check that I have all these →311

N.J.A. Sloane etc 5/16

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CF05

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SORTING NUMBERS FOR CYLINDERS AND OTHER CLASSIFICATION NUMBERS

(2)

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167–176

0. **Introduction.** Set partitions (corresponding to equivalence relations) are in [redacted]

MN

LN

* and mean values (eg of Stirling nos)

$L^N_<$ $L^N_>$ $L^N_=$

$L^N_<$ (in-valence ≤ 1 on L) the set of injections (N-ads in L),

[redacted]

$\ell_>$

[redacted]

$L^N_=$ the set of bijections,
$M! = M^M_=$ the set of permutations of M.

In analogy to $|M^N| = m^n$, $|L^N| = l^n$, $|M!| = m!$ we define

[redacted]

But this doesn't change the numbers

[redacted]

!

[redacted]

$$l^n_> = |L^N_>|, \qquad l^n_< = |L^N_<| = \binom{l}{n} n!,$$

and similarly in the sequel. Since

$$l^n_> = 0 \quad \text{for } l > n \qquad \text{and} \qquad l^n_< = 0 \quad \text{for } n > l,$$

the sums

$$\sum_l l^n_> \qquad \text{and} \qquad \sum_{n=0}^{l} l^n_<$$

are finite; such sums will be abbreviated to

$$\Sigma^n_> \qquad \text{and} \qquad l^\Sigma_<,$$

and their largest terms to

$$\max{}^n_> \qquad \text{and} \qquad l^{\max}_< = l!.$$

Deemphasizing the individuality of N, the sets L^N, $L^N_<$, $L^N_>$, $L^N_=$ can be regarded as sets of lists of elements of L (n-lists in L, n-ads in L, etc.) and denoted L^n, $L^n_<$, $L^n_>$,

3. **Classifications.** Deemphasizing the individuality of L, the set $L^N_{\geq}$ becomes [redacted]

$\ell^N_>$

$|\ell^N_>|$

[redacted]

$p(n) =$ $= \mathrm{Bell}(n)$

[redacted]

169.1

misprint

[redacted]

subsubclass, and if for $n>1$, not every 1-level class has only one element. The

[redacted]

number of proper setwise classifications of N is finite and will be denoted by h_n.

4. Sortings of a product. Among numbers connected with mappings from a product MN we mention

$$(1)\qquad !\,l^{!m\cdot n},\qquad !\,l^{\mathrm{cy}\,m\cdot n},\qquad !\,l^{\underline{!}m\cdot n},\qquad !\,l^{\underline{\mathrm{cy}}\,m\cdot n}.$$

For $l\geq mn$ they are independent of l and can be denoted by

$$!^{!m\cdot n},\qquad !^{\mathrm{cy}\,m\cdot n},\qquad !^{\underline{!}m\cdot n},\qquad !^{\underline{\mathrm{cy}}\,m\cdot n}.$$

[redacted]

identities in semigroups.

For $m=1$ the numbers in (1) become $!l^n$. For $m=2$ still $M!=M_{\mathrm{cy}}$, and

[redacted]

[redacted]

[redacted]

$$!^{(n+1)+} = \sum_{0}^{n} \binom{n}{\nu} (\nu+1)!\ !^{(n-\nu)+}$$

delete?

It can be shown that if an umbral series w generates the numbers of classifications of a certain level, then those of the next higher level are generated by e^{w-1} if the new classifications consist of sets of old ones, and by $1/(2-w)$ if they consist of

[redacted]

lists (ads

7. **Values.** The relative size of numbers of lists or sets is apparent from the following table where packing is, but covering is not assumed.

Euler numbers

dull

elements	n	0	1	2	3	4	5	For further values or references see
lists	$n^{!\Sigma}_{<}$ (from (2))	1	2	5	16	65	326	[11]
	$n^{!\max}_{<} = n!$	1	1	2	6	24	120	[1, pp. 272–273]; [11]
sets	$n^{!\Sigma}_{<} = 2^n$	1	2	4	8	16	32	see below
	$n^{!\max}_{<} = \binom{n}{[n/2]}$	1	1	2	3	6	10	see below

For level 2: numbers of lists (or sets) of lists (or sets or numbers) we reassume covering, and obtain by inspection (i.e., without machines) and use of recurrences

—2866

n	0	1	2	3	4	5	6	7	8	9	For further values or references see

[redacted]

#1190 Forests of ... + height

... arrangements

largest S. II

dull

partition

why did? leave this

it #100?

Add it!

old

[redacted]

where $\nu \log \nu = n$. A better approximation seems to be

$$\mu_n = \nu^n e^{\nu-n-1}\left(\frac{n}{\nu}+1\right)^{-1/2}\left(1-\frac{n(2n^2+7n\nu+10\nu^2)}{24\nu(n+\nu)^3}\right),$$ dd

given in [8] without error term or indication of the way in which it might be

[redacted]

superior to other asymptotic expressions. Setting $!^n=\mu_n(1-\lambda_n/n)$ we find

$$\lambda_{50} = .0015\ldots,$$
$$\lambda_{100} = .0008\ldots,$$
$$\lambda_{150} = .0006\ldots,$$
$$\lambda_{200} = .0004\ldots.$$

*** By [7], p. 413 last line (at whose end an exponent 1/2 should be appended) old
and p. 412, line 17 (both only with hints to proofs). $!\max^n_{\geq}$ is asymptotic to

