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TITLE— The Enumeration of Permutations With Three-Ply Staircase Restrictions

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DATE – October 11, 1963

AUTHOR(S) – John Riordan

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Also two triangles of numbers,

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ABSTRACT

An inquiry from M. R. Schroeder has revived interest in permutations with restricted positions specified by truncated three-ply staircases, studied originally because of their relation to four-line Latin rectangles. Two truncated staircases are studied here: (i) first and last columns removed, and (ii) last two columns removed. Asymptotic expressions for the two "hit" distributions, and a number of recurrences and relations for hit polynomials are derived.

Contains many sequence to be extended

Note – get all of JR's MM's. !

SUBJECT: The Enumeration of Permutations With Three-Ply Staircase Restrictions - [redacted]

DATE: October 11, 1963

FROM: John Riordan

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MEMORANDUM FOR FILE

1. Introduction

An inquiry from M. R. Schroeder has revived interest in permutations with restricted positions specified by truncated three-ply staircases; they seem to be involved in his work on design of a private communication system. The three-ply staircases appear in problems 21 to 28 of chapter 8 of my book (An Introduction to Combinatorial Analysis, New York, 1958). Nevertheless there is something to be done on the truncations of interest to Mr. Schroeder, which are (i) first and last columns removed, and (ii) last two columns removed; for n = 5 the two staircases are, in order,

```
×  ×                   ×  ×  ×
×  ×  ×                   ×  ×  ×
   ×  ×  ×                   ×  ×  ×
      ×  ×  ×                   ×  ×
         ×  ×                      ×
```

The rook polynomials, in the notation of the book, are $T_n(x)$ and $s_n(x)$. The corresponding hit polynomials (enumerating permutations of n elements by the number in forbidden positions) are taken as $A_n(t)$ and $B_n(t)$, respectively.

The main results developed below are, first, the asymptotic expressions for the coefficients A_{nk} and B_{nk} of the hit polynomials, namely

$$(1) \qquad \frac{A_{nk}}{n!} = \frac{3^k e^{-3}}{k!}\left[1 - \frac{k^2-5k+3}{3n} + \frac{a(k)}{54n(n-1)}\right] + O(n^{-3})$$

$$(2) \qquad \frac{B_{nk}}{n!} = \frac{3^k e^{-3}}{k!}\left[1 - \frac{k^2-4k}{3n} + \frac{b(k)}{54n(n-1)}\right] + O(n^{-3})$$

where, with $(k)_j = k(k-1)\ldots(k-j+1)$,

$$a(k) = 3(k)_4 - 20(k)_3 + 30(k)_2 + 36k - 81$$

$$b(k) = 3(k)_4 - 14(k)_3 - 12(k)_2 + 126k - 135.$$

Next, the "mixed" recurrences found are

(3)

$$A_n(t) = (n-2+2t)A_{n-1}(t) + (1-t)A'_{n-1}(t) - (n-1)(1-t)A_{n-2}(t)$$
$$- (1-t)^2A'_{n-2}(t) + (1-t)^3A_{n-3}(t), \quad n > 1$$

(4)

$$B_n(t) = (n-2+2t)B_{n-1}(t) + (1-t)B'_{n-1}(t) - (n-1)(1-t)B_{n-2}(t)$$
$$- (1-t)^2B'_{n-2}(t) + (1-t)^3B_{n-3}(t) + (-1)^n(1-t)^{n-1}, \quad n > 2.$$

The prime denotes a derivative. Next, there is an almost pure recurrence for $A_n(t)$, that is, without derivatives, namely

(5)

$$A_n(t) = (n+1-t)A_{n-1}(t) - (1-t)(n-3+4t)A_{n-2}(t) - (n-4)(1-t)^2A_{n-3}(t)$$

$$+ (1-t)^3(n-4+t)A_{n-4}(t) + (1-t)^5A_{n-5}(t) + (t-1)^n a_n$$

where $a_0 = 1$, $a_1 = 2$, $a_2 = -1$, $a_3 = a_4 = 1$ and

$$a_n = a_{n-1} + a_{n-2}, \quad n > 4.$$

Otherwise stated

$$a_n - a_{n-1} - a_{n-2} = \delta_{n0} + \delta_{n1} - 4\delta_{n2} + \delta_{n4} + \delta_{n5}$$

with δ_{nm} the Kronecker delta. Of course, a_n may be eliminated to produce a genuinely pure recurrence, at the cost of more terms. The chief interest of (5) is in the instance $t = 0$: $A_n = A_n(0)$ is the number of permutations with no elements in forbidden positions; the recurrence becomes

$$(6) \quad A_n = (n+1)A_{n-1} - (n-3)A_{n-2} - (n-4)A_{n-3} + (n-4)A_{n-4}$$

$$+ A_{n-5} + (-1)^n a_n.$$

I have not found a similar result for $B_n(t)$ but there is a moderately simple expression relating the two hit polynomials, namely

