check that & leave all these →311

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SORTING NUMBERS FOR CYLINDERS AND OTHER CLASSIFICATION NUMBERS

(2)

167–176

THEODORE S. MOTZKIN

0. **Introduction.** Set partitions (corresponding to equivalence relations) are in [redacted]

functions, values and congruence properties [redacted]

MN

LN

[redacted]

* and next values (eg of Stirling nos)

$L^N_<$, $L^N_>$, $L^N_=$ $L^N_<$ (in-valence ≤ 1 on L) the set of injections (3 N-ads in L),

[redacted]

$\ell_>$

[redacted]

//

[redacted]

$L^N_=$ the set of bijections,

$M! = M^M_=$ the set of permutations of M.

In analogy to $|MN| = mn$, $|L^N| = l^n$, $|M!| = m!$ we define

[redacted]

But this doesn't change the numbers

[redacted]

!

[redacted]

$$l^n_> = |L^N_>|, \qquad l^n_< = |L^N_<| = \binom{l}{n} n!,$$

and similarly in the sequel. Since

$$l^n_> = 0 \quad \text{for } l > n \qquad \text{and} \qquad l^n_< = 0 \quad \text{for } n > l,$$

the sums

$$\sum_l l^n_> \qquad \text{and} \qquad \sum_{n=0}^{l} l^n_<$$

are finite; such sums will be abbreviated to

$$\Sigma^n_> \qquad \text{and} \qquad l^{\Sigma}_<,$$

and their largest terms to

$$\max{}^n_> \qquad \text{and} \qquad l^{\max}_< = l!.$$

Deemphasizing the individuality of N, the sets L^N, $L^N_<$, $L^N_>$, $L^N_=$ can be regarded as

3. **Classifications.** Deemphasizing the individuality of L, the set $L^N_{\geq}$ becomes [redacted]

$\ell^N_>$

$|\ell^N_>|$

[redacted]

p(n) = [redacted] = Bell(n)

[redacted]

169.1

[redacted]

misprint

[redacted]

subsubclass, and if for $n>1$ not every 1-level class has only one element. The [redacted]

number of proper setwise classifications of N is finite and will be denoted by h_n. [redacted]

4. **Sortings of a product.** Among numbers connected with mappings from a product MN we mention

$$(1) \qquad !l^{!m\cdot n}, \qquad !l^{\mathrm{cy}\,m\cdot n}, \qquad !l^{\underline{!}m\cdot n}, \qquad !l^{\underline{\mathrm{cy}}\,m\cdot n}.$$

For $l \geq mn$ they are independent of l and can be denoted by

$$!^{!m\cdot n}, \qquad !^{\mathrm{cy}\,m\cdot n}, \qquad !^{\underline{!}m\cdot n}, \qquad !^{\underline{\mathrm{cy}}\,m\cdot n}.$$

of these latter combinatorial functions arise in the study of the number of [redacted]

Euler nos

Pref. arrangements

$(n-1)\ldots+$

$E_n = nE_{n-1} + 1$

[redacted]

identities in semigroups.

For $m=1$ the numbers in (1) become $!l^n$. For $m=2$ still $M! = M_{\mathrm{cy}}$, and

$$!l^{!2\cdot n} = \tfrac{1}{2}(!l^{2n} + !l^{\underline{!}2\cdot n}).$$

[redacted]

$$!^{(n+1)+} = \sum_{0}^{n} \binom{n}{\nu}(\nu+1)!\ !^{(n-\nu)+}$$

delete?

It can be shown that if an umbral series w generates the numbers of classifications of a certain level, then those of the next higher level are generated by e^{w-1} if the new classifications consist of sets of old ones, and by $1/(2-w)$ if they consist of [redacted]

lists (ad[illegible]

7. **Values.** The relative size of numbers of lists or sets is apparent from the following table where packing is, but covering is not assumed.

Euler numbers

diff

elements	n	0	1	2	3	4	5	For further values or references see
lists	$n^{!\Sigma}_{<}$ (from (2))	1	2	5	16	65	326	**[11]**
	$n^{!\max}_{<}=n!$	1	1	2	6	24	120	**[1**, pp. 272–273]; **[11]**
sets	$n^{!\Sigma}_{<}=2^n$	1	2	4	8	16	32	see below
	$n^{!\max}_{<}=\binom{n}{[n/2]}$	1	1	2	3	6	10	see below

For level 2: numbers of lists (or sets) of lists (or sets or numbers) we reassume covering, and obtain by inspection (i.e., without machines) and use of recurrences

2866

2867

n	0	1	2	3	4	5	6	7	8	9	For further values or references see

[redacted]

Bell n

largest S. II

[redacted]

partition

why don't I have this

[redacted]

#100.5?

Add it!

