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Computation of Tangent, Euler, and Bernoulli Numbers*

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Abstract. Some elementary methods are described which may be used to calculate tangent numbers, Euler numbers, and Bernoulli numbers much more easily and rapidly on electronic computers than the traditional recurrence relations which have been used for over a century. These methods have been used to prepare an accompanying table which extends the existing tables of these numbers. Some theorems about the periodicity of the tangent numbers, which were suggested by the tables, are also proved.

1. Introduction. The tangent numbers T_n, Euler numbers E_n, and Bernoulli numbers B_n, are defined to be the coefficients in the following power series:

$$\ldots = T_0/0! + T_1 z/1! + T_2 z^2/2! + \cdots = \sum T_n z^n/n!,$$

then the coefficient of $z^n/n!$ in $(\tan z)\,(\cos z)$ is

$$\sum_k \binom{n}{k} T_k C_{n-k}$$

and in $(\sec z)\,(\cos z)$ it is

$$\sum_k \binom{n}{k} E_k C_{n-k}\,.$$

Hence, making use of the fact that $T_{2n} = E_{2n+1} = 0$, we have the recurrence re-

[redacted]

(4) $$\binom{2n+1}{1}T_1 - \binom{2n+1}{3}T_3 + \cdots + (-1)^n\binom{2n+1}{2n+1}T_{2n+1} = 1\,, \qquad n \geqq 0\,;$$

(5) $$\binom{2n}{0}E_0 - \binom{2n}{2}E_2 + \cdots + (-1)^n\binom{2n}{2n}E_{2n} = 0\,, \qquad n > 0\,.$$

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large number T_{-+} and $n/2$ additions of large numbers. Since we are interested only

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to be retained in the computer memory at any one time. A further technique can be employed when the memory size has been exceeded; for example, suppose we start with the computation of T_{nk} for $n \leq 4$:

[redacted]

$n = 3$	2	0	8	0	6		
$n = 4$	0	16	0	40	0	24	

and suppose that very little memory space is available, so that we cannot completely evaluate all of the entries for $n = 5$; we might obtain

$n = 5$	16	0	136	0	240	0	*

where "*" denotes an unknown value. The calculation may still proceed, keeping track of unknown values:

$n = 6$	0	272	0	1232	0	*
$n = 7$	272	0	3968	0	*	

[redacted]

$n = 9$	7936	0	*	etc.

In this way we may compute the values of about twice as many tangent numbers

(a) Set area B to k times the next value indicated by P, and move P to the right.

(b) Store the value of $A + B$ into the locations indicated by Q, and move Q to the right.

(c) Transfer the contents of B to area A.

(d) Increase k by 2.

In the case of (13) we would change the memory configuration to

(14)

6	.	1	6	,	4	0	,	2	4	.	1	6	,
↑ Q					↑ P								

$k = 3 \quad A = 16 \quad B = 16$

also has advantages over (10) for the same reason.

It remains to consider the [redacted]

For $T_{n,k}$ can contribute nothing to any subsequent value of T_n when $k \geqq p$.

We will show below that the minimum polynomial equation satisfied by A is [redacted]

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$$+\ 4\cdot5\cdot6\cdot7 + 4\cdot5\cdot7\cdot1 + 5\cdot6\cdot7\cdot1$$

(modulo 7). Let us say two terms $a_1(a_1 + 1) \cdots a_k(a_k + 1)$ and $a_1'(a_1' + 1) \cdots a_k'(a_k' + 1)$ are "equivalent" if, for some r and t and for all j, $a_j \equiv a'_{(j+r) \bmod p} + t$; thus, in the above example the terms $1\cdot2\cdot4\cdot5$, $2\cdot3\cdot5\cdot6$, $3\cdot4\cdot6\cdot7$, $4\cdot5\cdot7\cdot1$, $5\cdot6\cdot1\cdot2$, $6\cdot7\cdot2\cdot3$, $7\cdot1\cdot3\cdot4$ are mutually equivalent. It is impossible for a term to be equivalent to itself when $0 < t < p$, since this would imply $a_1 + \cdots + a_k \equiv a_1 + \cdots + a_k + kt$, and $t \equiv 0$. Therefore, each equivalence class has precisely p terms in it. When $k < (p - 1)/2$ the sum over an equivalence class has the form

[redacted]

by Kummer make it possible to establish further results about the period-length:

[redacted]

$$E_{n+\lambda p^{k-1}} \equiv E_n \ (modulo\ p^k)\ , \qquad n \geqq k\ . \tag{35}$$