(7)

$$B_n(t) - (1-t)^2B_{n-2}(t) = A_n(t) + (1-t)A_{n-1}(t) - (1-t)^2A_{n-2}(t)$$

$$- \delta_{n1}(1-t)^n.$$

Iteration of (7) and mathematical induction show that

(8)

$$B_{2n}(t) = A_{2n}(t) + (1-t)A_{2n-1}(t) + (1-t)^3A_{2n-3}(t) + \dots + (1-t)^{2j-1}A_{2n-2j+1}(t) + \dots + (1-t)^{2n-1}A_1(t)$$

$$B_{2n+1}(t) = A_{2n+1}(t) + (1-t)A_{2n}(t) + (1-t)^3A_{2n-2}(t) + \dots + (1-t)^{2j+1}A_{2n-2j} + \dots + (1-t)^{2n-1}A_2(t).$$

Table 1 gives the polynomials $A_n(t)$ and $B_n(t)$ in detached coefficient form for $n = 0(1)9$.

Finally, it should be noticed that the hit polynomials, $C_n(t)$, for the circularized staircase are expressible by

$$C_n(t) = A_n(t) + (1-t)^2A_{n-2}(t) - 2(1-t)B_{n-1}(t).$$

2. Rook Polynomials

Consider first the rook polynomials $T_n(x)$ for three-ply staircases with first and last column removed. The expression

(9) $$T_n(x) = S_n(x) - 2xS_{n-1}(x) + x^2S_{n-2}(x)$$

given in problem 22 of chapter 8, with $S_n(x)$ the rook polynomial for a three-ply staircase, is found by developing $S_n(x)$ with respect to the single cells in first and last columns. The development is sufficiently clear in the case

$n = 3$. Using the convention that brackets on an array indicate its rook polynomial, this is as follows:

$$S_3(x) = \begin{bmatrix} \times & \times & \times & \\ & \times & \times & \times \\ & & \times & \times & \times \end{bmatrix}$$

$$= \begin{bmatrix} \times & \times & \\ \times & \times & \times & \\ & \times & \times & \times \end{bmatrix} + xS_2(x)$$

$$= T_3(x) + xS_2(x) + x \begin{bmatrix} \times & \times & \\ \times & \times & \times \end{bmatrix}$$

$$= T_3(x) + xS_2(x) + x[S_2(x) - S_1(x)].$$

Hence $T_0(x) = 1$, $T_1(x) = 1 + x$, $T_2(x) = 1 + 4x + 2x^2$.

It follows from (9) that the generating functions $T(x,y) = \Sigma T_n(x)y^n$, $S(x,y) = \Sigma S_n(x)y^n$, both sums starting at $n = 0$, are related by

(10) $$S(x,y) = T(x,y)(1-xy)^{-2}$$

and by problem 8.23 (problem 23 of chapter 8)

(11) $$T(x,y) = (1-xy)[1-y-2xy-xy^2+x^3y^3]^{-1}$$

$$S(x,y) = (1-xy)^{-1}[1-y-2xy-xy^2+x^3y^3]^{-1}.$$

Equation (11) implies the recurrence

(12)

$$T_n(x) - (1+2x)T_{n-1}(x) - xT_{n-2}(x) + x^3T_{n-3}(x) = \delta_{n0} - x\delta_{n1}$$

which is simpler than the one in problem 8.22. Writing

$$T_n(x) = \sum_{k=0}^{n} T_{nk}x^k,$$

equation (12) in turn implies

(13)

$$T_{nk} = T_{n-1,k} + 2T_{n-1,k-1} + T_{n-2,k-1} - T_{n-3,k-3}, \quad n > 1.$$

For the second truncation (last two columns removed) the rook polynomials $s_n(x)$ have the generating function $s(x,y)$, which by a result in problem 8.24(a) is given by

