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DAVENPORT-SCHINZEL SEQUENCES

R.G. Stanton and P.H. Dirksen

1. Introduction.

There are two interesting dual problems in sequence construction. Problem A is to construct as short a sequence as possible which contains subsequences of a certain type. Problem B is to construct as long a sequence as possible which excludes subsequences of a certain type. Davenport-Schinzel (or, more briefly, DS) sequences are a case of Problem B, one of the few combinatorial problems to arise from a problem in differential equations. This paper is intended as an up-to-date expository survey of current knowledge of DS sequences.

Davenport and Schinzel [3] explain that, if $F(D)f(x) = 0$ is a homogeneous, linear, differential equation of degree d, and if $f_1(x), f_2(x), \ldots, f_n(x)$, are n distinct (but not necessarily independent) solutions of $F(D)f(x) = 0$, then a dissection of the real line into intervals

$$(-\infty, x_1), (x_1, x_2), \ldots, (x_{N-1}, \infty)$$

is determined so that, in any one of these intervals, exactly one of the functions $f_i(x)$ dominates all others. The problem is then, given d and n, to maximize N.

This problem need not be introduced from differential equations, and reference [3] describes a completely combinatorial form of the problem as follows. One has the integers $1,2,3,\ldots,n$, and a preassigned integer d. A *DS sequence* is defined to be a sequence built up from $1,2,\ldots,n$, subject to the constraints that (a) no two adjacent

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elements are equal, and (b) no subsequence of elements of the form ...ababa... has length greater than d(the elements in the subsequence are ordered, but not necessarily adjacent). Thus, for $d = 4$, $n = 5$, the sequence

$$1\ 2\ 1\ 3\ 4\ 1\ 5\ 2$$

is a DS sequence, but

$$1\ 2\ 1\ 3\ 4\ 1\ 5\ 2\ 1$$

is not.

We denote the maximal length of a DS sequence by $N(d,n)$.

2. *Normal Sequences and* $N(3,n)$.

It is convenient to adopt a convention that all sequences considered be *normal*, that is, the symbols are renamed so as to appear in natural order. Thus 1251431 is not normal for $d = 4$, $n = 5$; we would make this sequence normal by writing it as 1231451.

With this convention, one can easily determine small values of $N(d,n)$ by an actual tree search. For example, in Figure 1, we show that $N(3,4) = 7$, and there are five maximal normal sequences.

The values of $N(d,n)$, for either n or d equal to 1 or 2, follow easily from the definition. Figure 1 leads us to the general result that $N(3,n) \geq 2n - 1$, since

$$1\ 2\ 3 \ldots n\text{-}1\ n\ n\text{-}1 \ldots 3\ 2\ 1$$

is a DS sequence. The converse is most readily deduced from an ingenious Lemma due to V. Turan.

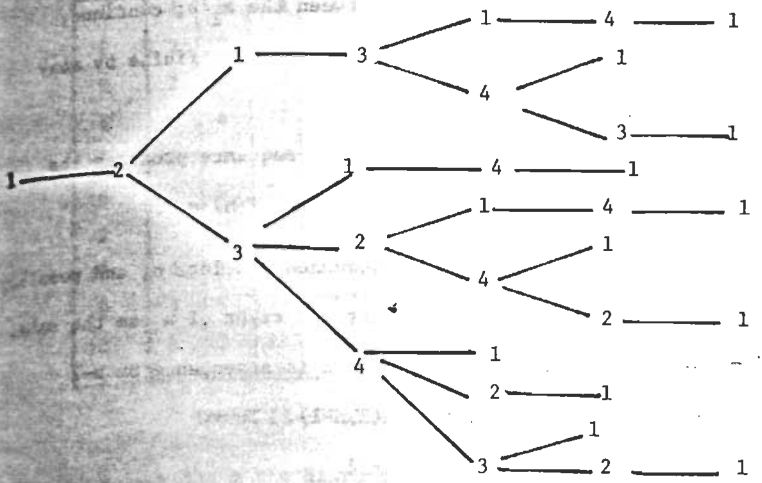

Figure 1 – Sequences for $d = 3$, $n = 4$.

LEMMA. *In a maximal DS sequence, there exists an element i whose frequency $f(i) = 1$.*

Proof. If possible, suppose $f(i) \geq 2$ for all i. Let us consider an element a_1. Between 2 occurrences of a_1, we must have some element a_2. If the second occurrence of a_2 precedes the first a_1, then we have a subsequence $a_2a_1a_2a_1$; if it follows the second a_1, then we have a subsequence $a_1a_2a_1a_2$. Hence the order must be

$$\ldots a_1 \ldots a_2 \ldots a_2 \ldots a_1 \ldots \ .$$

The same argument puts two elements a_3 between the a_2's; contin[ue] and one reaches a contradiction (since there are only finite by distinct symbols).

Application of the Turan Lemma to a normal sequence produces

COROLLARY. *In a normal maximal (3,n) sequence, f(n) = 1.*

Now suppose one has a maximal (3,n) sequence. Delete n, and p[ossibly] one other element (if the elements to left and right of n are the [same,] one of them must be eliminated). The result is a sequence on n-1 symbols, and so its length is at most N(3,n-1). Thus

$$N(3,n) \le N(3,n-1) + 2.$$

An easy induction shows that $N(3,n) \le 2n-1$, and this determines o[ne of] the main results of [3], namely, that $N(3,n) = 2n-1$.