Proof. Assume $n \geqq k$ and define the sequence $\langle u_m \rangle$ by the rule

$$u_m = (-1)^{(p-1)m/2} T_{n+(p-1)m}\ , \qquad m \geqq 0\ . \tag{36}$$

Kummer's congruence for the tangent numbers may be written

[redacted]

TABLE 1. *The first 60 nonzero tangent numbers*

5342976.				
5112832.				
3124096.				
6852352.				
2148019	0207983616.			
0160394	3959887872.			
8780959	7841473536.			
2516929	6170811392.			
4077037	2052957109	7509625856.		
6934508	8878007175	0349094912.		
7850591	1237939970	7952152576.		
3100439	4319164162	5494290432.		
5983772	0830318280	1462400217	5135645696.	
1907431	9093041833	1629578783	7183229952.	
9930886	4886002843	3631616460	3920777216.	
0025215	7859031027	1604996431	9565816634	0757225472.
6465792	8428175235	4450739249	2859204777	5873499136.
1567180	8914892880	7269899849	6749880539	8829268992.
6159128	0258358007	9642459877	5270244950	5540243456.
4112879	0088327017	8745605712	6484887310	5817055122
31232512.				
13357185	8035017444	9297951396	8470814438	2678544895
7122176.				
13421559	7452258201	0891145370	3030136589	7345704226
21068032.				
37958001	3937369998	9682771463	4659411015	0902726842
83100116	1763127296.			
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]			
[illegible]	[illegible]	[illegible]		
[illegible]	[illegible]	[illegible]		
5763027	8735849943	4326866459	2938843795	4320994101
7258153	1136084889	9380084736.		
6789446	5395544961	6369630081	3886273130	1380715918
2218698	6494595788	9839726592.		
9227344	4371901652	3516486805	2457457867	9114870464
0696607	0734990348	6020216502	7058221056.	
9829116	7370744681	8719138436	0603858737	8520405728
[illegible]	[illegible]	[illegible]	8174170112.	
[illegible]	[illegible]	[illegible]		

[redacted]

6817088804
0509816834
2529915556
6539875280
3638482311
6029941974
0384517339
9248162391
6628003769
6336438272.
2111631319
1015654470
9952935936.
4430020735
7007093225
8372641792.
4375913985
9280405845
5414739516
0048288233
0653272241
0669176471
3804707134
6390752650
6193750216
8558730200
5964664757
6527175740

		6738445312.				
117	5723585022	9555589879	7004078003	1278606958	7871036923	3804707134
		9116874639	2466184489	3499287007	8763836938	6390752650
		8633084261	1483758302	7014497286	3537856412	6193750216
		2020990976.				
119	3257	2969544137	3711110813	9491520587	0894578681	8558730200
		7333881055	9724342116	8172307776	6222847786	5964664757
		3601851664	4828218413	9690510871	7176120451	6527175740
		8580920993	7947000832.			

364

n	E_n					
0	1.					
2	1.					
4	5.					
6	61.					
8	1385.					
10	50521.					
12	2702765.					
14	199360981.					
16	1	9391512145.				
18	240	4879675441.				
20	37037	1188237525.				
22	6934887	4393137901.				
24	1551453416	3557086905.				
26	40	8707250929	3123892361.			
28	12522	5964140362	9865468285.			
30	4415438	9324902310	4553682821.			
32	1775193915	7953928943	6664789665.			
34	80	7232992358	8789806216	8247453281.		
36	41222	0603395177	0212234707	9671259045.		
38	23489580	5270431082	5201782857	6198947741.		
40	1	4851150718	1149800178	7715678140	5826684425.	
42	1036	4622733519	6121193979	5730474518	5976310201.	
44	794757	9422597592	7036080405	1008807061	9519273805.	
46	666753751	6685544977	4350284747	7374819752	4107684661.	
48	60	9627864556	8542158691	6857428768	4315397653	9044435185.
50	60532	8524818862	1896314383	7851116490	8810349822	5146815121.
52	65061624	8668460884	7715870634	0808229834	8364423676	5385576565.
54	7	5466599390	0873909806	1432565889	7367442122	4002471169
		9858645581.				
56	9420	3218964202	4120420228	6237690583	2272093888	5259964600
		9394905945.				
58	12622019	2518062187	1990340923	7287489255	4823410611	9182559406
		9964920041.				
60	1	8108911496	5792304965	4580774165	2158688733	4873492363
		1410600809	5454231325.			
62	2775	7101702071	5805973669	8090837152	7449233019	5948009175
		7803378276	6889782501.			
64	4535810	3330017889	1747468878	7156776236	6351861519	4703688814