(14)

$$s(x,y) = (1-2xy-xy^2+x^3y^3)(1-xy)^{-1}(1-y-2xy-xy^2+x^3y^3)^{-1}$$

or

(14a)

$$(1-y-2xy-xy^2+x^3y^3)s(x,y) = 1 - xy - x^2y^2 - xy^2(1-xy)^{-1}.$$

Equation (14a) implies the recurrence (compare (12))

(15) $$s_n(x) - (1+2x)s_{n-1}(x) - xs_{n-2}(x) + x^3s_{n-3}(x)$$

$$= \delta_{n0} - x\delta_{n1} - x^2\delta_{n2} - (1-\delta_{n0}-\delta_{n1})x^{n-1}.$$

The first few values are: $s_0(x) = 1$, $s_1(x) = 1 + x$, $s_2(x) = 1 + 3x + x^2$. If

$$s_n(x) = \sum_{k=0}^{n} s_{nk}x^k,$$

equation (15) implies

(16)

$$s_{nk} = s_{n-1,k} + 2s_{n-1,k-1} + s_{n-2,k-1} - s_{n-3,k-3} - \delta_{n-1,k}, \quad n > 2.$$

It is convenient to have results also for the associated rook polynomials $t_n(x) = x^nT_n(-x^{-1})$, $s_n^*(x) = x^ns_n(-x^{-1})$. Their generating functions are given by

(17)
$$\Delta(x,y)t(x,y) = 1 + y$$

$$\Delta(x,y)s^*(x,y) = 1 + y - y^2 + xy^2(1+y)^{-1}$$

$$= (\Delta(x,y) + xy)(1+y)^{-1}$$

where

$$\Delta(x,y) = 1 + 2y - y^3 - xy + xy^2.$$

2. Asymptotic Expressions

Asymptotic expressions are obtained from expressions for factorial moments. Consider the first truncation. The kth factorial moment $(m)_k$ of the distribution with probability generating function $A_n(t)/n!$ is given by $T_{nk}\binom{n}{k}^{-1}$, in the usual notation for binomial coefficients. Then if

(18) $$T_{nk} = a_{k0}\binom{n}{k} + a_{k1}\binom{n-1}{k-1} + \dots + a_{kj}\binom{n-j}{k-j} + \dots$$

$$(m)_k = a_{k0} + a_{k1}k/n + a_{k2}k(k-1)/n(n-1) + \dots$$

which leads to an asymptotic expression through the relation

(19) $$\frac{A_{nk}}{n!} = p_k(n) = \sum_{j=0} \frac{(-1)^j (m)_{k+j}}{j!k!}.$$

A_{nk} is of course the coefficient of t^k in $A_n(t)$.

Equation (13) implies

(20) $$a_{kj} = 3a_{k-1,j} - a_{k-1,j-1} - a_{k-3,j-2}$$

which with boundary conditions supplied by

$$T_{n0} = 1 \qquad T_{n2} = 9\binom{n}{2} - 9\binom{n-1}{1} + 2$$

$$T_{n1} = 3n - 2 \qquad T_{n3} = 27\binom{n}{3} - 36\binom{n-1}{2} + 14\binom{n-2}{1} - 2$$

leads to

$$a_{k0} = 3^k \qquad a_{k2} = 3^{k-3}\left(3\binom{k}{2} + 2k - 1\right)$$

$$a_{k1} = -3^{k-1}(k+1) \qquad a_{k3} = -3^{k-4}\left(3\binom{k}{3} + \binom{k}{2} + \delta_{k3}\right).$$

Using the first three of these in (19) gives (1).

In the same way, writing

$$s_{nk} = b_{k0}\binom{n}{k} + b_{k1}\binom{n-1}{k-1} + \dots + b_{kj}\binom{n-j}{k-j} + \dots$$

it is found that the b_{kj} also satisfy equation (20), but

$$s_{n0} = 1 \qquad s_{n2} = 9\binom{n}{2} - 12\binom{n-1}{1} + 4$$

$$s_{n1} = 3n - 3 \qquad s_{n3} = 27\binom{n}{3} - 45\binom{n-1}{2} + 23\binom{n-2}{1} - 5 + \delta_{n3}$$

so that

$$b_{k0} = 3^k \qquad b_{k2} = 3^{k-3}\left(3\binom{k}{2} + 5k - 1\right)$$

$$b_{k1} = -\,3^{k-1}(k+2).$$

Using these in (19) gives (2).

