

Genocchi Numbers

1869.1

→ 144

- Ref
1. MMA G 33 21 59-60
 2. Lucas, *Th des Nos* I p 250-263 (1891)
 3. Bell, *Messenger of Math*, (2) 52 56-68
 4. Nörlund, *Vorles: ub. differenzrechnung*, 1924

Defn G_N :

$$\frac{2x}{e^x + 1} = \sum_{N=1}^{\infty} G_N \frac{x^N}{N!}$$

$$G_N = -2(2^N - 1) B_N \text{ (ib Bernoulli)}$$

$$C_{N-1} = 2^{N-1} \frac{G_N}{N}$$

Fagant Coeffs of [4].

$$\frac{2x}{e^x + 1} = \frac{2x}{2 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots} = \frac{x}{1 + \frac{x}{2} + \frac{x^2}{2 \cdot 2!} + \frac{x^3}{2 \cdot 3!} + \dots}$$

$$\begin{aligned} \text{Ans } \frac{1}{1+x} &= \frac{1-x}{1-x^2} = (1-x)(1+x^2)(1+x^4)(1+x^8)(1+x^{16}) \\ &= (1-x)(1+x^2+x^4+x^6+x^8+x^{10}+x^{12}+x^{14}+\dots) \\ &= 1-x+x^2-x^3+x^4-x^5+\dots \end{aligned}$$

$$\therefore = x \left(1 - \frac{x}{2} - \frac{x^2}{4} - \frac{x^3}{12} - \frac{x^4}{48} - \frac{x^5}{240} \right)$$

$$\cdot \left(1 + \frac{x^2}{4} + \frac{x^3}{4} + \frac{7}{48}x^4 + \frac{1}{16}x^5 \right)$$

$$\cdot \left(1 + \frac{x^4}{16} + \frac{x^5}{8} \right)$$

$$= x \left(1 - \frac{x}{2} + 0x^2 + \frac{1}{24}x^3 - \frac{1}{16}x^4 - \frac{47}{480}x^5 \right) \left(1 + \frac{x^4}{16} + \frac{x^5}{8} \right)$$

$$= x \left(1 - \frac{x}{2} + \cancel{\frac{x^2}{24}} \cdot \cancel{\frac{1}{24}} \cdot \frac{x^3}{24} + 0x^4 \dots \right)$$

$$= 1 \cdot \frac{x}{1!} - 1 \cdot \frac{x^2}{2!} + 1 \cdot \frac{x^4}{4!} + O(x^6)$$

$$2x = (e^x + 1) \sum_{n=1}^{\infty} \frac{G_n x^n}{n!} = (e^x + 1) G(x) \quad (A)$$

$$2 = (e^x + 1) \sum_{n=0}^{\infty} G_{n+1} \frac{x^n}{n!} + e^x \sum_{n=1}^{\infty} \frac{G_n x^n}{n!}$$

$$= \frac{2x}{G(x)} \sum_{n=0}^{\infty} G_{n+1} \frac{x^n}{n!} + \cancel{2x} 2x - G(x)$$

$$2G(x) = 2x G'(x) + 2x G(x) - G(x)'$$

$$(2x - 1) G'(x) + \cancel{2x} 2(x-1) G(x) = 0 \quad (B)$$

From (A): $2x = \left(2 + \sum_{n=1}^{\infty} \frac{x^n}{n!} \right) \sum_{n=1}^{\infty} \frac{G_n x^n}{n!}$

$$2x = 2 \sum_{n=1}^{\infty} \frac{G_n x^n}{n!} + \sum_{n=2}^{\infty} \frac{\alpha_n x^n}{n!}$$

$$\frac{\alpha_n}{n!} = \sum_{r+s=n} \frac{G_r}{r!} \frac{G_s}{s!}$$

$$\alpha_n = \sum_{r=1}^{n-1} \binom{n}{r} G_r$$

~~2x = 2 + \sum_{n=1}^{\infty} \frac{x^n}{n!}~~

Equate % : $G_1 = 1$, $G_0 = 0$, and

$$0 = 2G_n + \sum_{r=1}^{n-1} \binom{n}{r} G_r$$

$$\boxed{G_n = -\frac{1}{2} \sum_{r=1}^{n-1} \binom{n}{r} G_r} = -\frac{1}{2} \left\{ n + \sum_{r=2}^{n-2} \binom{n}{r} G_r \right\}$$

{n must be even, ≥ 2 }

$$G_2 = -\frac{1}{2} \binom{2}{1} G_1 = -1$$

$$G_3 = -\frac{1}{2} \{ 3 \cdot 1 + 3(-1) \} = 0$$

$$G_4 = -\frac{1}{2} \{ 4 \cdot 1 - 6 + 4 \cdot 0 \} = 1$$

$$G_5 = -\frac{1}{2} \{ 5 \cdot 1 - 10 \cdot 1 + 10 \cdot 0 + 5 \cdot 1 \} = 0$$

$$G_6 = -\frac{1}{2} \{ 6 \cdot 1 - 15 \cdot 1 + 20 + 15 \cdot 1 + 6 \} = -3$$

$$G_7 = -\frac{1}{2} \{ 7 \cdot 1 - 21 \cdot 1 + 35 + 35 + 7 \cdot 1 - 3 \cdot 7 \} = 0$$

$$G_8 = -\frac{1}{2} \{ 8 \cdot 1 - 28 \cdot 1 + 70 + 70 - 3 \cdot 28 + 8 \} = 17$$

n	T_{2n-1}	$2T_{2n-1}$	$2^n T_{2n-1}$	$\div 2^{2n-1}$	
1	1	2	2 ¹	1	8 544
2	2	4	2 ²	-1	4 68
3	16	32	2 ⁵	-3	17
4	272	544	2 ⁵	17	2 15872
5	7936	15872	2 ⁹	155	8 7936
6	353792	707584			8 992
					4 124
					31

$$G_n = \frac{2^n}{2^{2n-1}} T_{2n-1}$$

↑
 G_n