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Sums of products of integers
type 1

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[UWP 1(3) 1926]ns
Ref [LE1 10]n to Lehmer

ON THE SUM OF PRODUCTS OF n CONSECUTIVE INTEGERS.

By Robert E. Moritz.

Isolated theorems relating to sums of products of the first n integers were established by Lagrange, Wolstenholme, Ferrers, Nielson, Allardice, and many others.[1] Glaisher in an exhaustive investigation,[2] summarized the results of his predecessors, supplied a long list of new theorems, and published the numerical values of such sums for values of n from 1 to 23.

The present paper deals with properties relating to sums of products of any n consecutive numbers, special consideration being given to the case when n is a prime number. While some of these properties will appear as more or less obvious extensions of known results, which, though not previously recorded, could hardly have escaped the attention of earlier investigators, other results, such as the generalization of Glaisher's extension of Wilson's theorem and the determinant method of dealing with sigma congruences, it is hoped, justify this permanent record of them.

To assist in the verification of the results here arrived at, and to facilitate further investigations, the tables of numerical values, which are appended to this paper, have been computed.

1. Let us denote by ${}^{m}_{n}P_k$ the sum of all possible products, k at a time, of the numbers

$$m+1,\ m+2,\ m+3,\ \cdots,\ m+n,$$

where m is any positive or negative integer or zero.

The sum of the partial products of ${}^{m}_{n}P_k$, each of which contains $m+1$ as a factor, is obviously $(m+1)\cdot{}^{m+1}_{n-1}P_{k-1}$; the sum of the partial products which do not contain $m+1$ as a factor is ${}^{m+1}_{n-1}P_k$, hence

$$(1)\qquad {}^{m}_{n}P_k = (m+1)\,{}^{m+1}_{n-1}P_{k-1} + {}^{m+1}_{n-1}P_k .$$

In like manner, by considering the partial products which make up ${}^{m}_{n}P_k$ separated into two sets, according as they do or do not contain $m+n$ as a factor, we obtain the formula

$$(2)\qquad {}^{m}_{n}P_k = (m+n)\,{}^{m}_{n-1}P_{k-1} + {}^{m}_{n-1}P_k .$$

[1] For a complete bibliography the reader is referred to Dickson's History of the Theory of Numbers, vol. 1, chap. 3.

[2] Quarterly Journal of Mathematics, vol. 31 (1900), pp. 1-36, pp. 321-354.

Just like Mitrinovich

$$
\begin{aligned}
{}^{10}_{6}P_1 &= 16\cdot{}^{10}_{5}P_0 + {}^{10}_{5}P_1 = 16\times 1 + 65 = 81,\\
{}^{10}_{6}P_2 &= 16\cdot{}^{10}_{5}P_1 + {}^{10}_{5}P_2 = 16\times 65 + 1\,685 = 2\,725,\\
{}^{10}_{6}P_3 &= 16\cdot{}^{10}_{5}P_2 + {}^{10}_{5}P_3 = 16\times 1\,685 + 21\,775 = 48\,735,\\
{}^{10}_{6}P_4 &= 16\cdot{}^{10}_{5}P_3 + {}^{10}_{5}P_4 = 16\times 21\,775 + 140\,274 = 488\,674,\\
{}^{10}_{6}P_5 &= 16\cdot{}^{10}_{5}P_4 + {}^{10}_{5}P_5 = 16\times 140\,274 + 360\,360 = 2\,604\,744,\\
{}^{10}_{6}P_6 &= 16\cdot{}^{10}_{5}P_5 = 16\times 360\,360 = 5\,765\,760.
\end{aligned}
$$

A convenient check on the results in any one column is obtained by applying formula (18),

$$n\,{}^{m}_{n}P_n = (1+m)(1 - {}^{m}_{n}P_1 + {}^{m}_{n}P_2 + \cdots - {}^{m}_{n}P_{n-1}).$$

