DAVENPORT-SCHINZEL SEQUENCES

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1. *Introduction.*

There are two interesting dual problems in sequence construction. Problem A is to construct as short a sequence as possible which contains subsequences of a certain type. Problem B is to construct as long a [redacted]

Davenport-Schinzel (or, more briefly, DS) sequences are a case of Problem B, one of the few combinatorial problems to arise from a problem in differential equations. This paper is intended as an up-to-date expository survey of current knowledge of DS sequences.

Davenport and Schinzel [3] explain that, if F(D)f(x[redacted]) = 0 is a homogeneous, linear, differential equation of degree d, and if $f_1(x), f_2(x), \ldots, f_n(x)$, are n distinct (but not necessarily independent) utions of F(D)f(x) = 0, then a dissection of the real line into elements are equal, and (b) no subsequence of elements of the form ...ababa... has length greater than d (the elements in the subsequence are ordered, but not necessarily adjacent). Thus, for d = 4, n = 5 [redacted]

the sequence

1 2 1 3 4 1 5 2

is a DS sequence, but

1 2 1 3 4 1 5 2 1

is not.

We denote the maximal length of a DS sequence by N(d,n).

2. *Normal Sequences and* N(3,n).

It is convenient to adopt a convention that all sequences [redacted] considered be *normal*, that is, the symbols are renamed so as to appear in natural order. Thus 1251431 is not normal for d = 4, n = 5; we [redacted] would make this sequence normal by writing it as 1231451.

With this convention, one can easily determine small values of [redacted]

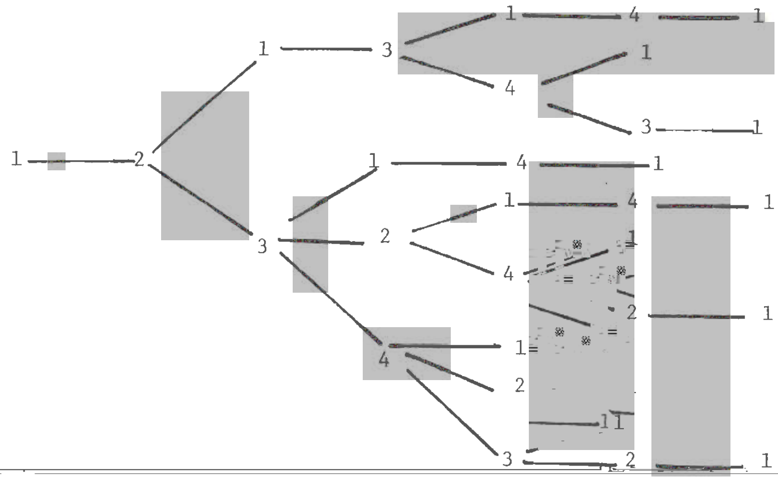

Figure 1 - Sequences for d = 3, n = 4.

LEMMA. *In a maximal DS sequence, there exists an element i whose frequency $f(i) = 1$.*

Proof. If possible, suppose $f(i) \geq 2$ for all i. Let us consider an

The same argument puts two elements a_3 between the a_2's; continue, and one reaches a contradiction (since there are only finite by many distinct symbols).

Application of the Turan Lemma to a normal sequence produces the

COROLLARY. *In a normal maximal (3,n) sequence, f(n) = 1.*

[illegible]

one other element (if the elements to left and right of n are the same, one of them must be eliminated). The result is a sequence on n-1 symbols, and so its length is at most N(3,n-1). Thus

$$N(3,n) \le N(3,n-1) + 2.$$

An easy induction shows that $N(3,n) \le 2n-1$, and this determines one of

[illegible]

n \ d	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9	10
3	1	3	5	8	10	14	16	20	22	26
4	1	4	7	12	16	23	28	35	40	47
5	1	5	9	17	22	34	41	53	661	73
6	1	6	11	22	29					
7	1	7	13	27						
8	1	8	15	32						
9	1	9	17	37						
10	1	10	19	42						

Table II - Values of N(d,n).

