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Hey I'll send you some interesting sequences, some of them are accompanied by some references. For other sequences I don't have references at the moment. If you are writing your book please send me (at cwi.nl!coster) or Frits Beukers (in Utrecht e-mail address unknown) a mail.

Matthew. Coster

All the sequences are based on recurrences of the form:

$$(1) \quad u(n+1) = p1(n)*u(n) + p2(n)*u(n-1).$$

The degree of the recurrence is max degree(denominator(p1), numerator(p1), denominator(p2), numerator(p2)).

Degree 1:

$$(2a) \quad (n+1)*u(n+1) = a*(2*n+1)*u(n) - b*n*u(n-1)$$

$$(2b) \quad u(0) = 1, u(1) = a.$$

These sequences can be derived from the the Legendre polynomials and they appear as the coefficients of special hypergeometric functions.

Let $f(x) = \sum_{n=0}^{\infty} u(n)*x^n$. The differential equation which satisfies $f(x)$ is

$$(3) \quad (b*x^2 - 2*a*x + 1)*y' + (b*x - a)*y = 0$$

Solving this equation gives:

$$(4) \quad y = (1 - 2*a*x + b*x^2)^{-1/2}.$$

The definition of the Legendre polynomials can be given in several ways (see [5], pp. 90-91). One definition is

$$(5) \quad (1 - 2*t*x + x^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(t)*x^n.$$

One important sequence is the sequence which we get with $a = 3$ and $b = 1$. In this case

$$(6) \quad u(n) = \sum_{k=0}^n \text{Choose}(n, k) * \text{Choose}(n+k, k).$$

with $u(0) = 1, u(1) = 3$.

Start of sequence: 1, 3, 13, 63, 321, 1683, 8989, 48639, 265729, 1462563, 8097453, 45046719, 251595969, 1409933619, 7923848253, ...

Let $v(n)$ satisfy (2a) with a, b as above and $v(0) = 0, v(1) = 1$. However the numbers $v(n)$ are not integers, the quotients $v(n)/u(n)$ are important to find a measure of irrationality of $\log(2)$.

Degree 2.

now consider sequences which satisfy the recurrence-equation

$$(7a) \quad (n+1)^2*u(n+1) = (a*n^2 + a*n + b)*u(n) - c*n^2*u(n-1),$$

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$$(7b) \quad u(0) = 1, u(1) = b.$$

There are only a few sequences with integer coefficients which satisfy (7) are sequences which are related to power series which appear as fibers in some surfaces which were described by Beauville [3]. Unfortunately I forgot some of those sequences. But those I remember are:

$$a=7, b=2, c=8: u(n) = \sum_{k=0}^n \binom{n}{k}^3.$$

Start of sequence: 1, 2, 10, 56, 346, 2252, 15184, 104960, 739162, 5280932, 38165260, 278415920, 2046924400, 15148345760, ... ✓

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$$a=11, b=3, c=-1: u(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}.$$

Start of sequence: 1, 3, 19, 147, 1251, 11253, 104959, 1004307, 9793891, 96918753, 970336269, 9807518757, 99912156111, ... ✓

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These numbers play a main role in finding a measure of irrationality of π^2 . See [2] and [4].

$$a=10, b=3, c=9: u(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k}.$$

Start of sequence: 1, 3, 15, 93, 639, 4653, 35169, 272835, 2157759, 17319837, 14066065, 1153462995, 9533639025, ...

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See [4].

$a=9, b=3, c=27$: Start of sequence: 1, 3, 9, 21, 9, -297, -2421, -12933, -52407, -145293, -35091, 2954097, 25228971, 142080669, ...

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Degree 3

Sequences which satisfy the recurrence-equation

$$(8a) \quad (n+1)^3 u(n+1) = (a n^3 + (3/2) a n^2 + b n + c) u(n) - (d n^3 + e n) u(n-1),$$

$$(8b) \quad u(0) = 1, u(1) = c.$$

$$a=34, b=27, c=5, d=1, e=0: u(n) = \sum_{k=0}^n \binom{n}{k}^2 \binom{n+k}{k}^2.$$

Start of sequence: 1, 5, 73, ...

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These numbers were used by Ape'ry proving the irrationality of $\zeta(3)$. See [2] and [4].

$$a=12, b=10, c=2, d=-64, e=4: u(n) = \sum_{k=0}^n \binom{n}{k}^4.$$

Start of sequence: 1, 2, 18, 164, ... ✓

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References.

- [1] K. Alladi and M.L. Robinson, On certain values of the logarithm, Lecture Notes 751, 1-9.
- [2] R. Ape'ry, Irrationalite' de $\zeta(2)$ et $\zeta(3)$, Aste'risque 61 (1979), 11-13.
- [3] A. Beauville, Les familles stables de courbes elliptiques sur P^2 admettant quatre fibres singulie'res, C.R. Acad. Sci. Paris 294 (1982), 657.
- [4] F. Beukers, Another Congruence for the Ape'ry Numbers, J. Num. Th. 25 (1987).
- [5] M.J. Coster, Supercongruences, Thesis, Leiden 1988.

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