Thus

$$(1+10)(1 - 81 + 2\,725 + 48\,735 + 488\,674 + 2\,604\,744) = 11\times 3\,144\,960 = 34\,594\,560$$

and

$$6\times 5\,765\,760 = 34\,594\,560.$$

${}^{0}_{n}P_k \quad (n = 1, 2, \ldots, 14;\ k = 1, 2, \ldots, 14).$

✓217
✓914
✓1303

k \ n	1	2	3	4	5	6	7	8	9	10
1	1	3	6	10	15	21	28	36	45	55
2	..	2	11	35	85	175	322	546	870	1 320
3	..	..	6	50	225	735	1 960	4 536	9 450	18 150
4	..	..	..	24	274	1 624	6 769	22 449	63 273	157 773
5	..	..	..	..	120	1 764	13 132	67 284	269 325	902 055
6	..	..	..	..	..	720	13 068	118 124	723 680	3 416 930
7	..	..	..	..	..	..	5 040	109 584	1 172 700	8 409 500
8	..	..	..	..	..	..	..	40 320	1 026 576	12 753 576
9	..	..	..	..	..	..	..	..	362 880	10 628 640
10	..	..	..	..	..	..	..	..	..	3 628 800

k \ n	11	12	13	14
1	66	78	91	105
2	1 925	2 717	3 731	5 005
3	32 670	55 770	91 091	143 325
4	357 423	749 463	1 474 473	2 749 747
5	2 637 558	6 926 634	16 669 653	37 312 275
6	13 339 535	44 990 231	135 036 473	368 411 615
7	45 995 730	206 070 150	790 943 153	2 681 453 775
8	105 258 076	657 206 836	3 336 118 786	14 409 322 928
9	150 917 976	1 414 014 888	9 957 703 756	56 663 366 760
10	120 543 840	1 931 559 552	20 313 753 096	159 721 605 680
11	39 916 800	1 486 442 880	26 596 717 056	310 989 260 400
12	..	479 001 600	19 802 759 040	392 156 797 824
13	..	..	6 227 020 800	283 465 647 360
14	..	..	..	87 178 291 200

Stirling I ✓

${}^{1}_{n}P_k \quad (n = 1, 2, \ldots, 14;\ k = 1, 2, \ldots, 14).$

1705 1706

✓96
✓1701
✓1702
✓1703

k \ n	1	2	3	4	5	6	7	8	9	10
1	2	5	9	14	20	27	35	44	54	65
2	..	6	26	71	155	295	511	826	1 266	1 860
3	..	..	24	154	580	1 665	4 025	8 624	16 884	30 810
4	..	..	..	120	1 044	5 104	18 424	54 649	140 889	326 613
5	..	..	..	..	720	8 028	48 860	214 676	761 166	2 310 945
6	..	..	..	..	..	5 040	69 264	509 004	2 655 764	11 028 590
7	..	..	..	..	..	..	40 320	663 696	5 753 736	34 967 140
8	..	..	..	..	..	..	..	362 880	6 999 840	70 290 936
9	..	..	..	..	..	..	..	..	3 628 800	80 627 040
10	..	..	..	..	..	..	..	..	..	39 916 800

Generalized Stirling Nos ✓

k \ n =	11	12	13	14
1	77	90	104	119
2	2 640	3 641	4 901	6 461
3	53 130	87 450	138 424	211 939
4	696 333	1 387 023	2 611 323	4 687 683
5	6 230 301	15 282 630	34 700 952	73 870 797
6	38 759 930	119 753 843	333 710 663	854 224 943
7	167 310 220	671 189 310	2 347 743 112	7 353 403 057
8	489 896 616	2 664 929 476	12 061 579 816	47 277 726 496
9	924 118 272	7 292 774 280	44 601 786 944	225 525 484 184
10	1 007 441 280	13 020 978 816	115 119 818 736	784 146 622 896
11	479 001 600	13 575 738 240	195 869 441 664	1 922 666 722 704
12	..	6 227 020 800	196 287 356 160	3 134 328 981 120
13	..	..	87 178 291 200	3 031 488 633 600
14	..	..	..	1 307 674 368 000

${}^{2}_{n}P_k \quad (n = 1, 2, \ldots, 13;\ k = 1, 2, \ldots, 13).$

1710 1711

new

k \ n =	1	2	3	4	5	6	7	8	9
1	3	7	12	18	25	33	42	52	63
2	..	12	47	119	245	445	742	1 162	1 734
3	..	..	60	342	1 175	3 135	7 140	14 560	27 342
4	..	..	..	360	2 754	12 154	40 369	111 769	271 929
5	..	..	..	..	2 520	24 552	133 938	537 628	1 767 087
6	..	..	..	..	..	20 160	241 128	1 580 508	7 494 416
7	..	..	..	..	..	..	181 440	2 592 720	19 978 308
8	..	..	..	..	..	..	..	1 814 400	30 334 320
9	..	..	..	..	..	..	..	..	19 958 400