3. Mixed Recurrences for Hit Polynomials

The hit polynomials $A_n(t)$ and $B_n(t)$ are given by

(21)
$$A_n(t) = \Sigma T_{nk}(n-k)!(t-1)^k$$

$$B_n(t) = \Sigma s_{nk}(n-k)!(t-1)^k.$$

Hence, by (13), for $n > 1$,

(22)
$$U_{nk} = (n-k)!T_{nk}$$

$$= (n-k)(U_{n-1,k} + U_{n-2,k-1}) + 2U_{n-1,k-1} - U_{n-3,k-3}.$$

Since

$$A_n'(t) = \Sigma k U_{nk}(t-1)^{k-1},$$

(22) is equivalent to (3). A similar procedure gives (4).

4. Another Recurrence for $A_n(t)$

Return to the first of equations (17), the expression for the generating function of the associated rook polynomials of the first truncation, namely

$$\Delta(x,y)t(x,y) = 1 + y.$$

Then, using the suffix notation for partial derivatives, and omitting arguments

$$\text{(23)} \qquad \Delta t_x + \Delta_x t = 0$$

$$\Delta t_y + \Delta_y t = 1.$$

The second of equations (23) is equivalent to

$$(1+y)\Delta t_y = (\Delta - (1-y)\Delta_y)t$$

$$= - (1 - 3y^2 - 2y^3 - x + 2xy + xy^2)t.$$

Hence, since $\Delta_x = -y(1-y)$, equations (23) may be replaced by

$$\text{(24)} \qquad g(y)\Delta t_x - y(1-y)g(y)t = 0$$

$$(1+y)f(y)\Delta t_y + f(y)[1 - 3y^2 - 2y^3 - x(1-2y-y^2)]t = 0.$$

Adding

(25)

$$g(y)\Delta t_x + (1+y)f(y)\Delta t_y + [f(y)(1-3y^2-2y^3) - xf(y)(1-2y-y^2) - y(1-y)g(y)]t = 0.$$

Equation (25) will be simpler if

$$f(y)[1-3y^2-2y^3-x(1-2y-y^2)] - y(1-y)g(y) = h(y)\Delta$$

$$= h(y)(1-2y-y^3) - xy(1-y)h(y).$$

Then first, equating coefficients of x

$$f(y)(1-2y-y^2) = y(1-y)h(y)$$

or if $f(y) = y(1-y)f_1(y)$

$$f_1(y)(1-2y-y^2) = h(y).$$

Next

$$y(1-y)(1-3y^2-2y^3)f_1(y) - y(1-y)g(y) = (1-2y-y^2)(1+2y-y^3)f_1(y)$$

or

$$-[1-y-4y^2+y^4-y^5]f_1(y) - y(1-y)g(y) = 0$$

and if $f_1(y) = y(1-y)$

$$f(y) = y^2(1-y)^2$$

$$g(y) = -1 + y + 4y^2 - y^4 + y^5$$

$$h(y) = y(1-y)(1-2y-y^2).$$

The simpler form of equation (25) is

(26)

$$-(1-y-4y^2+y^4-y^5)t_x + y^2(1+y)(1-y)^2 t_y + y(1-y)(1-2y-y^2)t = 0.$$

Equating coefficients of y^n, this corresponds to, with primes denoting derivatives,

(27)
$$nt_{n-1} - (n+1)t_{n-2} - (n-4)t_{n-3} + (n-3)t_{n-4}$$
$$= t_n' - t_{n-1}' - 4t_{n-2}' + t_{n-4}' = t_{n-5}'.$$

Now equation (21) may be rewritten in compressed form as

(21a)
$$A_n(t) = (1-t)^n t_n[E(1-t)^{-1}]0!$$

with E the shift operator: $E^k 0! = k!$. Also

(28)
$$A_n(t) - (1-t)^n t_n(0) = (1-t)^{n-1} t_n'[E(1-t)^{-1}]0!.$$

Using (21a) and (28) in (27), multiplied by $(1-t)^{n-1}$, leads to (5) when it is noted that

$$t_n(0) = (-1)^n f_n$$

with f_n a Fibonacci number: $f_0 = f_1 = 1$, $f_n = f_{n-1} + f_{n-2}$. In the first place, the numbers a_n of equation (5) are given by