[redacted]

[redacted]	7689327097
[redacted]	2858314932
[redacted]	8255239879
[redacted]	2954929765
[redacted]	1912367260
[redacted]	7068070281
[redacted]	0790510830
[redacted]	1945185560
[redacted]	5396878225.
[redacted]	2205397659
[redacted]	9951554801.
[redacted]	5845492837
[redacted]	7502043638
[redacted]	8391683907
[redacted]	8636544057
[redacted]	6610030678
[redacted]	9174800620
[redacted]	4538867236
[redacted]	3021532175
[redacted]	4884315911
[redacted]	5507146314
[redacted]	1341392681
[redacted]	8819075342
[redacted]	4449970053
[redacted]	[illegible]
[redacted]	1810672327
[redacted]	2770950810
[redacted]	1589450801
[redacted]	[illegible]

[illegible]

ernoulli numbers

$\leqq n \leqq 125$ appear below in the form $C_{2n} - \{p_1, p_2, \cdots, p_k\}$. 2, 3, 5} = 1 − 1/2 − 1/3 − 1/5 = −1/30. The Bernoulli C_{2n} have not been tabulated before.

47}				
5,7,13,17}				
11}				
5,53}				
7,19}				
643290	− {2,3,5,29}			
192193	− {2,3,59}			
191096	− {2,3,5,7,11,13,31,61}			
231026	− {2,3}			
701846	− {2,3,5,17}			
035276	9264581471	− {2,3,7,23,67}		
[illegible]	[illegible]	− {2,3,5}		
[illegible]	2435219412	− {2,3,11,71}		
235893	6689603010	− {2,3,5,7,13,19,37,73}		
586444	8051297182	− {2,3}		
346861	7333014550	8927628859	− {2,3,5}	
970289	5524177196	3644484776	− {2,3,7,79}	
132987	6872422013	2825915914		

7,11,13,31,61}			
17}			
1471	— {2,3,7,23,67}		
9412	— {2,3,11,71}		
03010	— {2,3,5,7,13,19,37,73}		
97182	— {2,3}		
14550	8927628859	— {2,3,5}	
77196	3644484776	— {2,3,7,79}	
22013	2825915914		
89471	9384367233	— {2,3,83}	
65106	9049904703		
163051	1444223148		
732987	3916346921		
368511	8091091449		
891627	1995959100		
0659751	4025337827		
4596737	3697590579		
0365954	7507479181		
8092801	2882128228		
4176879	7317880887		
3,7,103}			
6867270	8225328766		
3,5,53}			
1946828	7510420621		
3,107}			
96261472	5464727402		
3,5,7,13,19,37,109}			
54812442	3742649032		
91785388	— {2,3,11,23}		
82740978	4127789638		
00945120	— {2,3,5,17,29,113}		
24084923	5930550859		
76632445	— {2,3,71}		
47616182	6544872627		

ABLE 3—*Continued*

384	0015332666	6438279520	—{2,3,5,59}	
558	2452526426	4167780767	7268467832	
747	5076344103	1489529605	9086182634	—{2,3}
883	2345671293	2445573185	0549877801	
635	0025726591	0252803139	1154956835	
062	4599845957	3120465051	8433566283	
350	0718881721	8561301633	9661427406	—{2,3}
369	1808148735	2627667109	9112273184	
531	4804543981	2034228242	2969820299	
231	0024302926	6798669571	9179638977	
2073	1833362242	1938478819	1283226347	
9,43,127}				
4028	2462456517	5446919894	0377552432	
8183	3065745383	0640452814	1149421273	
17}				
2981	9883872814	3738272150	8758785424	
5171	2245962893	1773876814	5763813725	
,131}				
3398	9802393011	6690267498	5678971000	
2984	4759158434	4882999447	8018574251	
7,13,23,67}				
7227	2622874813	1691918757	5426552811	
2942	6528175678	7997886065	2087390581	
08362	—{2,3}			
2933	7727583015	4864565966	9040083595	
04264	3022208918	6918602388	7468948154	
77286	—{2,3,5,137}			
29696	0209752104	1491857990	7241070558	
19391	1585658026	7855114057	4147212665	
64679	—{2,3,7,47,139}			
67099	6341276113	0549942324	6203851158	
21525	5521783095	3721687111	4312353272	
09228	—{2,3,5,11,29,71}			
09783	4674040886	9039967369	5039984404	
57676	9302738510	9499436486	8624711701	
13467	8897524866	—{2,3}		
[illegible]	[illegible]	2181183117	1196320118	
82645	3276869447	0578038003	7383050883	
32674	5245777737	—{2,3,5,7,13,17,19,37,73}		
88859	8464400436	0924268134	7568589956	
13920	7141426350	0143698420	6381706690	
21194	73[illegible]1637478	—{2,3}		
16949	78[illegible]9541637	9556814489	5492650402	
[illegible]	[illegible]	[illegible]	[illegible]	