[redacted]

old

[redacted]

[redacted]

where $\nu \log \nu = n$. A better approximation seems to be

$$\mu_n = \nu^n e^{\nu-n-1}\left(\frac{n}{\nu}+1\right)^{-1/2}\left(1-\frac{n(2n^2+7n\nu+10\nu^2)}{24\nu(n+\nu)^3}\right),$$ dd

given in [8] without error term or indication of the way in which it might be

[redacted]

superior to other asymptotic expressions. Setting $!^n = \mu_n(1-\lambda_n/n)$ we find

$$\begin{aligned} \lambda_{50} &= .0015\ldots, \\ \lambda_{100} &= .0008\ldots, \\ \lambda_{150} &= .0006\ldots, \\ \lambda_{200} &= .0004\ldots. \end{aligned}$$

*** By [7], p. 413 last line (at whose end an exponent 1/2 should be appended) old and p. 412, line 17 (both only with hints to proofs), $!\max^n_>$ is asymptotic to

[redacted]

$$\exp\left(((\log \nu)^2 - \log \nu + 1)\nu - \tfrac{1}{2}\log \nu - 1 - \tfrac{1}{2}\log 2\pi\right) = \nu^{n-1/2}e^{\nu-n-1}/\sqrt{2\pi}.$$

Similarly we obtain these numbers of certain sets of subsets of direct products:

n	0	1	2	3	4	5	6	7	8	9
$!^{!\cdot n}$	1	2	6	22	94	454	2430	14214	89918	
$!\max_>^{!\cdot n}$	1	2	4	12	48	200	1040	5600	33600	
$!^{!2\cdot n}$	1	2	7	31	164	999	6841	51790	428131	3929021

[redacted]

of degree n and with constant coefficients [6]. A combinatorial proof of (9) is the,

[illegible]

by far simplest, case $t=1$ of the proof of (10).

From (9) follows, first letting $n=0$, then using (4) for $n=p-1$,

$$!^p \underset{p}{=} 2, \qquad \sum_{2}^{p-1} (-1)^k \, !^k \underset{p}{=} 2.$$

The recurrence (9) holds also for α^n if $\alpha^p=\alpha+1$. In the field $GF(p^p)$, the characteristic polynomial $g(x)=x^p-x-1$ has p roots $\alpha_k=\alpha_p+k$, no proper subset of which has its sum in the prime field $GF(p)$, over which $g(x)$ is therefore irre-

[illegible]

ducible. The α_k^n are the p fundamental solutions of (9); the linear combination that represents $!^n$ can be shown to be

$$!^n \underset{p}{=} \sum f(\alpha_k)\alpha_k^n, \qquad f(x) = x^p - \sum_{0}^{p} !^j x^j.$$

[illegible]

$$p' = 1+p+\cdots+p^{p-1} = (p^p-1)/(p-1);$$

hence

$$!^{n+p'} \underset{p}{=} !^n.$$

It is unknown where $!^n$ mod p can have a smaller period. The prime decompositions of the first values of p' are, according to J. L. Selfridge,

2	3
3	13
5	11·71

[illegible]

belong to the same class in σ (case I) or they belong to p different classes (case II); the latter contain no element of N nor any case I-element. Finally it is easily seen that the number of sortings of $N+P^t$ for which exactly j of the p-sets (11) belong to case I ($j=0,\ldots,p^{t-1}$) is

$$s_{p^{t-1}-j}(p)\binom{p^{t-1}}{j}\,!^{n+j}.$$

His prove (10)

For $j\neq_p 0$ we have $\binom{p^{t-1}}{j}=_{p^{t-1}}0$, hence $s_{p^{t-1}-j}(p)$ in (10) can then be replaced by its constant coefficient 1. For $j=_p 0$, $\neq_{p^2}0$ we have $\binom{p^{t-1}}{j}=_{p^{t-2}}0$, but

$$s_{p^{t-1}-j}(p)=\binom{p^{t-1}-j}{\;}p+1=1+0^{p-2}\cdot 2\quad(\text{unless } p=t=j=2).$$

where $0^0=1$; thus $s_{p^{t-1}-j}(p)$ in (10) can be replaced by $1+0^{p-2}\cdot 2$. There follows in particular

$$!^{n+p^2}\underset{p^2}{=}\sum_0^p\binom{p}{j}\,!^{n+j}+0^{p-2}\cdot 2\;!^n.$$

For $p>2$ this can be written

$$!^{n+p^2}\underset{p^2}{=}(1+!^{n+1})^p.$$

A more detailed analysis shows that for every $t\geq 1$ and prime $p>2$ we have

(12) $$!^{n+p^t}\underset{p^t}{=}(1+!^{n+1})^{p^{t-1}}.$$

Setting $n=0$ we obtain

$$!^{p^t}\underset{p^t}{=}!^{p^{t-1}+1}.$$

The characteristic polynomial of the recurrence (10) is

$$x^{p^t}-\sum s_{p^{t-1}-j}(p)\binom{p^{t-1}}{j}x^j.$$

For $p\neq 2$ we can replace it by

$$x^{p^t}-(x+1)^{p^{t-1}}.$$

For $t=2$, $p=2$ the characteristic polynomial is x^4-x^2-2x-3.

A similar (and simpler) proof than that of (10) shows that

$$!^{\underline{\mathrm{cy}}2\cdot(n+p)}\underset{p}{=}(2-0^{p-2})\;!^{\underline{\mathrm{cy}}2\cdot n}+!^{\underline{\mathrm{cy}}2\cdot(n+1)}.$$

It follows from the fact that the highest and lowest coefficients of the recurrence

[redacted]

(10) are $\neq_p 0$ that $!^n$ is periodic, without preperiod, for p^t and hence for every modulus m, and that the period is $\leq m^{\mu}$, where μ is the largest prime power dividing m. The periodicity, with possible preperiods, follows also from (4′) and Fujiwara's [redacted]

$$F(z, w, w', \ldots, w^{(d)}) = 0$$

with integer coefficients for which

$$\frac{\partial F}{\partial w^{(d)}}(0, a_0, a_1, \ldots, a_d) = 1.$$

[redacted]