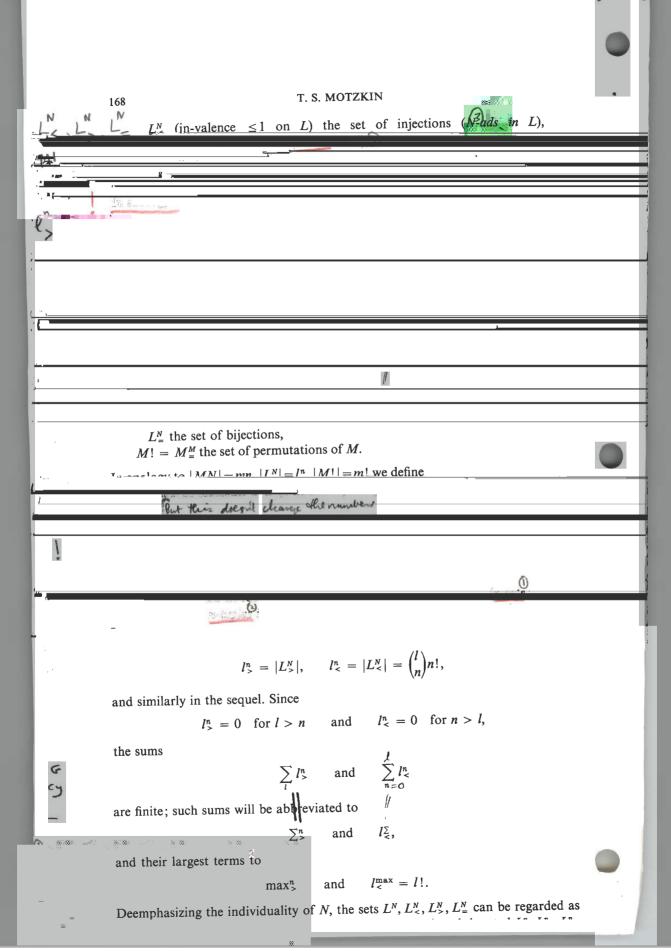
ches that I have all these ->311

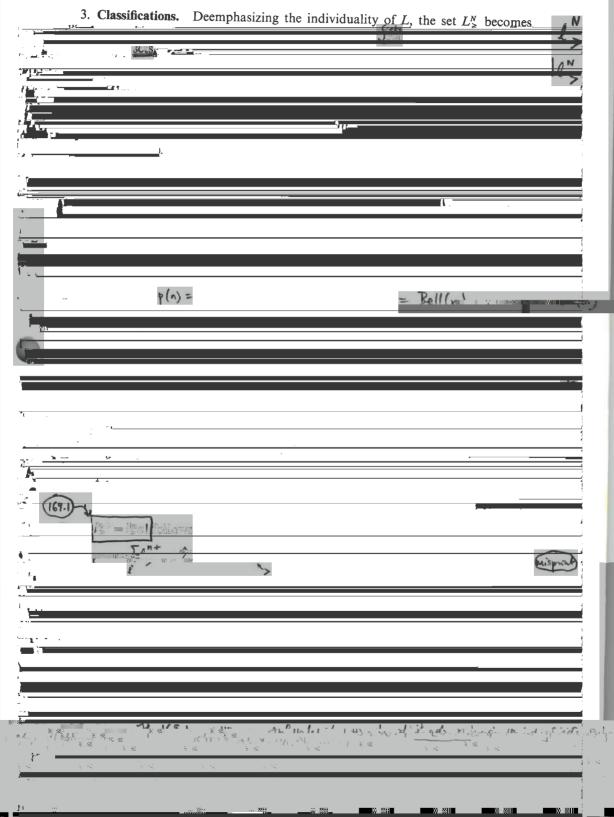
Sold of the set of th

SORTING NUMBERS FOR CYLINDERS AND OTHER CLASSIFICATION NUMBERS.