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G. S. Nos

k \ n =	10	11	12	13
1	75	88	102	117
2	2 490	3 465	4 697	6 227
3	48 150	80 520	129 030	199 485
4	600 033	1 225 983	2 353 263	4 288 713
5	5 030 235	12 830 664	29 994 426	65 293 571
6	28 699 460	94 092 515	273 721 811	723 638 291
7	109 911 300	483 004 280	1 800 299 490	5 966 126 625
8	270 074 016	1 698 920 916	8 460 980 836	37 165 473 186
9	383 970 240	3 894 932 448	27 679 825 272	154 594 357 822
10	239 500 800	5 231 113 920	59 760 168 192	474 957 547 272
11	..	3 113 510 400	76 349 165 280	972 754 628 160
12	..	..	43 589 145 600	1 188 825 721 800
13	..	..	..	653 837 184 000

$^{3}_{n}P_k \quad (n = 1, 2, \cdots, 13;\ k = 1, 2, \cdots, 13).$

k	$n = 1$	2	3	4	5	6	7	8	9
1	4	9	15	22	30	39	49	60	72
2	..	20	74	179	355	625	1 015	1 554	2 274
3	..	..	120	638	2 070	5 265	11 515	22 680	41 328
4	..	..	..	840	5 944	24 574	77 224	203 889	476 049
5	..	..	..	..	6 720	60 216	305 956	1 155 420	3 602 088
6	..	..	..	..	..	60 480	662 640	4 028 156	17 893 196
7	..	..	..	..	..	..	604 800	7 893 840	56 231 712
8	..	..	..	..	..	..	..	6 652 800	101 378 880
9	..	..	..	..	..	..	..	..	79 833 600

k	$n =$ 10	11	12	13
1	85	99	114	130
2	3 210	4 400	5 885	7 709
3	70 890	115 830	181 830	275 990
4	1 013 313	2 005 773	3 743 223	6 652 503
5	9 790 725	23 977 107	54 063 702	113 955 270
6	64 720 340	201 790 490	561 447 095	1 426 466 327
7	288 843 260	1 194 928 020	4 221 785 370	13 204 938 890
8	832 391 136	4 876 196 776	22 800 117 076	90 348 682 996
9	1 397 759 040	13 051 234 944	86 494 186 584	450 996 059 800
10	1 037 836 800	20 606 463 360	216 374 987 520	1 595 481 972 864
11	..	14 529 715 200	323 626 665 600	3 785 626 465 920
12	..	..	217 945 728 000	5 395 972 377 600
13	..	..	..	3 487 131 648 000

$^{4}_{n}P_k \quad (n = 1, 2, \cdots, 13;\ k = 1, 2, \cdots, 13).$

k	$n = 1$	2	3	4	5	6	7	8	9
1	5	11	18	26	35	45	56	68	81
2	..	30	107	251	485	835	1 330	2 002	2 886
3	..	..	210	1 066	3 325	8 175	17 360	33 320	59 346
4	..	..	..	1 680	11 274	44 524	134 449	342 769	775 929
5	..	..	..	..	15 120	127 860	617 624	2 231 012	6 687 009
6	..	..	..	..	..	151 200	1 557 660	8 969 148	37 972 304
7	..	..	..	..	..	..	1 663 200	20 355 120	136 954 044
8	..	..	..	..	..	..	..	19 958 400	284 574 960
9	..	..	..	..	..	..	..	..	259 459 200

k	$n =$ 10	11	12	13
1	95	110	126	143
2	4 020	5 445	7 205	9 347
3	99 750	160 050	247 170	369 655
4	1 606 773	3 103 023	5 663 823	9 865 713
5	17 550 015	41 651 610	91 299 978	187 584 969
6	131 590 430	394 840 655	1 061 266 415	2 613 366 041
7	668 566 300	2 642 422 750	8 959 873 230	27 001 402 285
8	2 201 931 576	12 230 426 076	54 509 190 076	206 827 034 986
9	4 243 508 640	37 272 482 280	232 959 299 496	1 159 615 530 788
10	3 632 428 800	67 285 058 400	663 644 774 880	4 623 952 866 312
11	..	54 486 432 000	1 131 047 366 400	12 413 008 539 360
12	..	..	871 782 912 000	20 099 588 140 800
13	..	..	..	14 820 309 504 600