$$a_n = f_n + f_{n-1} - 4f_{n-2} + f_{n-4} + f_{n-5}.$$

Next the generating function a(x) of numbers a_n is given by

$$a(x) = (1+x-4x^2+x^4+x^5)f(x)$$

where $f(x) = \Sigma f_n x^n = (1-x-x^2)^{-1}$; hence

$$(1-x-x^2)a(x) = (1+2x-x^2)(1-x-x^2) + x^3 + x^5$$

or

$$a(x) = 1 + 2x - x^2 + (x^3+x^5)f(x)$$

which shows that $a_0 = 1$, $a_1 = 2$, $a_2 = -1$, $a_n = f_{n-3} + f_{n-5}$, $n > 2$. A short table may be interesting; it is

n	0	1	2	3	4	5	6	7	8	9	10
a_n	1	2	-1	1	1	3	4	7	11	18	29

Lucas nos ✓

5. A Relation for $A_n(t)$ and $B_n(t)$

In problem 8.24(a) it is noted that the generating function for rook polynomials $s_n(x)$ may be written as

$$s(x,y) = yS(x,y) + (1-xy)^{-1}.$$

Using equation (10), this is the same as

$$(1-xy)^2 s(x,y) = yT(x,y) + 1 - xy$$

or, shifting to the associated polynomials ($s^*(x,y) = s(-x^{-1},xy)$, $t(x,y) = T(x^{-1},xy)$)

$$(1+y)^2 s^*(x,y) = xyt(x,y) + 1 + y.$$

But

$$xy(1-y)t(x,y) = (1+2y-y^3)t(x,y) + (1+y).$$

Hence

$$(1+y)^2(1-y)s^*(x,y) = xy(1-y)t(x,y) + 1 - y^2$$

$$= (1+2y-y^3)t(x,y) - y(1+y)$$

which has a removable factor 1+y, so that

$$(1-y^2)s^*(x,y) = (1+y-y^2)t(x,y) - y$$

or

(29) $s_n^*(x) - s_{n-2}^*(x) = t_n(x) + t_{n-1}(x) - t_{n-2}(x) - \delta_{n1}$.

Using (21)a and its correspondent

$$B_n(t) = (1-t)^n s_n^*[E(1-t)^{-1}]0!$$

in (29) produces (7).

Now, turn to the relations with the circularized staircase (the triangle removed by removal of the last two columns is reinserted in the first two columns). The rook polynomials, in the notation of problem 8.24, are $R_n(x)$ and $r_n(x) = x^n R_n(-x^{-1})$, and the hit polynomial is $C_n(t)$. Then first, by problem 8.24(b)

(30)

$$R_n(x) = s_n(x) + x^2 s_{n-2}(x) + xT_{n-1}(x)$$

$$+ 2x[S_{n-2}(x) - xS_{n-3}(x) + xT_{n-2}(x)].$$

But (problem 8.24(a))

$$s_n(x) - xs_{n-1}(x) = S_{n-1}(x) - xS_{n-2}(x)$$

so that

(31)

$$R_n(x) = s_n(x) + 2xs_{n-1}(x) - x^2s_{n-2}(x) + xT_{n-1}(x) + 2x^2T_{n-2}(x) - x\delta_{n1}$$

(the additional term, $-x\delta_{n1}$, is inserted to suit the convention $R_1(x) = 1 + 3x$). Equation (31) implies the generating function relation

(32) $$R(x,y) = (1+2xy-x^2y^2)s(x,y) + (xy+2x^2y^2)T(x,y) - xy.$$

Combining this with one of the equations above, namely

$$(1-xy)^2s(x,y) = yT(x,y) + 1 - xy$$

leads to

$$R(x,y) = 2s(x,y) - [y(1-x) - 2x^2y^2]T(x,y) - 1$$

or, shifting to associated rook polynomials

(33) $$r(x,y) = 2s^*(x,y) - (xy+y-2y^2)t(x,y) - 1.$$

This is the same as, using the first of equations (17),

$$\begin{aligned}(1-y)r(x,y) &= 2(1-y)s^*(x,y) - y(1-y)(1-2y)t(x,y) - (1-y) \\ &\quad - (1+2y-y^3)t(x,y) + 1 + y \\ &= 2(1-y)s^*(x,y) - (1+3y-3y^2+y^3)t(x,y) + 2y.\end{aligned}$$