[redacted]

078	872701906
920	004565489
607	177774046
,11,13,1	1,37,61,181}
047	3371527694
453	5232578272
108	8555947991
988	1600926988
157	7645148677
536	1359613968
016	2,3,5,47}
733	[illegible]
636	036916065
322	971803519
941	2,3,7}
757	492683808
437	055520607
878	859946730
699	2,3,5}
552	981012117
584	766966132
883	678744934
476	921781652
147	560005423
870	033629100
544	020534114
248	410299256
821	264722872
353	294345595
812	525149224
963	889003542
302	904344422
338	245017672
802	548317388
055	284840257
414	596881957
914	587611834
140	213909430
835	071239515
468	191058737
006	387496808
800	516180654
084	451297095
585	118603710
820	298817680
299	768640406
533	283250606

	3429355657	1403660905
	1178927312	4036225303
	9281199287	3335083984
	8224372602	4150845715
03}		
	8242731374	8983148899
	8334448739	3615611074
	4679418664	6682265708
	1840875841	2535715340
	9370917366	3618746795
	9308402125	9096461499
	4384998570	4423384432
	9360250776	4120246691
3}		
4	9676853975	9962892161
2	7204489612	7090496935
4	5210024031	0159699351
5	1689483118	7470399162
4	— {2,3,7,11,31,43,71,211}	
1	4251219952	0385256050
7	8826924331	4587153964
2	1550902842	4409830023
7	1858879998	4318266834
76	— {2,3,5,107}	
34	5107953790	8103711340
31	6499243600	5693167818
19	7867121979	7119475720
47	9856004904	0741960957
02	— {2,3}	
82	2957000240	1395817760
80	2936510368	0530514353
51	7344439893	1506681060
45	4283324504	8399788068
61	— {2,3,5,7,13,19,37,73,109}	
762	4852939477	9681195006
284	5981956117	8018896163
871	3098394359	2743108529
041	7772745635	4356010999
869	5021253934	— {2,3}
925	4835379696	4714139788
934	9974452549	7367883712
803	1307012711	0695460686
8426	3152138601	3855375811
7092	5355305441	— {2,3,5,11,23}
8971	7991300396	6452873285
8091	5783619081	8912755517
9614	1975027633	6559804985
0276	5386744234	4443377036
6656	7512564136	— {2,3,7,223}

[illegible]

094314992
166852319
355300226
590200953
776824872
761158145
468153629
482278121
065707036
831390856
7267467117
0999964338
7787555531
4946257333
3517067124
6901932057
3262555126
6800349276
6235632262
5120926143
2790039834
5285569627
4780786847
1564731368
7378396322
9967717969
8585332169
5426701817
4601261827
0829389397
4047848115
1660668994
5047192006
9790766520
3314709961
9537111655
8220377591
4013811373

[illegible]

[redacted]

54667652	7378396322
3,23}	
49699881	9967717969
41813083	8585332169
15624519	5426701817
05318122	4601261827
33353488	0829389397
3,5}	
824410S2	4047848115
72502170	1660668994
41553115	5047192006
61938087	9790766520
64605129	3314709961
,3,7,83}	
04178389	9537111655
13341139	8220377591
29831158	4013811373
55456887	6734383380
21479246	8850325243
58331599	− {2,3,5}
39649247	7000034429
55844047	4285697821
80448568	1973565915
56333382	8622890759
81295360	9407215690
54881572	− {2,3,11,251}

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1. Thomas Clausen, "Theorem." *Astr. Nachrichten*, v. 17, 1840, cols. 351–352.
2. S. A. Joffe, "Calculation of the first thirty-two Eulerian numbers from central differences of zero," *Quart. J. Math.*, v. 47, 1916, pp. 103–126.
3. S. A. Joffe, "Calculation of eighteen more, fifty in all, Eulerian numbers from central differences of zero," *Quart. J. Math.*, v. 48, 1917–1920, pp. 193–271.