$^{5}_{n}P_k \quad (n = 1, 2, \cdots, 13;\ k = 1, 2, \cdots, 13).$

k	$n = 1$	2	3	4	5	6	7	8	9
1	6	13	21	30	40	51	63	76	90
2	..	42	146	335	635	1 075	1 687	2 506	3 570
3	..	..	336	1 650	5 000	11 985	24 885	46 816	81 900
4	..	..	..	3 024	19 524	74 524	218 344	541 849	1 197 273
5	..	..	..	..	30 240	245 004	1 139 292	3 977 764	11 563 650
6	..	..	..	..	..	332 640	3 272 688	18 083 484	73 772 180
7	..	..	..	..	..	..	3 991 680	46 536 624	299 705 400
8	..	..	..	..	..	..	..	51 891 840	703 404 576
9	..	..	..	..	..	..	..	..	726 485 760

k	$n =$ 10	11	12	13
1	105	121	138	156
2	4 920	6 600	8 657	11 141
3	135 450	214 170	326 370	482 196
4	2 425 773	4 592 973	8 233 863	14 108 523
5	29 522 745	68 335 113	146 415 654	294 625 188
6	247 226 930	719 590 850	1 881 287 771	4 516 769 543
7	1 406 288 100	5 361 918 980	17 594 963 430	51 458 143 368
8	5 198 985 576	27 699 595 176	118 852 217 836	435 561 559 576
9	11 277 554 400	94 461 323 616	565 351 441 608	2 704 694 362 656
10	10 897 286 400	191 338 156 800	1 797 180 658 272	11 973 560 607 216
11	..	174 356 582 400	3 427 105 248 000	35 776 357 096 896
12	..	..	2 961 061 900 800	64 651 956 364 800
13	..	..	..	53 353 114 214 400

$^{6}_{n}P_k \quad (n = 1, 2, \cdots, 12;\ k = 1, 2, \cdots, 12).$

k	$n = 1$	2	3	4	5	6	7	8
1	7	15	24	34	45	57	70	84
2	..	56	191	431	805	1 345	2 086	3 066
3	..	..	504	2 414	7 155	16 815	34 300	63 504
4	..	..	..	5 040	31 594	117 454	336 049	816 242
5	..	..	..	..	55 440	434 568	1 961 470	6 666 156
6	..	..	..	..	..	665 280	6 314 664	33 775 244
7	..	..	..	..	..	..	8 648 640	97 053 [illegible]
8	..	..	..	..	..	..	..	121 080 960

k	$n =$ 9	10	11	12
1	99	115	132	150
2	4 326	5 910	7 865	10 241
3	109 494	178 710	279 180	420 750
4	1 768 809	3 520 713	6 558 783	11 584 023
5	18 909 891	47 210 835	107 062 956	225 121 050
6	133 767 584	436 325 840	1 238 910 035	3 166 043 243
7	603 682 596	2 743 963 940	10 161 503 220	32 461 883 850
8	1 576 890 000	11 235 811 536	57 883 198 516	240 791 256 476
9	1 816 214 400	27 046 454 400	218 055 250 512	1 259 952 823 800
10	..	29 059 430 400	488 849 155 200	4 413 843 664 416
11	..	..	494 010 316 800	9 293 295 110 400
12	..	..	..	8 892 185 702 400

$^{7}_{n}P_k \quad (n = 1, 2, \cdots, 12;\ k = 1, 2, \cdots, 12).$

k	n = 1	2	3	4	5	6	7	8
1	8	17	27	38	50	63	77	92
2	..	72	242	539	995	1 645	2 527	3 682
3	..	..	720	3 382	9 850	22 785	45 815	83 720
4	..	..	..	7 920	48 504	176 554	495 544	1 182 769
5	..	..	..	..	95 040	725 592	3 197 348	10 630 508
6	..	..	..	..	..	1 235 520	11 393 808	59 354 028
7	..	..	..	..	..	..	17 297 280	188 204 400
8	..	..	..	..	..	..	..	259 459 200

k	n = 9	10	11	12
1	108	125	143	162
2	5 154	6 990	9 240	11 957
3	142 632	230 250	356 070	531 630
4	2 522 289	4 947 033	9 091 533	15 856 863
5	29 554 812	72 433 725	161 480 319	334 219 446
6	229 442 156	731 873 960	2 035 681 010	5 103 807 071
7	1 137 868 848	5 038 385 500	18 212 116 780	56 890 055 970
8	3 270 729 600	22 614 500 016	113 305 439 016	459 335 657 836
9	4 151 347 200	59 753 750 400	466 814 750 688	2 619 618 091 992
10	..	70 472 902 400	1 146 140 409 600	10 015 620 672 672
11	..	..	1 270 312 243 200	23 046 980 025 600
12	..	..	..	24 135 932 620 800

