

ON POINT-SYMMETRIC TOURNAMENTS

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1. Introduction. A *tournament* is a directed graph in which there is exactly one arc between any two distinct vertices. Let $a(T)$ denote the automorphism group of T . A tournament T is said to be *point-symmetric* if $a(T)$ acts transitively on the vertices of T . Let $g(n)$ be the maximum value of $|a(T)|$ taken over all tournaments of order n . In [3] Goldberg and Moon conjectured that $g(n) \leq \sqrt{3}^{n-1}$ with equality holding if and only if n is a power of 3. The case of point-symmetric tournaments is what prevented them from proving their conjecture. In [2] the conjecture was proved through the use of a lengthy combinatorial argument involving the structure of point-symmetric tournaments. The results in this paper are an outgrowth of an attempt to characterize point-symmetric tournaments so as to simplify the proof employed in [2].

The construction discussed in §2 was used in [1] as a means of producing regular tournaments. The analogous construction for graphs was employed by J. Turner in [6] independent of any knowledge of [1]. There is an obvious generalization to directed graphs.

We list some of the terminology used in this paper. If there is an arc from the vertex u to the vertex v in T , we write $(u, v) \in T$. If S is a subset of the vertex set of T , then $\langle S \rangle$ denotes the subtournament whose vertex set is S . We use the symbol " $\simeq$ " to denote isomorphism between tournaments. The sets $\mathcal{O}(u) = \{v \in T : (u, v) \in T\}$ and $\mathcal{I}(u) = \{v \in T : (v, u) \in T\}$ are called the *outset* and *inset* of u , respectively. The *score* $s(u)$ of the vertex u is given by $s(u) = |\mathcal{O}(u)|$. The *score sequence* of T is the sequence $(s_1, s_2, \dots, s_{|T|})$ of scores of the respective vertices of T written so that $s_1 \leq s_2 \leq \dots \leq s_{|T|}$. Throughout this paper all subscripts are understood modulo $2n+1$.

2. Main results. Consider a fixed integer of the form $2n+1$, $n \geq 1$. Let $S = \{\alpha_1, \dots, \alpha_n\}$ be a set of n distinct integers chosen from $1, 2, \dots, 2n$ with the property that $\alpha_i + \alpha_j \neq 2n+1$ for any two α_i, α_j in S . Construct a directed graph T with vertices $v_0, v_1, \dots, v_{2n}$ as follows: There is an arc from v_i to v_j if and only if $j - i \equiv \alpha_k \pmod{2n+1}$ for some $\alpha_k \in S$. It is not difficult to see that T is a regular tournament of degree n . Any tournament that is constructible in the above manner is called a *rotation tournament* and S is called the *symbol* of the rotation tournament.

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It is easy to see that the permutation $(v_0 v_1 \dots v_{2n})$ is an automorphism of T and this proves the following result.

PROPOSITION 1. *A rotation tournament is a point-symmetric tournament.*

Moreover, if T is a tournament of order m and the automorphism group of T possesses an m -cycle $(v_0 v_1 \dots v_{m-1})$, then T is a rotation tournament with symbol $S = \{j: (v_0, v_j) \in T\}$. The following proposition has been proved.

PROPOSITION 2. *A tournament T of order m is a rotation tournament if and only if $a(T)$ possesses an m -cycle.*

We are interested in the question of whether or not the construction given above produces all the point-symmetric tournaments. By Propositions 1 and 2 an equivalent question is the following: If T is a point-symmetric tournament of order $2n+1$, does $a(T)$ possess a $(2n+1)$ -cycle? In the case that $2n+1$ is a prime the latter question is easy to answer. For if $2n+1$ is a prime and $a(T)$ acts transitively on the vertices of T , then $2n+1$ divides $|a(T)|$ and, thus, $a(T)$ contains a $(2n+1)$ -cycle [6, Theorem 3.2 and Exercise 3.12]. We have proved the following result.

THEOREM 1. *A tournament T of prime order is point-symmetric if and only if it is a rotation tournament.*

The first non-prime case to consider is $2n+1=9$. There are 15 regular tournaments of order 9 of which three are point-symmetric. It can be verified directly that all three of them are also rotation tournaments. However, we shall give a proof that every point-symmetric tournament of order 9 is a rotation tournament as it employs a technique that is useful for point-symmetric tournaments of larger composite order.

