ C_{6,n} &= (-1)^n[5^{2n} + 1^{2n}], \\ C_{7,n} &= (-1)^n[3^{2n} + 1^{2n} - 5^{2n}], \\ C_{10,n} &= (-1)^n[9^{2n} - 7^{2n} + 3^{2n} + 1^{2n}], \\ D_{2,n} &= (-1)^{n-1}, \\ D_{3,n} &= (-1)^{n-1}2^{2n-1}, \\ D_{5,n} &= (-1)^{n-1}[1^{2n-1} + 3^{2n-1}], \\ D_{6,n} &= (-1)^{n-1}[5^{2n-1} + 1^{2n-1}], \\ D_{7,n} &= (-1)^{n-1}[6^{2n-1} + 4^{2n-1} - 2^{2n-1}], \\ D_{10,n} &= (-1)^{n-1}[9^{2n-1} + 7^{2n-1} - 3^{2n-1} + 1^{2n-1}]. \end{aligned}$$

... simple recurrences we express  $c_{b,n}$  ( $d_{b,n}$ ) as a linear combina-

2143, 3142, 3241, 4132, and 4231

are the five ways in which 1234 may be permuted in which successive differences

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CORRIGENDA

DANIEL SHANKS & JOHN W. WRENCH, JR., "The calculation of certain Dirichlet

REVIEW

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