THEODORE S. MOTZKINSE 167-176

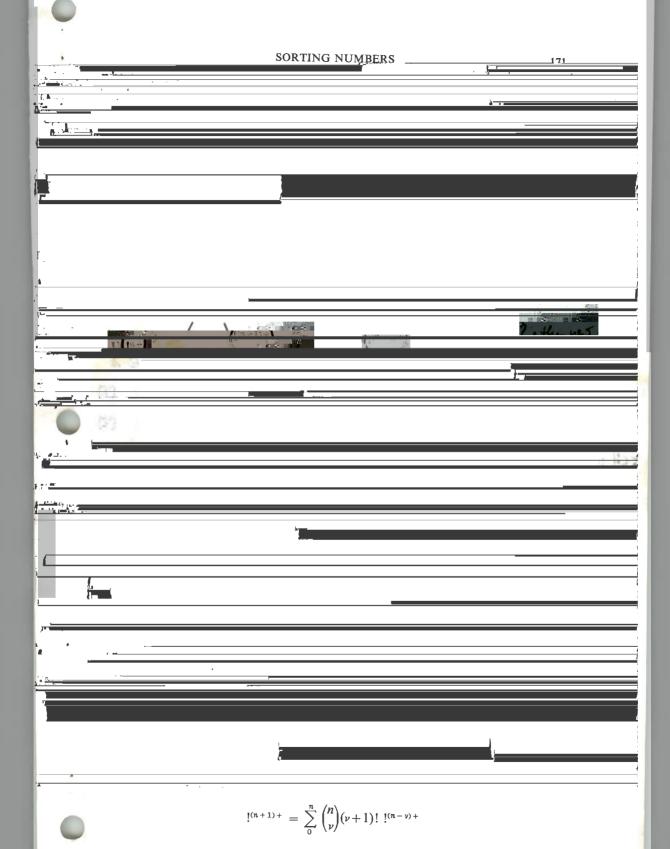
_	on. Set partitions (corresponding to equivalence relation	
1000 6 2 2 2	in the suring thinkers become (discharatical) ;	partition,
to a 11 the struct		
lists;	to (of Ren) sie. Whitely ma 28 5 5	
tarrotons, value		I.
<i>3</i>	SCH - The second	ara
1 1	Small At the of the best of the sectorday	MN
and maturalis, les	al Herd now)	
		ipo :





T. S. MOTZKIN

	1. 5. MOTZAIN
	puboubalon and if for no 1 not every 1-level class has only one element. The
3	
,	···
	pumber of proper setwise classifications of N is finite and will be denoted by h_n .
=-	
	4. Sortings of a product. Among numbers connected with mappings from a
	product MN we mention
	$(1) ! l^{!m \cdot n}, ! l^{\operatorname{cy}_{m \cdot n}}, ! l^{!m \cdot n}, ! l^{!m \cdot n}.$
	`´
	For $l \ge mn$ they are independent of l and can be denoted by
	$!^{lm\cdot n}$ $!^{cy}m\cdot n$ $!^{\underline{l}m\cdot n}$ $!^{\underline{cy}m\cdot n}$
	G Calego lotter combinatorial functions arise in the study of the number of
TILL .	J
	1.
William A	E . A E . 1
NO	En = VEn - , +1
mumb	
	(mores
7.11	
	· · · · · · · · · · · · · · · · · · ·
	•
	identities in semigroups.
<u> </u>	identities in semigroups. For $m=1$ the numbers in (1) become $!l^n$. For $m=2$ still $M!=M_{cy}$, and



where $\nu \log \nu = n$. A better approximation seems to be

$$\mu_n = \nu^n e^{\nu - n - 1} \left(\frac{n}{\nu} + 1 \right)^{-1/2} \left(1 - \frac{n(2n^2 + 7n\nu + 10\nu^2)}{24\nu(n + \nu)^3} \right),$$

old

given in [8] without error term or indication of the way in which it might be

superior to other asymptotic expressions. Setting $!^n = \mu_n(1 - \lambda_n/n)$ we find

$$\lambda_{50} = .0015\ldots,$$

$$\lambda_{100} = .0008...,$$

$$\lambda_{150} = .0006...,$$

$$\lambda_{200} = .0004 \dots$$

*** By [7], p. 413 last line (at whose end an exponent 1/2 should be appended) and p. 412, line 17 (both only with hints to proofs). !maxⁿ is asymptotic to

 $\exp\left(\left((\log \nu)^2 - \log \nu + 1\right)\nu - \frac{1}{2}\log \nu - 1 - \frac{1}{2}\log 2\pi\right) = \nu^{n-1/2}e^{\nu - n - 1}/\sqrt{2\nu^2}e^{\nu - n - 1}$

Similarly we obtain these numbers of certain sets of subsets of direct products:

n $! \max_{>}^{! \cdot n}$ of degree n and with constant coefficients [6]. A combinatorial proof of (9) is the,

by far simplest, case t=1 of the proof of (10).