Let T be a point-symmetric tournament of order 9. Let u_0 be a fixed vertex of T and let a_{u_0} denote the stabilizer of u_0 , i.e., $a_{u_0} = \{\sigma \in a(T): \sigma(u_0) = u_0\}$. Notice that a_{u_0} is, in a very natural way, the automorphism group of $\langle T - u_0 \rangle$. Since $\mathcal{O}(u_0)$ contains four elements and the orbits of the automorphism group of any tournament have odd cardinality because every permutation in an odd order permutation group is a product of disjoint cycles of odd length and any automorphism group of a tournament has odd order by [4], a_{u_0} must fix at least one element of $\mathcal{O}(u_0)$. Let u_1 be a vertex of $\mathcal{O}(u_0)$ fixed by a_{u_0} . Therefore, if $H = \{\sigma \in a(T): \sigma(u_0) = u', u' \text{ a fixed vertex of } T\}$, then each $\sigma \in H$ maps u_1 to the same vertex of T . In particular, every $\sigma \in a(T)$ that maps u_0 to u_1 maps u_1 to the same vertex of T , call it u_2 . Since $\langle T - u_0 \rangle \simeq \langle T - u_1 \rangle$, each $\sigma \in a(T)$ that maps u_1 to u_2 maps u_2 to the same vertex of T , call it u_3 . Either $u_3 = u_0$ or we can continue this process until we obtain a sequence $u_0, u_1, \dots, u_j$ of distinct vertices of T such that every $\sigma \in a(T)$ for which $\sigma(u_0) = u_1$ also satisfies $\sigma(u_1) = u_2$, $\sigma(u_2) = u_3$, $\dots$, $\sigma(u_{j-1}) = u_j$, and $\sigma(u_j) = u_0$. If $u_0, u_1, \dots, u_j$ does not exhaust all the vertices of T , pick a $u'_0 \in T$ not appearing in the sequence. For every $\tau \in a(T)$ such that $\tau(u_0) = u'_0$ we also have $\tau(u_1) = u'_1, \dots$,

$\tau(u_j)=u'_j$ with $\{u_0, u_1, \dots, u_j\} \cap \{u'_0, u'_1, \dots, u'_j\} = \emptyset$. Hence, either $u_0, u_1, \dots, u_j$ exhausts all the vertices of T or the vertices of T can be decomposed into mutually disjoint sequences of vertices of T having the same property as $u_0, u_1, \dots, u_j$ with respect to automorphisms and such that $\langle\{u_0, u_1, \dots, u_j\}\rangle \simeq \langle\{v_0, v_1, \dots, v_j\}\rangle$ via the isomorphism $u_0 \rightarrow v_0, \dots, u_j \rightarrow v_j$, where $v_0, v_1, \dots, v_j$ denotes any of the other sequences.

If $u_0, u_1, \dots, u_8$ exhausts all nine vertices of T , then $\sigma=(u_0 u_1 \dots u_8)$ is an automorphism of T . There is only one possibility remaining, namely, $\alpha(T)$ is imprimitive with three blocks (see [7]), say $\{u_0, u_1, u_2\}$, $\{u_3, u_4, u_5\}$, and $\{u_6, u_7, u_8\}$. Every $\sigma \in \alpha_{u_0}$, the stabilizer of u_0 , fixes u_0, u_1 , and u_2 . Suppose some $\sigma \in \alpha_{u_0}$ moves one of the other vertices. Without loss of generality assume $\sigma(u_3) \neq u_3$. Then either $\sigma(u_3)=u_4$ or $\sigma(u_3)=u_5$ for otherwise σ would contain an even cycle in its disjoint cycle decomposition which cannot happen.

By considering what happens to the remaining vertices of T under any automorphism containing the 3-cycle (u_0, u_1, u_2) , we see that all arcs between two distinct 3-blocks must have the same orientation. Therefore, $(u_0 u_3 u_6 u_1 u_4 u_7 u_2 u_5 u_8) \in \alpha(T)$ and T is a rotation tournament.

We may assume every $\sigma \in \alpha_{u_0}$ fixes every vertex of T , that is, $|\alpha_{u_0}|=1$ which implies $|\alpha(T)|=9$. To within isomorphism there are two transitive permutation groups of order nine in S_9 . One is cyclic and generated by a 9-cycle and, hence, corresponds to a rotation tournament. The other is generated by two permutations σ_1, σ_2 , of the form $\sigma_1=(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)$ and $\sigma_2=(1\ 4\ 7)(2\ 5\ 8)(3\ 6\ 9)$. By using the process described in the addendum to [3] it can be shown that the latter permutation group is not the automorphism group of a tournament of order nine. Therefore, every point-symmetric tournament of order nine is a rotation tournament.

