Scan A5631

RWR "Bishops"

Add to 10 segs

f91

899-903
→ 5631
-5635

"Combinatorial Mathematics IV. (Proc. of the Fourth Australian Conf., L.R.A. Casse and W.D. Wallis eds) Lecture Notes in Mathematics 560, Springer, Berlin (1976) 198 - 214.

05 A 15

COUNTING ARRANGEMENTS OF BISHOPS

ROBERT W. ROBINSON

The problem of the bishops is to determine the number of arrangements of n bishops on an n × n chessboard such that no bishop threatens another and every unoccupied square is threatened by at least one bishop. Two arrangements are considered equivalent if they are isomorphic by way of one of the eight symmetries of the chessboard. The total number of inequivalent solutions to the problem of the bishops is found, as well as the numbers of solutions which have each of the possible automorphism groups. The values up to n = 16 are tabulated, and asymptotic formulas are found. A review of analogous results for the problem of the rooks is included, since they are made use of in studying the problem of the bishops.

1. INTRODUCTION

The *problem of the rooks* is to find the number of inequivalent arrangements of n rooks on an n × n chessboard such that no rook attacks another. Each such arrangement is said to be a *solution* to the problem of the rooks. Every solution to the problem of the rooks has the property that every square of the chessboard is dominated by some rook. If the chessboard is considered to be fixed in place, it is clear that there are n! solutions. However two solutions will be considered *equivalent* if one can be obtained from the other by one of the eight symmetries of the chessboard.

The symmetries of the chessboard are named as follows: e is the identity, c is the rotation by π radians, q and q' are the rotations by ±π/2 radians, d and d' are the reflections about the main diagonals, and m and m' are the reflections about the horizontal and vertical mid-lines. We denote this group of eight symmetries by A. Each solution to the problem of the rooks has some subgroup of A as its group of automorphisms. It is not hard to see that for n > 1 the possible automorphism groups are <e>, <c>, <q>, <d>, <d'>, and <d,d'>. Lucas [7] found the number of inequivalent solutions to the problem of the rooks having each automorphism group. Kraitchik [6, Chapter 10] rediscovered the problem of the rooks and found the numbers for some of the automorphism groups. In the next section we state Lucas' results and give an asymptotic analysis of the numbers he found.

The *problem of the bishops* is to find the number of inequivalent arrangements of n bishops on an n × n chessboard so that no bishop threatens another and so that together they dominate the entire board. This problem is solved in Section 3, building on Lucas' results. For n > 1 the possible automorphism groups for solutions to the problem of the bishops are <e>, <c>, <q>, <m>, <m'>, and <m,m'>. As before the number of inequivalent solutions with each automorphism group is found both exactly and asymptotically. The values are tabulated up to $n = 16$.

In the last section we discuss variations on the problem of the bishops, and the outstanding unsolved problem of the queens.

2. THE PROBLEM OF THE ROOKS

It has long been known that the number of orbits of a finite permutation group is the average number of fixed points of the permutations. This fact is called Burnside's lemma [2, p.191]. Let σ_n be the number of inequivalent $n \times n$ solutions to the problem of the rooks. If A is regarded as permuting the n! different $n \times n$ solutions among themselves, then σ_n is just the number of orbits of A on this object set. Let P_n, G_n, R_n, and D_n be the numbers of $n \times n$ solutions left fixed by e, c, q, and d respectively. Clearly D_n is also the number left fixed by d' and R_n is the number left fixed by q'. For $n > 1$ there are no solutions left fixed by m or m'. Thus, by Burnside's lemma

(1) $$\sigma_n = \tfrac{1}{8}P_n + \tfrac{1}{8}G_n + \tfrac{1}{4}R_n + \tfrac{1}{4}D_n.$$

The restriction $n > 1$ is necessary for (1), and we assume $n > 1$ implicitly henceforth.

Obviously $P_n = n!$. For $n = 2k$ or $2k+1$ the number of $n \times n$ solutions invariant under c is $2k(2k-2)\ldots 2$ [7, pp.66-67]. Thus we have

(2) $$G_{2k} = G_{2k+1} = k!\,2^k.$$

For $n \times n$ solutions invariant under q we have $(4m-2)(4m-6)\ldots 2$ when $n = 4m$ or $4m+1$, and none otherwise [7, pp.67-68]. It follows that

(3) $$R_n = \begin{cases} 0 & \text{if } n \equiv 2 \text{ or } 3 \pmod 4, \\ \dfrac{(2m)!}{m!} & \text{if } n = 4m \text{ or } 4m+1. \end{cases}$$

Finally, it can be shown [7, p.215] that D_n satisfies the recurrence

(4) $$D_{n+1} = D_n + nD_{n-1},$$

which is valid for $n > 0$ if we take $D_0 = 1$.

On the basis of equations (1)-(4) the values of σ_n for $1 < n \leq 16$ shown in Table 1 were computed. These values are in agreement with those obtained by Lucas [7, p.222] for $1 < n \leq 12$.

Let α_n, γ_n, ρ_n, δ_n, and β_n be the number of inequivalent solutions to the problem of the rooks which have <e>, <c>, <q>, <d> or <d'>, and <d,d'>, respectively, as automorphism groups. The method of Lucas for finding σ_n is to find α_n, γ_n, ρ_n, δ_n, and β_n and then sum them. There is surprisingly little extra trouble involved in this approach. One first needs to be able to evaluate the number B_n of $n \times n$ solutions which are invariant under both d and d'. It is found [7, pp.217-218] that

(5) $$B_{2k+1} = B_{2k} = 2B_{2k-2} + (2k-2)B_{2k-4},$$

which is valid for $k > 0$ on the assumption that $B_0 = B_{-2} = 1$.