From (9) follows, first letting n=0, then using (4) for n=p-1,

$$!^{p} = 2, \qquad \sum_{p=1}^{p-1} (-1)^{k} !^{k} = 2.$$

The recurrence (9) holds also for α^n if $\alpha^p = \alpha + 1$. In the field $GF(p^p)$, the characteristic polynomial $g(x) = x^p - x - 1$ has p roots $\alpha_k = \alpha_p + k$, no proper subset of which has its sum in the prime field GF(p), over which g(x) is therefore irre-

ducible. The α_k^n are the p fundamental solutions of (9); the linear combination that represents !n can be shown to be

$$!^{n} = \sum_{p} f(\alpha_{k})\alpha_{k}^{n}, \qquad f(x) = x^{p} - \sum_{k=0}^{p} !^{j}x^{j}.$$

Simon of Live house of - 1 where

$$p' = 1 + p + \cdots + p^{p-1} = (p^p - 1)/(p-1);$$

hence

$$!^{n+p'} = !^n.$$

It is unknown where $!^n \mod p$ can have a smaller period. The prime decompositions of the first values of p' are, according to J. L. Selfridge,

belong to the same class in σ (case I) or they belong to p different classes (case II); the latter contain no element of N nor any case I-element. Finally it is easily seen that the number of sortings of $N+P^t$ for which exactly j of the p-sets (11) belong to case I $(j=0,\ldots,p^{t-1})$ is

$$S_{p^{t-1}-j}(p) \binom{p^{t-1}}{j} !^{n+j}$$
. His pose (10)

For $j \neq_p 0$ we have $\binom{p^{t-1}}{j} = p^{t-1}0$, hence $s_p^{t-1} = j(p)$ in (10) can then be replaced by its constant coefficient 1. For $j = p^0$, $\neq_p \geq_0 0$ we have $\binom{p^{t-1}}{j} = p^{t-2}0$, but

$$s_{n^{t-1}} - (p_i) = (p^{t-1} - j) p_i + 1 = 1 + 0^{p-2} \cdot 2 \quad \text{(unless } p = t = j = 2).$$

where $0^0 = 1$; thus $s_{p^{t-1}-1}(p)$ in (10) can be replaced by $1 + 0^{p-2} \cdot 2$. There follows in particular

$$!^{n+p^2} = \sum_{p^2}^{p} \binom{p}{j} !^{n+j} + 0^{p-2} \cdot 2 !^n.$$

For p > 2 this can be written

$$!^{n+p^2} = (1+!^{n+1})^p.$$

A more detailed analysis shows that for every $t \ge 1$ and prime p > 2 we have

(12)
$$!^{n+p^t} = (1+!^{n+1})^{p^{t-1}}.$$

Setting n=0 we obtain

$$!^{p^t} = !^{p^{t-1}+1}.$$

The characteristic polynomial of the recurrence (10) is

$$x^{p^t} - \sum s_{p^{t-1}-j}(p) \binom{p^{t-1}}{j} x^j.$$

For $p \neq 2$ we can replace it by

$$x^{p^t}-(x+1)^{p^{t-1}}$$

For t=2, v=2 the characteristic polynomial is x^4-x^2-2x-3 .

A similar (and simpler) proof than that of (10) shows that

$$!^{\underbrace{\operatorname{cy}_{2\cdot(n+p)}}_{p}} = (2-0^{p-2}) !^{\underbrace{\operatorname{cy}_{2\cdot n}}_{p}} + !^{\underbrace{\operatorname{cy}_{2\cdot(n+1)}}_{n}}.$$

It follows from the fact that the highest and lowest coefficients of the recurrence

(10) are $\neq_p 0$ that !ⁿ is periodic, without preperiod, for p^t and hence for every modulus m, and that the period is $\leq m^{\mu}$, where μ is the largest prime power dividing m. The periodicity, with possible preperiods, follows also from (4') and Fujiwara's



$$F(z, w, w', \ldots, w^{(d)}) = 0$$

with integer coefficients for which

$$\frac{\partial F}{\partial a_0(0)}(0, a_0, \underline{a_1}, \ldots, a_d) = 1.$$