We now consider the point-symmetric tournaments of order fifteen. Let α_u denote the stabilizer of the vertex u in a point-symmetric tournament T of order fifteen. If the orbits of α_u are $\{u\}$, $\mathcal{O}(u)$, and $\mathcal{J}(u)$, then the transitive constituents of α_u [see 7] in $\mathcal{O}(u)$ and $\mathcal{J}(u)$ must each contain a 7-cycle. Therefore, $\langle\mathcal{O}(u)\rangle$ and $\langle\mathcal{J}(u)\rangle$ are both rotation tournaments of order seven of which there are two to within isomorphism. By considering the four possible cases for $\langle\mathcal{O}(u)\rangle$ and $\langle\mathcal{J}(u)\rangle$ it can be shown through a tedious argument that it is impossible for the orbits of α_u to be $\{u\}$, $\mathcal{O}(u)$, and $\mathcal{J}(u)$. Therefore, α_u fixes a point of either $\mathcal{O}(u)$ or $\mathcal{J}(u)$. We assume without loss of generality that α_u fixes a point of $\mathcal{O}(u)$ since T and T^* have the same automorphism group where T^* denotes the converse tournament of T .

We proceed as before via some fixed point of $\mathcal{O}(u)$ under α_u . If $\alpha(T)$ is primitive, then T must be a rotation tournament. Otherwise there are three 5-blocks or five 3-blocks. Suppose we have the blocks $\{u_1, u_2, u_3, u_4, u_5\}$, $\{u_6, u_7, u_8, u_9, u_{10}\}$, and $\{u_{11}, u_{12}, u_{13}, u_{14}, u_{15}\}$. Let $B_1=\langle\{u_1, u_2, \dots, u_5\}\rangle$, $B_2=\langle\{u_6, u_7, \dots, u_{10}\}\rangle$, and $B_3=\langle\{u_{11}, u_{12}, \dots, u_{15}\}\rangle$. If any automorphism $\sigma \in \alpha_{u_1}$, the stabilizer of u_1 , moves some vertex in B_1 or B_2 , then following the argument used in the order nine case

we see that all arcs between two distinct B_i 's must have the same orientation and, therefore, the permutation $(u_1 u_6 u_{11} u_2 u_7 u_{12} \dots u_5 u_{10} u_{15}) \in \alpha(T)$. Now suppose we have the decomposition $\{u_1, u_2, u_3\}$, $\{u_4, u_5, u_6\}$, $\{u_7, u_8, u_9\}$, $\{u_{10}, u_{11}, u_{12}\}$, and $\{u_{13}, u_{14}, u_{15}\}$ where each set of three vertices forms a 3-block in T . Notice that $\alpha(T)$ induces an odd order transitive permutation group on the five 3-blocks as the object set. Since no odd order transitive subgroup of S_5 contains a 3-cycle, there is no $\sigma \in \alpha(T)$ that maps exactly three of the 3-blocks onto different 3-blocks. If any $\sigma \in \alpha_{u_1}$ moves some vertex in another 3-block, say $\sigma(u_4) \neq u_4$, then by the preceding remark and the fact there are no even cycles appearing in the cycle decomposition of σ , we have that $\sigma(u_4) = u_5$ or u_6 and all the arcs between $\langle\{u_1, u_2, u_3\}\rangle$ and $\langle\{u_4, u_5, u_6\}\rangle$ have the same orientation. By examining σ 's action on the remaining three 3-blocks we see that all the arcs between two distinct 3-blocks have the same orientation. Thus there is a 15-cycle in $\alpha(T)$. Therefore, we are left with the case that $|\alpha_{u_1}| = 1$, i.e., $|\alpha(T)| = 15$. However, to within isomorphism there is only one transitive permutation group in S_{15} of order fifteen and it is generated by a 15-cycle. Therefore, every point-symmetric tournament of order fifteen is a rotation tournament.

Consider the following three 7×7 matrices:

$$A_1 = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Let T be the tournament of order twenty-one whose incidence matrix is given by

$$\left(\begin{array}{c|c|c} A_1 & A_2 & A_3 \\ \hline A_3 & A_1 & A_2 \\ \hline A_2 & A_3 & A_1 \end{array} \right).$$

The score sequence of $\langle \mathcal{O}(u_1) \rangle$ is $(3, 3, 4, 4, 4, 5, 5, 6, 7)$ and the four vertices

with score four form a transitive quadruple implying the automorphism group of $\langle \mathcal{O}(u_1) \rangle$ is the identity group. Similarly, the score sequence of $\langle \mathcal{J}(u_1) \rangle$ is (2, 3, 4, 4, 5, 5, 5, 5, 6, 6) and the four vertices with score five form a strongly connected quadruple implying the automorphism group of $\langle \mathcal{J}(u_1) \rangle$ is the identity group. In particular, $|a_{u_1}| = 1$.