If a particular solution g has automorphism group G, then the number of solutions equivalent to g is [A;G], as they are in a 1-1 correspondence with the left cosets of G in A. Also, any solution equivalent to g has an automorphism group which is conjugate to G. Thus the automorphism group will be equal to G unless G is <d> or <d'>, in which case it could be the other of the two. Now <q> and <d,d'> are maximal automorphism groups, so we conclude at once that

(6) $$\rho_n = \frac{1}{2}R_n,$$

(7) $$\beta_n = \frac{1}{2}B_n.$$

Since <d> or <d'> is properly contained only in <d,d'> (from among the possible automorphism groups), we have

(8) $$\delta_n = \frac{1}{2}D_n - \frac{1}{2}B_n.$$

The group <c> may be properly contained in either <q> or <d,d'>, but never both at once. It follows that

(9) $$\gamma_n = \frac{1}{4}C_n - \frac{1}{4}R_n - \frac{1}{4}B_n.$$

Finally, any solution with a nontrivial automorphism group is either invariant under c or else has automorphism group <d> or <d'>, but not both. This leads to the relation

$$\alpha_n = \frac{1}{8}P_n - \frac{1}{8}C_n - \frac{1}{2}\delta_n,$$

or, in view of (8),

(10) $$\alpha_n = \frac{1}{8}P_n - \frac{1}{8}C_n + \frac{1}{4}B_n - \frac{1}{4}D_n.$$

From equations (1)-(10) the values of α_n, β_n, γ_n, δ_n, and ρ_n can all be computed. For $1 \leq n \leq 16$ these numbers are displayed in Table 1. For $1 \leq n \leq 12$ they are in agreement with the values given by Lucas [7, p.222].

902 901 900 899 903

n	ρ_n	β_n	γ_n	δ_n	α_n	σ_n
2	0	1	0	0	0	1
3	0	1	0	1	0	2
4	1	3	0	2	1	7
5	1	3	0	10	9	23
6	0	10	7	28	70	115
7	0	10	7	106	571	694
8	6	38	74	344	4,820	5,282
9	6	38	74	1,272	44,676	46,066
10	0	156	882	4,592	450,824	456,454
11	0	156	882	17,692	4,980,274	4,999,004
12	60	692	11,144	69,384	59,834,748	59,916,028
13	60	692	11,144	283,560	778,230,060	778,525,516
14	0	3,256	159,652	1,191,984	10,896,609,768	10,897,964,660
15	0	3,256	159,652	5,171,512	163,456,629,604	163,461,964,024
16	840	16,200	2,571,960	23,087,168	2,615,335,902,176	2,615,361,578,344

Numbers of inequivalent solutions to the problem of the rooks.

TABLE 1

LNM 560 2076

Asymptotic values of ρ_n, β_n, γ_n, δ_n, α_n, and σ_n as $n \to \infty$ are shown in Table 2. The value of ρ_n shown follows at once from (3) and (6) using Stirling's formula. Once the values for β_n and δ_n are known, the values of γ_n, α_n and σ_n follow similarly from (2), (9) and (10).

ρ_n	$\frac{1}{\sqrt{2}} \left(\frac{n}{e}\right)^{n/4} \cdot \begin{cases} 0 & \text{if } n \equiv 2 \text{ or } 3 \pmod 4 \\ 1 & \text{if } n \equiv 0 \pmod 4 \\ n^{-1/4} & \text{if } n \equiv 1 \pmod 4 \end{cases}$
β_n	$\frac{e^{\sqrt{n/2}}}{2\sqrt{2}\, e^{1/4}} \left(\frac{n}{e}\right)^{n/4} \cdot \begin{cases} 1 & \text{if } n \equiv 0 \pmod 2 \\ n^{-1/4} & \text{if } n \equiv 1 \pmod 2 \end{cases}$
γ_n	$\sqrt{2\pi} \left(\frac{n}{e}\right)^{n/2} \cdot \begin{cases} \sqrt{n} & \text{if } n \equiv 0 \pmod 2 \\ 1 & \text{if } n \equiv 1 \pmod 2 \end{cases}$
δ_n	$\frac{e^{\sqrt{n}}}{2\sqrt{2}\, e^{1/4}} \left(\frac{n}{e}\right)^{n/2}$
α_n and σ_n	$\frac{1}{4} \sqrt{\frac{\pi n}{2}} \left(\frac{n}{e}\right)^{n}$

Asymptotic numbers of inequivalent solutions to the problem of the rooks.

TABLE 2

The recurrence relation (4) for D_n has received considerable attention in the literature. Chowla *et al.* [3, Theorem 8] used elementary methods to show

(11)
$$D_n \sim \frac{e^{\sqrt{n}}}{\sqrt{2}\, e^{1/4}} \left(\frac{n}{e}\right)^{n/2}.$$

Moser and Wyman [8, equation 3.40] computed the next two terms in the asymptotic expansion of D_n. They also pointed out that (4) is equivalent to the generating function identity

(12)
$$\sum_{n=0}^{\infty} D_n x^n/n! = e^{x+x^2/2}.$$

Bender [1, p.507] analysed the coefficients of this generating function using a pair of general theorems which he had derived from more complicated results of Hayman [5]. In this way Bender obtained the equivalent of (11) on the basis of (12).