The values above the diagonal are due to Stanton and Roselle, who considered the case $d > n$. They proved [8] that $N(d,3) = 3d-4$ (d even) and $N(d,3) = 3d-5$ (d odd). They extended this in [9] to establish $N(d,4) = 6d-13$ (d even) or $6d-14$ (d odd). In this same paper, they show that

$$N(d,n) \geq \binom{n}{2}d - D_e(n), \text{ d even;}$$

$$N(d,n) \geq \binom{n}{2}d - D_e(n), \text{ d odd.}$$

These lower bounds are usually very close, since $D_e(n) = [(2n^3 + 9n^2 - 32n + 12)/12]$, and are attained for $n = 3$ or 4. They are also shown to hold for $n = 5$ [10], giving $N(d,5) = 10d-27$ or $10d-29$ (but, cf. [5]).

4. The case d = 4.

Roselle and Stanton used the inequality

$$5n-8 \leq N(4,n) \leq \frac{n}{n-1} N(4,n-1) + 2$$

to obtain small values of $N(4,n)$. However, the attractive conjecture that $N(4,n) = 5n-8$ breaks down for $n = 12$, and Davenport (with Conway) showed [2] that, for q and r positive,

$$N(4,qr+1) \geq 6qr-q-5r+2.$$

This result immediately applies to give N(4,13) ≥ 57. However, Davenport actually showed that

$$\lim_{n\to\infty} \frac{N(4,n)}{n} \geq 8$$

and a reasonable conjecture today is that this limit is infinite.

The number 4 exerts an irresistible fascination over W.H. Mills, and so it is not surprising to find that he has made the most extensive

[redacted]

the reference to Mills!). The Mills table continues on from Table II to give the following.

n	11	12	13	14	15	16	17	18	19	20	21
N(4,n)	47	53	58	64	69	75	81	86	92	98	104

5. *Numbers of DS Sequences.*

There has really only been a detailed study of the number of DS sequences for d = 3. Table 1 shows the number for (3,4), and the general result is given in [7], where Stanton and Mullin prove that the maximal numbers of (3,n) sequences are 1,1,2,5,14,..., that is, the Catalan numbers

$$\frac{1}{n}\binom{2n-2}{n-1}.$$

[redacted]

Roselle in reference [6].

6. *Remarks on Recent Work.*

The value N(5,5) was originally determined by computer in [9]. Peterkin [5] used a very efficient computer search to obtain N(5,6) = 29, and to show that there are 35 (5,6) sequences. He corrected the Stanton Roselle value N(6,5) to 34 (they had failed to distinguish between $x > 0$ and $x \geq 0$, and so had the incorrect value 33).

Peterkin's work also suggested better bounding sequences, and he was able to prove that $N(5,n) \geq 7n-13$, $N(6,n) \geq 13n-32$. These bounds are probably quite good for small n, if we use the analogy with N(4,n).

Very recently, Burkowski and Ecklund [1] have considered the numbers N(d,n,r). Here r is a regularity number which imposes the additional restriction that any symbol in the sequence can appear at most r times.

7. *Final Remarks.*

The first six sections of this paper are a slightly revised version of a survey given to the Australian Mathematical Society Annual Meeting in Newcastle in the winter of 1974. Two recent papers by Australian authors have added considerably to our knowledge of DS sequences. A.J. Dobson and S.O. Macdonald, in *Lower Bounds for the Lengths of Davenport-*

[redacted]

[redacted]

[redacted]

Roselle; they also give a very useful table, for n and d ranging from 5 to 12, which embodies the latest information known. The Rennie-Dobson upper bounds result from a recursion relation

$$\left(n-2+\frac{1}{d-3}\right) N_d(n) \leq n\, N_d(n-1) + \frac{2n-d+2}{d-3} .$$

REFERENCES

[1] F.J. Burkowski and E.F. Ecklund, *Exclusion Sequences with Regularity* [redacted]

[redacted]

[redacted]

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[7] R.G. Stanton and R.C. Mullin, *A Map-Theoretic Approach to Davenport-Schinzel Sequences*, Pacific Journal Of Mathematics, Vol 40 No. 1 (1972), pp. 167-172.

[8] R.G. Stanton and D.P. Roselle, *A Result on Davenport-Schinzel* [redacted] Combinatorial Theory and Its Applications: Balantonfured (Hungary) (1969), pp. 1023-1027.

[9] R.G. Stanton and D.P. Roselle, *Some Properties of Davenport-* [redacted]