$^{8}_{n}P_k \quad (n = 1, 2, \cdots, 12;\ k = 1, 2, \cdots, 12).$

k	n = 1	2	3	4	5	6	7	8
1	9	19	30	42	55	69	84	100
2	..	90	299	659	1 205	1 975	3 010	4 354
3	..	..	990	4 578	13 145	30 015	59 640	107 800
4	..	..	..	11 880	71 394	255 424	705 649	1 659 889
5	..	..	..	..	154 440	1 153 956	4 985 316	16 275 700
6	..	..	..	..	..	2 162 160	19 471 500	99 236 556
7	..	..	..	..	..	..	32 432 400	343 976 400
8	..	..	..	..	..	..	..	518 918 400

k	n = 9	10	11	12
1	117	135	154	174
2	6 054	8 160	10 725	13 805
3	181 818	290 790	445 830	660 330
4	3 492 489	6 765 213	12 290 223	21 206 823
5	44 493 813	107 358 615	235 897 662	481 702 122
6	375 923 456	1 176 812 090	3 216 625 775	7 934 579 015
7	2 030 997 852	8 797 620 060	31 157 049 770	95 489 565 270
8	6 366 517 200	42 924 478 536	210 079 259 676	833 220 255 076
9	8 821 612 800	123 418 922 400	938 984 014 584	5 140 569 208 104
10	..	158 789 030 400	2 503 748 556 000	21 283 428 847 680
11	..	..	3 016 991 577 600	53 091 962 697 600
12	..	..	..	60 339 831 552 000

$^{9}_{n}P_k \quad (n = 1, 2, \cdots, 12;\ k = 1, 2, \cdots, 12).$

k	n = 1	2	3	4	5	6	7	8
1	10	21	33	46	60	75	91	108
2	..	110	362	791	1 435	2 335	3 535	5 082
3	..	..	1 320	6 026	17 100	38 625	75 985	136 080
4	..	..	..	17 160	101 524	358 024	976 024	2 267 769
5	..	..	..	..	240 240	1 763 100	7 491 484	24 083 892
6	..	..	..	..	..	3 603 600	31 813 200	159 168 428
7	..	..	..	..	..	..	57 657 600	598 482 000
8	..	..	..	..	..	..	..	980 179 200

k	n = 9	10	11	12
1	126	145	165	186
2	7 026	9 420	12 320	15 785
3	227 556	361 050	549 450	808 170
4	4 717 209	9 040 773	16 261 773	27 800 223
5	64 903 734	154 530 705	335 346 165	676 843 398
6	592 678 484	1 825 849 430	4 916 463 530	11 958 732 995
7	3 463 513 704	14 724 404 900	51 241 393 500	154 487 127 630
8	11 752 855 200	77 559 615 576	372 047 743 576	1 448 116 977 076
9	17 643 225 600	240 947 474 400	1 792 139 785 920	9 605 141 771 016
10	..	335 221 286 400	5 154 170 774 400	42 789 106 278 720
11	..	..	6 704 425 728 000	114 942 011 990 400
12	..	..	..	140 792 940 288 000

$^{10}_{n}P_k \quad (n = 1, 2, \cdots, 12;\ k = 1, 2, \cdots, 12).$

k	n = 1	2	3	4	5	6	7	8
1	11	23	36	50	65	81	98	116
2	..	132	431	935	1 685	2 725	4 102	5 866
3	..	..	1 716	7 750	21 775	48 735	95 060	168 896
4	..	..	..	24 024	140 274	488 674	1 317 169	3 028 249
5	..	..	..	..	360 360	2 604 744	10 912 202	34 621 244
6	..	..	..	..	..	5 765 760	50 046 408	246 466 644
7	..	..	..	..	..	..	98 017 920	998 853 264
8	..	..	..	..	..	..	..	1 764 322 560

k	n = 9	10	11	12
1	135	155	176	198
2	8 070	10 770	14 025	17 897
3	280 350	441 750	667 920	976 459
4	6 237 273	11 844 273	21 121 023	35 815 263
5	92 157 975	216 903 435	465 633 168	930 295 674
6	904 269 680	2 747 429 180	7 302 401 315	17 546 331 011
7	5 681 708 100	23 767 101 700	81 463 114 480	242 115 943 459
8	20 742 534 576	134 376 696 576	633 485 832 276	2 425 674 590 856
9	33 522 128 640	448 372 820 160	3 270 283 448 256	17 206 971 758 328
10	..	670 442 572 800	10 086 271 796 160	82 032 307 657 792
11	..	..	14 079 294 028 800	235 977 275 344 320
12	..	..	..	309 744 468 633 600