From the recurrence relation (5) for B_{2n} it is straightforward to deduce the equivalent generating function identity

$$\sum_{n=0}^{\infty} B_{2n}x^n/n! = e^{2x+x^2}. \tag{13}$$

Theorems 6 and 7 of Bender [1, pp.506-7] can be applied to (13), with the result that

$$B_{2n+1} = B_{2n} \sim \frac{e^{\sqrt{n/2}}}{\sqrt{2}\,e^{1/4}}\left(\frac{2n}{e}\right)^{n/2}. \tag{14}$$

The asymptotic value for β_n given in Table 2 follows from (7) and (14). The value shown for δ_n can then be deduced from (8) and (11).

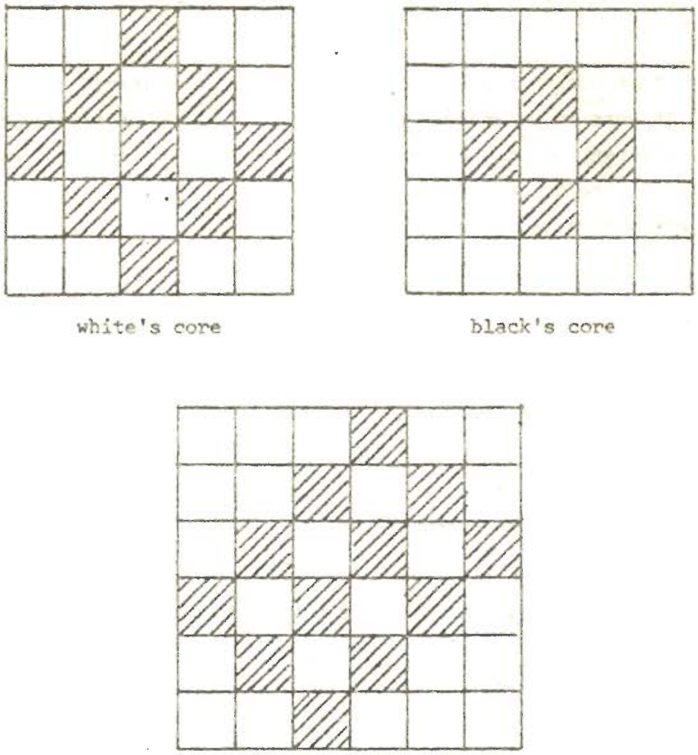
3. THE PROBLEM OF THE BISHOPS

A solution to the problem of the bishops is an arrangement of n bishops on the $n \times n$ chessboard such that no bishop threatens another and every square is threatened by some bishop. Let E_n, C_n, Q_n, M_n, and S_n be the number of $n \times n$ solutions left invariant by e, c, q, m, and both m and m', respectively. Then Q_n is also the number left fixed by q', and M_n is the number left fixed by m'. For $n > 1$ none will be left fixed by d or d'. Thus if we let τ_n be the total number of inequivalent solutions on the $n \times n$ chessboard, then by Burnside's lemma

$$\tau_n = \tfrac{1}{8}E_n + \tfrac{1}{8}C_n + \tfrac{1}{4}Q_n + \tfrac{1}{4}M_n. \tag{15}$$

Relation (15) for τ_n is analogous to relation (1) for σ_n. However for the problem of the rooks $P_n = n!$ is trivial, whereas for the problem of the bishops E_n is the most troublesome quantity to determine. It will be a convenient convention to colour the squares of the $n \times n$ chessboard alternately black and white in such a way that if $n = 2k+1$ then the central square is white when k is even and black when k is odd. As far as the moves of the biships are concerned, the black and white arrays are entirely independent. Within each we distinguish a unique part called thé *core*, as follows. For even $n = 2k$, the white core consists of the squares common to the central k+1 white diagonals in one direction and the central k diagonals in the other direction, the choice of directions determined so that the core is a rectangular $(k+1) \times k$ array of white squares. The black core for $n = 2k$ is isomorphic, but we think of it as a $k \times (k+1)$ array. For odd $n = 2k+1$, the core of either array is the unique maximal square sub-array. With our colouring convention, this results in the white core being $(k+1) \times (k+1)$ and the block core being $k \times k$. In Figure 1, one of the two isomorphic cores for the 6×6 chessboard is shown, along with the two cores for the

5 × 5 chessboard.



Cores of chessboards, for n = 5 and 6.

FIGURE 1

For $n = 2k$, any $n \times n$ solution to the problem of the bishops must have k on white and k on black, since the cores are $(k+1) \times k$ and $k \times (k+1)$ and so require at least k pieces each to be dominated. Similarly an $n \times n$ solution for $n = 2k+1$ must have k+1 on white and k on black.

