Pi in Golden Ratio Base Phi

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Abstract

π and ϕ (Golden Ratio) have fascinated mathematicians, scientists and artists for many centuries. Both are irrational numbers, but still related to integers in many ways. This article proves a new BBP-like formula for π in base ϕ.

1 Formula

$$\pi = \frac{4}{\phi} + \sum_{n=0}^{\infty} \frac{1}{\phi^{12n}} \left[\frac{8\phi^{-3}}{12n+3} + \frac{4\phi^{-5}}{12n+5} - \frac{4\phi^{-7}}{12n+7} - \frac{8\phi^{-9}}{12n+9} - \frac{4\phi^{-11}}{12n+11} + \frac{4\phi^{-13}}{12n+13} \right] \quad (1)$$

where

$$\phi = \frac{1+\sqrt{5}}{2} \quad (2)$$

The equation 1 gives value of π sequentially from left to right providing increasing precision of 12 fractional places in base ϕ with each iteration.

2 Identity

equation 1 is derived from the following geometric relation between π and ϕ

$$\frac{\pi}{4} = \arctan\left(\frac{1}{\phi}\right) + \arctan\left(\frac{1}{\phi^3}\right) \quad (3)$$

We will first prove the equation 3 and then derive equation 1

3 Proof

Formula for addition of tan is

$$\tan{(A+B)} = \frac{\tan{(A)} + \tan{(B)}}{1 - \tan{(A)}\tan{(B)}} \tag{4}$$

tan of right-hand side of equation 3 gives

$$= \frac{\frac{1}{\phi} + \frac{1}{\phi^3}}{1 - \frac{1}{\phi}\frac{1}{\phi^3}} \tag{5}$$

simplifying further

$$= \frac{\phi^3 + \phi}{\phi^4 - 1} \tag{6}$$

simplifying further

$$= \frac{\phi(\phi^2 + 1)}{(\phi^2 - 1)(\phi^2 + 1)} \tag{7}$$

since $\phi^2 - 1 = \phi$ hence

$$= \frac{\phi(\phi^2 + 1)}{\phi(\phi^2 + 1)} \tag{8}$$

simplifying further

$$= 1 = \tan{(\frac{\pi}{4})} \tag{9}$$

which is left-hand side of equation 3. Hence proving equation 3.

4 Derivation

Taylor series for $\arctan{(x)}$ is

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)} x^{2n+1} \tag{10}$$

substituting $\frac{1}{\phi}$ and $\frac{1}{\phi^3}$ in equation 10 gives

$$\arctan(\frac{1}{\phi}) = \frac{1}{\phi} - \frac{1}{3\phi^3} + \frac{1}{5\phi^5} - \frac{1}{7\phi^7} + \frac{1}{9\phi^9} - \frac{1}{11\phi^{11}} + \frac{1}{13\phi^{13}} - \frac{1}{15\phi^{15}} + \frac{1}{17\phi^{17}} - \frac{1}{19\phi^{19}} + \frac{1}{21\phi^{21}} - \frac{1}{23\phi^{23}} + \frac{1}{25\phi^{25}} - \cdots \tag{11}$$

$$\arctan(\frac{1}{\phi^3}) = \frac{1}{\phi^3} - \frac{1}{3\phi^9} + \frac{1}{5\phi^{15}} - \frac{1}{7\phi^{21}} + \frac{1}{9\phi^{27}} - \frac{1}{11\phi^{33}} + \frac{1}{13\phi^{39}} - \frac{1}{15\phi^{45}} + \frac{1}{17\phi^{51}} - \frac{1}{19\phi^{57}} + \frac{1}{21\phi^{63}} - \frac{1}{23\phi^{69}} + \frac{1}{25\phi^{75}} - \cdots \tag{12}$$

adding and arranging equation 11 and equation 12 gives

$$\frac{\pi}{4} = \frac{1}{\phi} + (\frac{1}{\phi^3} - \frac{1}{3\phi^3}) + \frac{1}{5\phi^5} - \frac{1}{7\phi^7} - (\frac{1}{3\phi^9} - \frac{1}{9\phi^9}) - \frac{1}{11\phi^{11}} + \frac{1}{13\phi^{13}} + (\frac{1}{5\phi^{15}} - \frac{1}{15\phi^{15}}) + \frac{1}{17\phi^{17}} - \frac{1}{19\phi^{19}} - (\frac{1}{7\phi^{21}} - \frac{1}{21\phi^{21}}) - \frac{1}{23\phi^{23}} + \frac{1}{25\phi^{25}} - \cdots \tag{13}$$

combining equal ϕ power terms from equation 13 gives

$$\frac{\pi}{4} = \frac{1}{\phi} + \frac{2}{3\phi^3} + \frac{1}{5\phi^5} - \frac{1}{7\phi^7} - \frac{2}{9\phi^9} - \frac{1}{11\phi^{11}} + \frac{1}{13\phi^{13}} + \frac{2}{15\phi^{15}} + \frac{1}{17\phi^{17}} - \frac{1}{19\phi^{19}} - \frac{2}{21\phi^{21}} - \frac{1}{23\phi^{23}} + \frac{1}{25\phi^{25}} - \cdots \tag{14}$$

multiplying both sides by 4 and arranging equation 14 in groups of 6 terms gives

$$\pi = \frac{4}{\phi} + (\frac{2}{3\phi^3} + \frac{1}{5\phi^5} - \frac{1}{7\phi^7} - \frac{2}{9\phi^9} - \frac{1}{11\phi^{11}} + \frac{1}{13\phi^{13}}) + \frac{1}{\phi^{12}}(\frac{2}{15\phi^3} + \frac{1}{17\phi^5} - \frac{1}{19\phi^7} - \frac{2}{21\phi^9} - \frac{1}{23\phi^{11}} + \frac{1}{25\phi^{13}}) - \cdots \tag{15}$$

which represented as series gives us equation 1.

$$\pi = \frac{4}{\phi} + \sum_{n=0}^{\infty} \frac{1}{\phi^{12n}} \left[\frac{8\phi^{-3}}{12n+3} + \frac{4\phi^{-5}}{12n+5} - \frac{4\phi^{-7}}{12n+7} - \frac{8\phi^{-9}}{12n+9} - \frac{4\phi^{-11}}{12n+11} + \frac{4\phi^{-13}}{12n+13} \right] \tag{16}$$

5 References

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