The other results of [3] are much more analytical than combinat[orial;] we briefly note that

$$N(4,n) \ge 5n-C;$$

$N(d,n) \ge n(d^2-4d+3) - C(d)$ for odd $d > 3$; $N(d,n) \ge n(d^2-5d+8) - C(d)$ [for] even $d > 4$. Also, [3] gives upper bounds

$$N(4,n) < 2n(1+\log n),$$

$$N(d,n) < An \exp (B\sqrt{\log n}),$$

for $d > 4$, A and B dependent on d.

3. *DS Sequences for d > n.*

We begin this section with Table II giving current knowledge of N(d,n). But see Section 7 for bounds on missing values.

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n \ d	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9	10
3	1	3	5	8	10	14	16	20	22	26
4	1	4	7	12	16	23	28	35	40	47
5	1	5	9	17	22	34	41	53	61	73
6	1	6	11	22	29					
7	1	7	13	27						
8	1	8	15	32						
9	1	9	17	37						
10	1	10	19	42						

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Table II - Values of N(d,n).

The values above the diagonal are due to Stanton and Roselle, who considered the case d > n. They proved [8] that N(d,3) = 3d-4 (d even) and N(d,3) = 3d-5 (d odd). They extended this in [9] to establish N(d,4) = 6d-13 (d even) or 6d-14 (d odd). In this same paper, they show that

$$N(d,n) \geq \binom{n}{2}d - D_e(n), \text{ d even;}$$

$$N(d,n) \geq \binom{n}{2}d - D_e(n), \text{ d odd.}$$

These lower bounds are usually very close, since $D_e(n) = [(n^3 + 9n^2 - 32n + 12)/12]$, and are attained for n = 3 or 4. They are also shown to hold for n = 5 [10], giving N(d,5) = 10d-27 or 10d-29 (but, cf. [5]).

4. *The case d = 4.*

Roselle and Stanton used the inequality

$$5n-8 \leq N(4,n) \leq \frac{n}{n-1} N(4,n-1) + 2$$

to obtain small values of N(4,n). However, the attractive conjecture that N(4,n) = 5n-8 breaks down for n = 12, and Davenport (with Conway) showed [2] that, for q and r positive,

$$N(4,qr+1) \geq 6qr-q-5r+2.$$

This result immediately applies to give $N(4,13) \geq 57$. However, Davenport actually showed that

$$\lim_{n\to\infty} \frac{N(4,n)}{n} \geq 8$$

and a reasonable conjecture today is that this limit is infinite.

The number 4 exerts an irresistible fascination over W.H. Mills, and so it is not surprising to find that he has made the most exte... determination of $N(4,n)$ in [4] (note our careful reservation of [4] ... the reference to Mills!). The Mills table continues on from Table ... to give the following.

n	11	12	13	14	15	16	17	18	19	20
N(4,n)	47	53	58	64	69	75	81	86	92	98

20	21
98	104

5. *Numbers of DS Sequences.*

There has really only been a detailed study of the number of DS sequences for $d = 3$. Table 1 shows the number for (3,4), and the general result is given in [7], where Stanton and Mullin prove that the maximal numbers of (3,n) sequences are 1,1,2,5,14,..., that is, the Catalan numbers

$$\frac{1}{n}\binom{2n-2}{n-1}.$$

The result for all (3,n) sequences is more complicated, namely,

$$f_n = \frac{1}{4\pi}\int_{-a}^{a}(t+3)^{n-2}\sqrt{8-t^2}\,dt,$$

$$= \sum_{k=0}^{\infty} 3^{n-3-2k}2^k\binom{n-2}{2k}\frac{2k!}{k!(k+1)!},$$

where $a = 2\sqrt{2}$.

The result for maximal (3,n) sequences is obtained more easily by Roselle in reference [6].

6. *Remarks on Recent Work.*

The value N(5,5) was originally determined by computer in [9]. Peterkin [5] used a very efficient computer search to obtain N(5,6) = 29, and to show that there are 35 (5,6) sequences. He corrected the Stanton Roselle value N(6,5) to 34 (they had failed to distinguish between $x > 0$ and $x \geq 0$, and so had the incorrect value 33).

Peterkin's work also suggested better bounding sequences, and he was able to prove that $N(5,n) \geq 7n-13$, $N(6,n) \geq 13n-32$. These bounds are probably quite good for small n, if we use the analogy with N(4,n).

Very recently, Burkowski and Ecklund [1] have considered the numbers N(d,n,r). Here r is a regularity number which imposes the additional restriction that any symbol in the sequence can appear at most r times.

7. *Final Remarks.*

The first six sections of this paper are a slightly revised version of a survey given to the Australian Mathematical Society Annual Meeting in Newcastle in the winter of 1974. Two recent papers by Australian authors have added considerably to our knowledge of DS sequences. A.J. Dobson and S.O. Macdonald, in *Lower Bounds for the Lengths of Davenport-Schinzel Sequences*, Utilitas Mathematica 6 (1974), 251-257, have considerably strengthened the known lower bounds. B.C. Rennie and A.J. Dobson, in *Upper Bounds for the Lengths of Davenport-Schinzel Sequences*, Utilitas Mathematica 8 (1975), 181-185, have derived new upper bounds which are usually better than those given by Stanton and Roselle; they also give a very useful table, for n and d ranging from 5 to 12, which embodies the latest information known. The Rennie-Dobson upper bounds result from a recursion relation

$$(n-2+\frac{1}{d-3})\ N_d(n) \le n\ N_d(n-1) + \frac{2n-d+2}{d-3} .$$

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