To evaluate E_{2k}, restrict attention to the k bishops to be placed on white. Thinking of the white core as k+1 rows by k columns, it is clear that each of these columns must contain a bishop in order that the row (or rows) without a bishop be completely dominated. Further, each of the central k-1 rows extends beyond the core, and consequently needs a bishop for complete domination. Within this $(k-1) \times k$ array, then, we must have k-1 bishops. The remaining white bishop must lie in one of the rows which are of length k or less. The number of squares among these rows is

(16)
$$2 \sum_{0 \le i < k/2} k - 2i = \begin{cases} \frac{k(k+2)}{2} & \text{if } k \text{ is even,} \\ \frac{(k+1)^2}{2} & \text{if } k \text{ is odd .} \end{cases}$$

Once the position of this bishop is fixed, deleting its column leaves a $(k-1) \times (k-1)$ array within the white core to be covered by k-1 bishops. This of course can be achieved in exactly $(k-1)!$ different ways, as there must be one bishop in each row and column of the array. For $n = 2k$ the number of ways to dispose k bishops on the black squares is equal to the number for white, so we have

(17)
$$E_{2k} = \begin{cases} \left(k! \frac{k+2}{2}\right)^2 & \text{if } k \text{ is even,} \\ \left((k-1)! \frac{(k+1)^2}{2}\right)^2 & \text{if } k \text{ is odd.} \end{cases}$$

To evaluate E_{2k+1}, consider first the $k \times k$ black core. Each row and column of the core extends beyond the core, and so must contain one of the k black bishops in order to be completely dominated. Thus all k black bishops must be situated in the black core, which can be accomplished in k! different ways.

Next, consider the $(k+1) \times (k+1)$ white core. The outer pair of rows and columns do not extend beyond the core, while the others do. We shall classify the solutions for white according to the number of white bishops not positioned in the core. The number of solutions in which all of the white bishops lie in the core is $(k+1)!$ Suppose just one white bishop lies outside of the core. Then it is in one of the four corners of the white array which remain after the core is removed. Each of these corners contains

$$\sum_{0 \le i < k/2} k-1-2i = \begin{cases} \frac{k^2}{4} & \text{if } k \text{ is even,} \\ \frac{k^2-1}{4} & \text{if } k \text{ is odd} \end{cases}$$

squares in all (see (16)). Once the position of the outside bishop is fixed, deletion of its row or column from the core leaves a $k \times (k+1)$ array, say, within the core to be covered by the k bishops to be placed in the core. The central k-1 columns must each contain a bishop, since they all extend beyond the core to squares not covered by the outside bishop and not contained in rows passing through the core. Thus one bishop must be placed on one of the 2k squares of the end-columns of this array, leaving k-1 bishops for the central k-1 columns and the remaining k-1 rows of the core. This gives $(k-1)!2k = 2(k!)$ ways to distribute the k bishops within the core given a fixed position for the outside bishop.

Now suppose we have two bishops placed in adjacent corners of the white array minus the core. That leaves a $k \times k$ sub-array of the core which needs to be covered by just k-1 bishops, which is impossible. Suppose on the other hand that two bishops are positioned in the same or opposing corners of the white array minus the core.

Then, say, a $(k-1)\times(k+1)$ sub-array of the core remains to be covered by $k-1$ bishops. Since the central $k-1$ columns extend beyond the core, each must contain a bishop. None of these bishops can lie outside the core, else there would be two in adjacent corners. Thus the $k-1$ remaining bishops are restricted to a $(k-1)\times(k-1)$ sub-array of the core. We conclude that two is the maximum number of bishops which can be placed on white outside the core. Also, once the positions of two outside bishops in the same or opposite corners are determined, there will be $(k-1)!$ ways to fill in the remaining $k-1$ bishops.

It remains to see how many ways there are to place two bishops in a particular pair of opposing corners, say consisting of the white columns which do not intersect the core. Suppose first that k is even. Then there are $k/2$ columns in each corner, of lengths $k-1, k-3, \ldots, 1$, for a total (see equation (16)) of $k^2/2$ cells in all. If a bishop is located at an end square of one of these columns, then it dominates exactly k of these white corner squares. In general, if a bishop is located i squares from the near end of its column, then it dominates $k+2i$ of the white corner squares. There are $2(k-1-2i)$ squares which are distance i from the near end of their columns. So the number of ways to place two bishops which do not threaten each other within these corner squares is

$$\frac{1}{2}\sum_{0\le i<k/2} 2(k-1-2i)(k^2/2-k-2i) = \frac{k}{24}(3k^3-8k^2+6k-4).$$

There are two pairs of opposing corners, so the total number of $n\times n$ solutions to the problem of the bishops for $n=2k+1$ and k even is

$$k!\left((k+1)! + k^2\cdot 2\cdot k! + 2(k-1)!\frac{k}{24}(3k^3-8k^2+6k-4)\right),$$

which simplifies as shown in (18). Similarly, when $n=2k+1$ and k is odd the number of ways to place two bishops in a pair of opposing corners without threatening each other is

$$\frac{1}{2}\sum_{0\le i<k/2} 2(k-1-2i)\left(\frac{k^2-1}{2}-k-2i\right) = \frac{k^2-1}{24}(3k^2-8k+3).$$

In that case the total number of $n\times n$ solutions to the problem of the bishops is

$$k!\left((k+1)! + (k^2-1)\cdot 2\cdot k! + 2(k-1)!\,\frac{k^2-1}{24}\,(3k^2-8k+3)\right)$$

and so in all we have

$$(18)\qquad E_{2k+1} = \begin{cases} \dfrac{k!^2}{12}(3k^3+16k^2+18k+8) & \text{for } k \text{ even},\\[2ex] \dfrac{(k+1)!(k-1)!}{12}(3k^3+13k^2-k-3) & \text{for } k \text{ odd}.\end{cases}$$