Hence,

(34)

$$(1-y^2)r(x,y) = 2(1-y^2)s^*(x,y) - (1+y)(1+3y-3y^2+y^3)t(x,y) + 2y(1+y)$$

$$= [2(1+y-y^2) - (1+y)(1+3y-3y^2+y^3)]t(x,y) + 2y^2$$

$$= (1-2y-2y^2+2y^3-y^4)t(x,y) + 2y^2$$

$$= (1-y^4)t(x,y) - 2y[(1+y-y^2)t(x,y) - y]$$

$$= (1-y^4)t(x,y) - 2y(1-y^2)s^*(x,y)$$

where the relation

$$(1-y^2)s^*(x,y) = (1+y-y^2)t(x,y) - y$$

has been used twice. Cancelling the common factor $1-y^2$ gives

(34a) $$r(x,y) = (1+y^2)t(x,y) - 2ys^*(x,y)$$

or

$$r_n(x) = t_n(x) + t_{n-2}(x) - 2s^*_{n-1}(x)$$

which gives immediately the equation cited in the introduction, namely

$$C_n(t) = A_n(t) + (1-t)^2A_{n-2}(t) - 2(1-t)B_{n-1}(t).$$

It may be noticed that (34a) and

$$y = (1+y-y^2)t(x,y) - (1-y^2s*(x,y))$$

imply

(35) $$D(y)t(x,y) = (1-y^2)r(x,y) - 2y^2$$

$$D(y)s*(x,y) = (1+y-y^2)r(x,y) - y - y^3$$

with $D(y) = 1 - 2y - 2y^2 + 2y^3 - y^4$.

Finally, the generating function $q(x,y)$ of polynomials $q_n(x)$, where $r_n'(x) = nq_{n-1}(x)$ the prime denoting a derivative, is given by

$$\Delta(x,y)q(x,y) = 1 - y$$

with $\Delta(x,y) = 1 + 2y - y^3 - xy + xy^2$, as in equations (17). Hence

(36) $$(1+y)q(x,y) = (1-y)t(x,y)$$

or

$$q_n(x) + q_{n-1}(x) = t_n(x) - t_{n-1}(x),$$

and, if as in problem 8.27,

$$M_n(t) = (1-t)^n q_n[E(1-t)^{-1}]0!, \quad E^k0! = k!$$

then

(37) $$M_n(t) + (1-t)M_{n-1}(t) = A_n(t) - (1-t)A_{n-1}(t).$$

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Since

$$(n+1)M_n(t) = C_{n+1}(t) - (1-t)^{n+1}r_{n+1}(0)$$

equation (37) may be rewritten as

$$nC_{n+1}(t) + (1-t)(n+1)C_n(t) + (t-1)^{n+1}(nc_{n+1} - (n+1)c_n)$$

$$= n(n+1)A_n(t) - (1-t)n(n+1)A_{n-1}(t)$$

with $c_n = (-1)^n r_n(0)$. A short table of the numbers is as follows

n	0	1	2	3	4	5	6	7	8	9	10
c_n	1	3	5	6	9	13	20	31	49	78	125

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Note that $c_n = c_{n-1} + c_{n-2} - 2$, $n > 2$.

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TABLE 1

The Polynomials $A_n(t)$ and $B_n(t)$

rest are type 1

$A_n(t)$

k\n	0	1	2	3	4	5	6	7	8	9	
0	1	0	0	0	1	4	29	206	1708	15702	type 1 — 1883
1		1	0	1	2	20	104	775	6140	55427	— 1884
2			2	2	10	28	207	1288	10366	91296	— 1885
3				3	6	44	180	1407	10384	92896	— 1886
4					5	16	151	830	7298	63140	
5						8	36	437	3100	31278	
6							13	76	1138	10048	
7								21	152	2744	
8									34	294	
9										55	(45)

Δ is A080018

$B_n(t)$

k\n	0	1	2	3	4	5	6	7	8	9	
0	1	0	0	0	1	5	33	236	1918	17440	— 1887
1		1	1	1	4	21	122	849	6719	59873	— 1888
2			1	4	8	38	209	1400	10849	95516	— 1889
3				1	10	34	206	1351	10543	92708	— 1890
4					1	21	109	836	6629	60284	
5						1	40	295	2821	26870	
6							1	72	715	8372	
7								1	125	1604	
8									1	212	1891
9										1	

Δ is A080061

A1887 – A1891