It is easy to check that the two permutations

$$\sigma = (u_1 u_8 u_{15})(u_2 u_9 u_{16}) \dots (u_7 u_{14} u_{21})$$

and

$$\tau = (u_1 u_7 u_6 u_5 u_4 u_3 u_2)(u_8 u_{13} u_{11} u_9 u_{14} u_{12} u_{10})(u_{15} u_{18} u_{21} u_{17} u_{20} u_{16} u_{19})$$

are in $a(T)$. Since $A_{u_1} = \{1\}$, the given permutations σ and τ , with $\tau^7 = \sigma^3 = 1$, generate a group of order 21. Observing that $(\sigma\tau)(u_1) = u_{14}$ while $(\tau\sigma)(u_1) = u_{13}$, we see that the group is non-abelian, hence non-cyclic, and, therefore, contains no element of order 21. Hence, T is an example of a point-symmetric tournament that is not a rotation tournament.

An *anti-automorphism* of a tournament T is a mapping σ of the vertex set of T onto itself satisfying $(u, v) \in T$ if and only if $(\sigma(u), \sigma(v)) \notin T$ for every pair of distinct vertices u and v belonging to T . A tournament T is said to be *self-converse* if it has an anti-automorphism, that is, if $T \simeq T^*$.

PROPOSITION 3. *A rotation tournament is self-converse.*

Proof. Let T be a rotation tournament with vertices $u_0, u_1, \dots, u_{2n}$. The permutation σ defined by $\sigma(u_i) = u_{2n-i+1}$ is easily seen to be an anti-automorphism of T . Thus $T \simeq T^*$.

An anti-automorphism of a tournament T composed with an automorphism of T results in an anti-automorphism of T . Therefore, if T is point-symmetric and self-converse, there exists an anti-automorphism of T that fixes any vertex one chooses. Let T denote the order twenty-one tournament exhibited above. If T is self-converse, there exists an anti-automorphism of T that maps $\mathcal{O}(u)$ onto $\mathcal{J}(u)$ with the four vertices of score four in $\langle \mathcal{O}(u) \rangle$ going onto the four vertices of score five in $\langle \mathcal{J}(u) \rangle$. But since one quadruple is transitive and the other is strongly connected we see that no such anti-automorphism exists. Therefore, T is not self-converse.

This suggests the following question: If T is a self-converse point-symmetric tournament, is T a rotation tournament?

3. Enumeration of rotation tournaments. We now consider the problem of enumerating the rotation tournaments of a given order. Let C_n denote the set of all symbols of the rotation tournaments of order $2n+1$ so that $|C_n| = 2^n$. For each integer m satisfying $1 \leq m < 2n+1$ with m and $2n+1$ relatively prime define $P_{n,m}$ by $P_{n,m}(S) = mS = \{x_i \equiv m\alpha_i \pmod{2n+1}: 1 \leq x_i \leq 2n\}$ where $S \in C_n$. It is easy to see $P_{n,m}$ is a permutation on C_n . The set of all such $P_{n,m}$ form a permutation group,

call it G_n , acting on C_n . The number of orbits in C_n under the group G_n is given by the result [5, Theorem 3.21]

$$\frac{1}{\varphi(2n+1)} \sum_{G_n} F(P_{n,m})$$

where $F(P_{n,m})$ is the number of symbols fixed by $P_{n,m}$ and φ denotes the Euler φ -function. Denote the number of orbits by $g(n)$.

If $S' = mS = P_{n,m}(S)$, then the mapping π defined on $(u_0, u_1, \dots, u_{2n})$ by $\pi(u_i) = u'_{mi}$ is an isomorphism between the tournaments corresponding to S and S' . Thus, $g(n)$ gives us an upper bound for the number of non-isomorphic rotation tournaments of order $2n+1$. The results obtained in [6] apply equally well to circulant tournament matrices so that in case $2n+1$ is a prime we have that two rotation tournaments are isomorphic if and only if their corresponding symbols are in the same orbit. Theorem 1 then proves the following result.

THEOREM 2. *If $2n+1$ is a prime, then the number of non-isomorphic point-symmetric tournaments of order $2n+1$ is*

$$g(2n+1) = \frac{1}{2n} \sum_{G_n} F(P_{n,m}).$$

Letting $r(n)$ denote the number of non-isomorphic rotation tournaments of order $2n+1$ and $t(n)$ denote the number of non-isomorphic point-symmetric tournaments of order $2n+1$ we have the following table.

n	$g(n)$	$r(n)$	$t(n)$
1	1	1	1
2	1	1	1
3	2	2	2
4	4	3	3
5	4	4	4
6	6	6	6
7	16	16	16
8	16	16	16
9	30	30	30
10	88		

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