Now consider solutions to the problem of the bishops which are invariant under c. For n even there are no solutions invariant under the rotation by π radians. This is because, as we saw in evaluating E_n for n even, there will be exactly one white bishop which lies in a white column of length $\leq k$. Then c maps the position of this bishop to the centrally opposite white square, which is distinct and also lies in a white column of length $\leq k$, and so must be unoccupied. We conclude that

(19) $$C_{2k} = 0.$$

Now suppose $n = 2k+1$. As we saw in the computation of E_{2k+1}, the arrangement of bishops on black is k pieces on the $k \times k$ core; i.e., essentially a $k \times k$ solution to the problem of the rooks. The action of c on the $n \times n$ board induces on the black core exactly the action of c on the $k \times k$ board, and so there are exactly G_k ways to arrange the k bishops on black with invariance under c. Similarly there are G_{k+1} arrangements of the k+1 bishops on the white core invariant under c. Clearly if just one bishop on white is to be outside the core, then the arrangement is not invariant under c. The number of ways to have two bishops in a pair of opposing corners of the white array, in positions which are centrally symmetric and not threatening each other, is just the number of ways to place one bishop in a particular corner but not on a main diagonal. When k is even there are $\frac{k^2}{4} - \frac{k}{2}$ ways to do this, and when k is odd there are $\frac{k^2-1}{4}$ ways. Then there are G_{k-1} ways in which to position the remaining k-1 bishops within the appropriate $(k-1) \times (k-1)$ sub-array of the white core so as to be invariant under c. To summarise, we have

$$C_{2k+1} = \begin{cases} G_k\left(G_{k+1} + \frac{k}{2}(k-2)G_{k-1}\right) & \text{if } k \text{ is even,} \\ G_k\left(G_{k+1} + \frac{k^2-1}{2}G_{k-1}\right) & \text{if } k \text{ is odd.} \end{cases}$$

Using equation (2) for G_k we find

(20) $$C_{2k+1} = \begin{cases} k\, 2^{k-1}\left(\frac{k}{2}\right)!^2 & \text{if } k \text{ is even,} \\ 2^k\left(\frac{k+1}{2}\right)!^2 & \text{if } k \text{ is odd.} \end{cases}$$

We take up next the consideration of the $n \times n$ solutions to the problem of the bishops which are invariant under the rotation of $\pi/2$ radians. It is clear from (19) that $Q_n = 0$ if n is even, since $q^2 = c$. If $n = 2k+1$, then the action of q on the black core is exactly the action of q on the $k \times k$ chessboard. Thus there will be R_k arrangements of the k bishops on black invariant under q, as they must all lie in the core. Similarly there are R_{k+1} arrangements of the k+1 bishops on white invariant under q with all the pieces in the core. Moreover there are no other solutions for white, since a bishop on white outside the core would be mapped by q to a bishop in an adjacent corner of the white array, and this we saw earlier was impossible. Thus we have

$$Q_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ R_{(n+1)/2}R_{(n-1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

In view of equation (3) for R_n, this amounts to

$$(21) \qquad Q_n = \begin{cases} 0 & \text{if } n \not\equiv 1 (\text{mod } 8), \\ \frac{(2m)!^2}{m!^2} & \text{if } n = 8m+1. \end{cases}$$

To compute M_n, take the case $n = 2k$ first. If we start with an arbitrary solution for the k bishops on white, the reflection m maps this to the symmetric solution for the k bishops on black and vice versa. Consequently from our evaluation of E_{2k} we conclude that $M_{2k}{}^2 = E_{2k}$, that is

$$(22) \qquad M_{2k} = \begin{cases} k!\ \frac{k+2}{2} & \text{if } k \text{ is even,} \\ (k-1)!\ \frac{(k+1)^2}{2} & \text{if } k \text{ is odd.} \end{cases}$$