[redacted]

$$\exp\left(((\log \nu)^2-\log \nu+1)\nu-\tfrac{1}{2}\log \nu-1-\tfrac{1}{2}\log 2\pi\right) = \nu^{n-1/2}e^{\nu-n-1}/\sqrt{2}$$ [redacted]

Similarly we obtain these numbers of certain sets of subsets of direct products: [redacted]

n	0	1	2	3	4	5	6	7	8	9
$!^{!\cdot n}$	1	2	6	22	94	454	2430	14214	89918	
$!\max^{!\cdot n}_{\geq}$	1	2	4	12	48	200	1040	5600	33600	
$!^{!2\cdot n}$	1	2	7	31	164	999	6841	51790	428131	3929021
$!\max^{!2\cdot n}_{\geq}$	1	1	3	10	53	265	1700			

[redacted]

of degree n and with constant coefficients [6]. A combinatorial proof of (9) is the,

by far simplest, case $t=1$ of the proof of (10).

From (9) follows, first letting $n=0$, then using (4) for $n=p-1$,

$$!^p \underset{p}{=} 2, \qquad \sum_{2}^{p-1} (-1)^k \, !^k \underset{p}{=} 2.$$

The recurrence (9) holds also for α^n if $\alpha^p=\alpha+1$. In the field GF(p^p), the characteristic polynomial $g(x)=x^p-x-1$ has p roots $\alpha_k=\alpha_p+k$, no proper subset of which has its sum in the prime field GF(p), over which $g(x)$ is therefore irre-

ducible. The α_k^n are the p fundamental solutions of (9); the linear combination that represents $!^n$ can be shown to be

$$!^n \underset{p}{=} \sum f(\alpha_k)\alpha_k^n, \qquad f(x) = x^p - \sum_{0}^{p} !^j x^j.$$

$$p' = 1+p+\cdots+p^{p-1} = (p^p-1)/(p-1);$$

hence

$$!^{n+p'} \underset{p}{=} !^n.$$

It is unknown where $!^n$ mod p can have a smaller period. The prime decompositions of the first values of p' are, according to J. L. Selfridge,

2	3
3	13
5	11·71

belong to the same class in σ (case I) or they belong to p different classes (case II); the latter contain no element of N nor any case I-element. Finally it is easily seen that the number of sortings of $N+P^t$ for which exactly j of the p-sets (11) belong to case I ($j=0, \ldots, p^{t-1}$) is

$$s_{p^{t-1}-j}(p)\binom{p^{t-1}}{j}\,!^{n+j}.$$

His proof (10)

For $j \neq_p 0$ we have $\binom{p^{t-1}}{j} =_{p^{t-1}} 0$, hence $s_{p^{t-1}-j}(p)$ in (10) can then be replaced by its constant coefficient 1. For $j =_p 0$, $\neq_{p^2} 0$ we have $\binom{p^{t-1}}{j} =_{p^{t-2}} 0$, but

$$s_{p^{t-1}-j}(p) = \binom{p^{t-1}-j}{p}p+1 = 1+0^{p-2}\cdot 2 \quad (\text{unless } p=t=j=2).$$

where $0^0=1$; thus $s_{p^{t-1}-j}(p)$ in (10) can be replaced by $1+0^{p-2}\cdot 2$. There follows in particular

$$!^{n+p^2} \underset{p^2}{=} \sum_0^p \binom{p}{j}\,!^{n+j} + 0^{p-2}\cdot 2\,!^n.$$

For $p>2$ this can be written

$$!^{n+p^2} \underset{p^2}{=} (1+!^{n+1})^p.$$

A more detailed analysis shows that for every $t \geq 1$ and prime $p>2$ we have

(12) $$!^{n+p^t} \underset{p^t}{=} (1+!^{n+1})^{p^{t-1}}.$$

Setting $n=0$ we obtain

$$!^{p^t} \underset{p^t}{=} !^{p^{t-1}+1}.$$

The characteristic polynomial of the recurrence (10) is

$$x^{p^t} - \sum s_{p^{t-1}-j}(p)\binom{p^{t-1}}{j}x^j.$$

For $p \neq 2$ we can replace it by

$$x^{p^t}-(x+1)^{p^{t-1}}.$$

For $t=2$, $p=2$ the characteristic polynomial is x^4-x^2-2x-3.

A similar (and simpler) proof than that of (10) shows that

$$!^{\underline{\mathrm{cy}}2\cdot(n+p)} = (2-0^{p-2})\,!^{\underline{\mathrm{cy}}2\cdot n} + !^{\underline{\mathrm{cy}}2\cdot(n+1)}.$$

It follows from the fact that the highest and lowest coefficients of the recurrence [redacted] (10) are $\neq_p 0$ that $!^n$ is periodic, without preperiod, for p^t and hence for every modulus m, and that the period is $\leq m^{\mu}$, where μ is the largest prime power dividing m. The periodicity, with possible preperiods, follows also from (4′) and Fujiwara's [redacted]

$$F(z, w, w', \ldots, w^{(d)}) = 0$$

with integer coefficients for which

$$\frac{\partial F}{\cdots}(0, a_0, a_1, \ldots, a_d) = 1.$$

[redacted]

By the same theorem, modular periodicity holds also for the coefficients