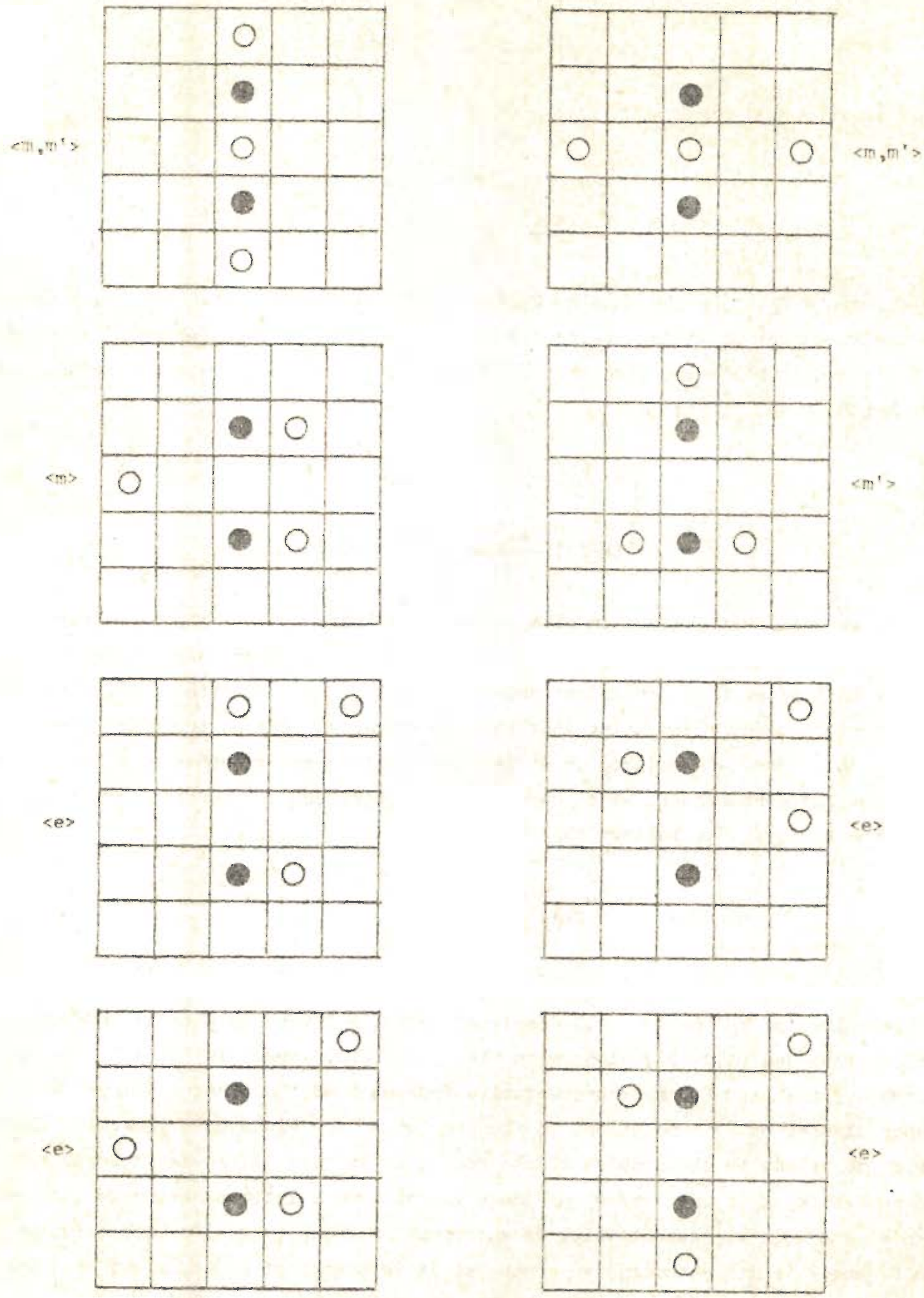
If $n = 2k+1$, we note that the action of m on the $n \times n$ board induces the action of d on the black $k \times k$ core and the white $(k+1) \times (k+1)$ core. Thus there are D_k arrangements of the bishops on black invariant under m, and D_{k+1} arrangements of the bishops on white invariant under m and using just the core squares. Again, for white there are no other solutions, as a bishop on white outside the core is mapped by m to a position in an adjacent corner, which cannot also be occupied in any solution to the problem of the bishops. It follows that

$$(23) \qquad M_{2k+1} = D_k D_{k+1},$$

with $D_1 = 1$.

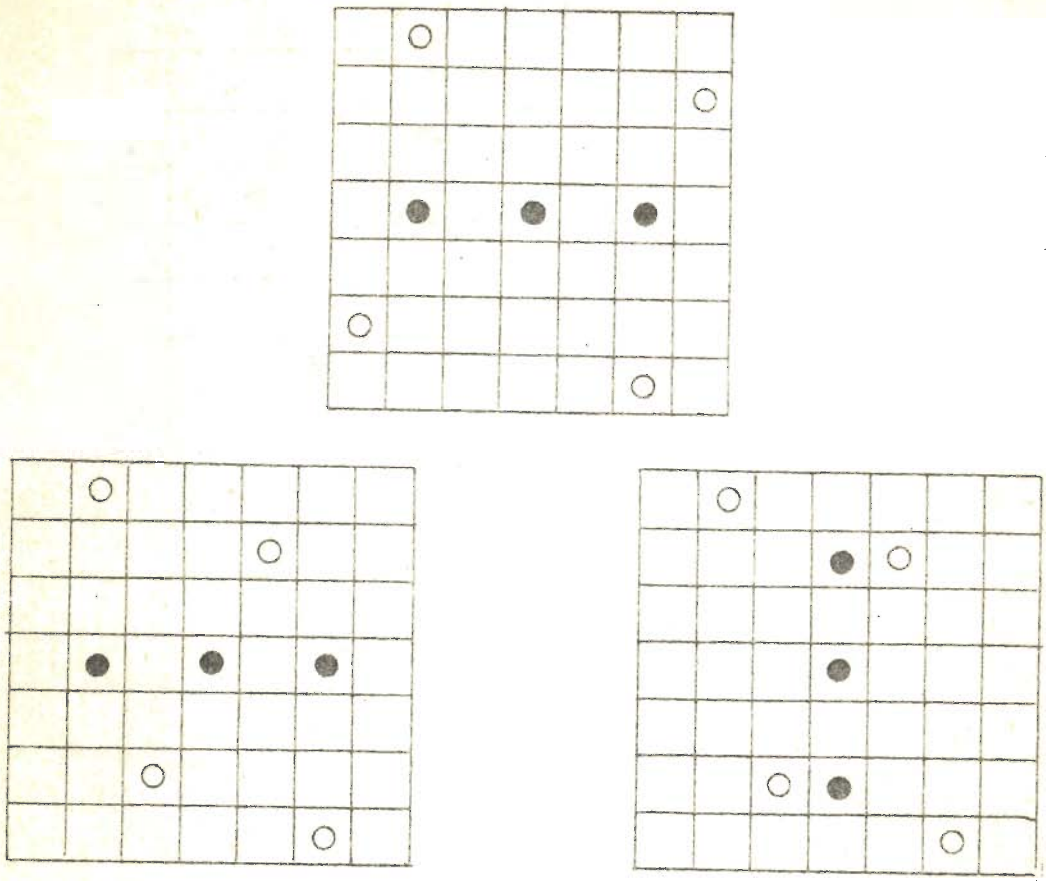
The values of τ_n for $1 < n \leq 16$ displayed in Table 3 were computed on the basis of equations (15) and (17)-(23) along with the values of D_k shown in Table 1. By way of illustrating that $\tau_5 = 8$, one representative from each of the eight different A-equivalence classes of 5×5 solutions to the problem of the bishops is shown in Figure 2. As with solutions to the problem of the rooks, it is very little extra trouble to find the numbers $\epsilon_n, \chi_n, \zeta_n, \mu_n$, and ψ_n of inequivalent $n \times n$ solutions having $\langle e \rangle, \langle c \rangle, \langle q \rangle$, $\langle m \rangle$ or $\langle m' \rangle$, and $\langle m, m' \rangle$, respectively, as automorphism groups. Beside each solution shown in Figure 2 is its automorphism group, so it is seen that $\epsilon_5 = 4$, $\chi_5 = \zeta_5 = 0$, and $\mu_5 = \psi_5 = 2$. In Figures 3 and 4 are illustrated the facts that $\psi_7 = 3$ and $\zeta_9 = 2$.

In order to evaluate ψ_n in general we need to find the number S_n of $n \times n$ solutions to the problem of the bishops which are invariant under both m and m'. Since $mm' = c$, it follows from (19) that $S_n = 0$ if n is even. If $n = 2k+1$ then just as for solutions invariant under m alone, the solutions must consist of k bishops in the $k \times k$



Inequivalent 5 × 5 solutions to the problem of the bishops.

FIGURE 2



Inequivalent 7 × 7 solutions to the problem of the bishops
which have automorphism group <c>.

FIGURE 3

black core and k+1 bishops in the (k+1) × (k+1) white core. Moreover the action of m and m' on the n × n board induces the action of d and d' on each of these cores. Thus we have

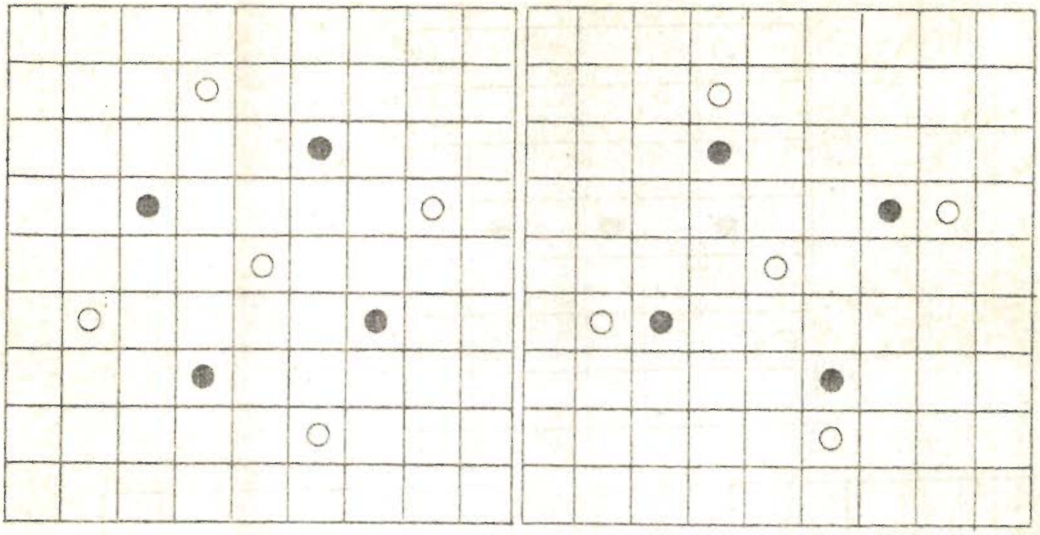
(24) $$S_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ B_k B_{k+1} & \text{if } n = 2k+1, \end{cases}$$

where $B_1 = 1$.

It is now straightforward to deduce the following relations;

(25) $$\psi_n = \tfrac{1}{2}S_n,$$

(26) $$\zeta_n = \tfrac{1}{2}Q_n,$$



Inequivalent 9 × 9 solutions to the problem of the bishops which have automorphism group <q>.

FIGURE 4

(27) $$\mu_n = \frac{1}{2}M_n - \frac{1}{2}S_n,$$

(28) $$\chi_n = \frac{1}{4}C_n - \frac{1}{4}S_n - \frac{1}{4}Q_n,$$

(29) $$\varepsilon_n = \frac{1}{8}E_n - \frac{1}{8}C_n + \frac{1}{4}S_n - \frac{1}{4}M_n.$$

The reasoning is entirely analogous to that involved in justifying (6)-(10) for the problem of the rooks. The values of $\psi_n, \zeta_n, \mu_n, \chi_n$, and ε_n for $1 < n \leq 16$ shown in Table 3 were computed on the basis of equations (17)-(29).

The asymptotic values of $\zeta_n, \psi_n, \chi_n, \mu_n, \varepsilon_n$ and τ_n as $n \to \infty$ are shown in Table 4. These follow at once from Stirling's formula and equations (11), (14), (15), and (17)-(29).

S9C
S to enter please!

n	ζ_n	ψ_n	χ_n	μ_n	ϵ_n	τ_n
2	0	0	0	1	0	1
3	0	1	0	0	0	1
4	0	0	0	2	1	3
5	0	2	0	2	4	8
6	0	0	0	8	28	36 ← 36
7	0	6	5	14	85	110
8	0	0	0	36	630	666
9	2	18	22	112	3,036	3,250
10	0	0	0	216	23,220	23,436
11	0	60	258	928	123,952	125,198
12	0	0	0	1,440	1,036,080	1,037,520
13	0	200	1,628	8,616	7,230,828	7,241,272
14	0	0	0	11,520	66,349,440	66,360,960
15	0	760	18,052	87,864	503,745,252	503,851,928
16	0	0	0	100,800	5,080,269,600	5,080,370,400

S631 S632 S633 S634 S635

Numbers of inequivalent solutions to the problem of the bishops.

TABLE 3

omit internal commas!

Bishops on an $n x n$ bd.

[NM 560 212 76]

ζ_n	0 if $n \not\equiv 1 \pmod 8$ $\sqrt[4]{\frac{2}{n}} \left(\frac{n}{2e}\right)^{n/4}$ if $n \equiv 1 \pmod 8$
ψ_n	0 if n is even $\frac{1}{4\sqrt{e}} \sqrt[4]{\frac{2}{n}}\ e^{\sqrt{n/2}} \left(\frac{n}{2e}\right)^{n/4}$ if n is odd
χ_n	0 if n is even $\frac{\pi n}{16} \sqrt{\frac{n}{2}} \left(\frac{n}{2e}\right)^{n/2}$ if n is odd
μ_n	$\frac{n\sqrt{\pi n}}{8} \left(\frac{n}{2e}\right)^{n/2}$ if n is even $\frac{1}{4\sqrt{e}}\ e^{\sqrt{2n}} \left(\frac{n}{2e}\right)^{n/2}$ if n is odd
ε_n and τ_n	$\frac{\pi n^3}{128} \left(\frac{n}{2e}\right)^n$

Asymptotic numbers of inequivalent solutions to the problem of the bishops.

TABLE 4

4. EXTENSIONS AND UNSOLVED PROBLEMS

It is not hard to see that for $n > 1$ the maximum number of bishops which can be arranged on the $n \times n$ chessboard so as not to threaten one another is $2n-2$. Call such an arrangement a *solution to the maximum bishops problem.* Any such solution will of course have the property that the bishops dominate the entire board. It is not hard to see that the number of $n \times n$ solutions to the maximum bishops problem is 2^n. Moreover, the only possible invariance such a solution can have is under m or m', and there are just $2^{[\frac{n+1}{2}]}$ invariant under each. Thus there are exactly $2^{n-3} + 2^{[\frac{n-3}{2}]}$ inequivalent $n \times n$ solutions to the maximal problem of the bishops for $n > 1$, of which $2^{[\frac{n-1}{2}]}$ are invariant under m or m' and $2^{n-3} - 2^{[\frac{n-3}{2}]}$ have the trivial automorphism group.

A problem which follows on from this is obtained by allowing for solutions with any number k of bishops on the $n \times n$ chessboard such that $n \le k \le 2n-2$ (for $n > 1$). It is not difficult to see that the number of such solutions will always be positive, but for $n < k < 2n-2$ no other general results on the numbers of solutions are known to the author.

The *problem of the queens* is to find the number of inequivalent arrangements of n queens on the n × n chessboard such that no queen threatens another. This problem was posed and discussed by Kraitchik [6, Chapter 10], and more recently by Harary and Palmer [4, Chapter 10]. It can be varied by allowing fewer queens while requiring that they still dominate the entire chessboard. In either variation the problem of the queens is an outstanding unsolved problem.

REFERENCES

[1] E.A. Bender, Asymptotic methods in enumeration, *SIAM Review* 16(1974), 485-515.

[2] W. Burnside, *Theory of Groups of Finite Order.* (Second edition, Cambridge University Press, London, 1911; reprinted by Dover, New York, 1955.)

[3] S. Chowla, I.N. Herstein and K. Moore, On recursions connected with symmetric groups I, *Canadian J. Math.* 3 (1951), 328-334.

[4] F. Harary and E.M. Palmer, *Graphical Enumeration.* (Academic Press, New York, 1973.)

[5] W.K. Hayman, A generalisation of Stirling's formula, *J. Reine Angew. Math.* 196 (1956), 67-95.

[6] M. Kraitchik, *Mathematical Recreations.* (Second revised edition, W.W. Norton, New York, 1942; reprinted by Dover, New York, 1953.)

[7] E. Lucas, *Théorie des Nombres,* V.I. (Gauthier-Villars, Paris, 1891; reprinted by Albert Blanchard, Paris, 1961.)

[8] L. Moser and M. Wyman, On the solution of $X^d = 1$ in symmetric groups, *Canadian J. Math.* 7 (1955), 159-168.

Department of Mathematics,
University of Newcastle,
New South Wales.