

New series for old functions

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1 Introduction

One way to derive Mercator's series for the natural logarithm function starts from the integral expression

$$(1.1) \quad \log(1+x) = \int_0^1 \frac{x}{1+xt} dt.$$

Expanding the integrand as a Taylor series in t and integrating term by term yields Mercator's expansion

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots.$$

We can get more rapidly converging series for $\log(1+x)$ by a simple modification of the above approach; the idea is to rewrite the integrand in (1.1) before carrying out the expansion and term by term integration.

Example

The quadratic polynomial $1 + \frac{x^2}{1+x}t(1-t)$ in t vanishes when $t = -\frac{1}{x}$, and hence is divisible by the linear polynomial $1+xt$ in t . It follows that the quotient

$$\frac{1 + \frac{x^2}{1+x}t(1-t)}{1+xt} = \frac{x}{1+x}(1+x-xt),$$

a linear polynomial in t . We can thus write the integrand of (1.1) in the form

$$\frac{x}{1+xt} = \frac{x^2}{1+x} \left\{ \frac{1+x-xt}{1 + \frac{x^2}{1+x}t(1-t)} \right\}.$$

Integrating both sides between 0 and 1 gives

$$\begin{aligned} \log(1+x) &= \int_0^1 \frac{x}{1+xt} dt \\ &= \frac{x^2}{1+x} \int_0^1 \left\{ \frac{1+x-xt}{1 + \frac{x^2}{1+x}t(1-t)} \right\} dt \\ &= \frac{x^2}{1+x} \sum_{k=0}^{\infty} \left(\frac{-x^2}{1+x} \right)^k \int_0^1 (1+x-xt)t^k(1-t)^k dt. \end{aligned}$$

The integrals in the series can be evaluated making use of the beta function result

$$(1.2) \quad B(p+1, q+1) = \int_0^1 t^p (1-t)^q dt = \frac{p!q!}{(p+q+1)!},$$

where p and q are nonnegative integers. After a short calculation we obtain the expansion

$$\log(1+x) = \frac{x+2}{x} \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{x^2}{1+x} \right)^n \frac{1}{n \binom{2n}{n}},$$

which converges provided $\left| \frac{x^2}{1+x} \right| < 4$.

More generally, if m and n are nonnegative integers we can write the integrand of (1.1) in the form

$$(1.3) \quad \frac{x}{1+xt} = \left\{ \frac{P_{m,n}(t)}{1 + R_{m,n}(x)t^m(1-t)^n} \right\},$$

where $P_{m,n}(t)$ is a polynomial in t (with coefficients rational functions in x) and

$$R_{m,n}(x) = (-1)^{m+1} \frac{x^{m+n}}{(1+x)^n}.$$

Integrating both sides of (1.3) between 0 and 1 gives

$$\begin{aligned} \log(1+x) &= \int_0^1 \frac{x}{1+xt} dt \\ &= \int_0^1 \left\{ \frac{P_{m,n}(t)}{1 + R_{m,n}(x)t^m(1-t)^n} \right\} dt \\ &= \sum_{k=0}^{\infty} (-R_{m,n}(x))^k \int_0^1 P_{m,n}(t) t^{mk} (1-t)^{nk} dt \end{aligned}$$

The integrals can be evaluated using (1.2) to produce a series expansion for $\log(1+x)$. Some examples of these expansions for small values of m and n are listed in the next section.

Once we have these new expansions for the logarithmic function $\log(1+x)$ we can obtain new series expansions for the inverse tangent function $\tan^{-1}(x)$ and the functions $\frac{\sin^{-1}(x)}{\sqrt{1-x^2}}$ and $(\sin^{-1}(x))^2$ by means of the relations

$$(1.4) \quad \tan^{-1}(x) = \frac{i}{2}(\log(1-ix) - \log(1+ix)),$$

$$(1.5) \quad \frac{\sin^{-1}(x)}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right),$$

and

$$(1.6) \quad (\sin^{-1}(x))^2 = \int_0^x \frac{\sin^{-1}(t)}{\sqrt{1-t^2}} dt.$$

Some examples are listed in Section 4.

It turns out that the expansions we obtain by this method for the functions $(\sin^{-1}(x))^2$ and $\frac{\sin^{-1}(x)}{\sqrt{1-x^2}}$ are in fact power series in x , and hence must be just the usual Maclaurin expansions for these functions in a disguised form. Nevertheless, these equivalent expansions are useful for finding new series for the constants π , $\zeta(2)$, $\zeta(3)$, $\zeta(4)$ and Catalan's constant G . Some examples of these new representations for these constants may be found in Sections 5 through 9.

2 Series expansions for $\log(1+x)$

$$(2.1) \quad \log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2+x)}{n \binom{2n}{n}} \frac{x^{2n-1}}{(1+x)^n} \quad (m=n=1)$$

$$(2.2) \quad \log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n(9+15x+4x^2) - (3+6x+2x^2)}{2n(2n-1) \binom{3n}{n}} \frac{x^{3n-2}}{(1+x)^{2n}} \quad (m=1, n=2)$$

$$(2.3) \quad \log(1+x) = \sum_{n=1}^{\infty} \frac{n(9+3x-2x^2) - (3-x^2)}{2n(2n-1) \binom{3n}{n}} \frac{x^{3n-2}}{(1+x)^n} \quad (m=2, n=1)$$

$$(2.4) \quad \log(1+x) = \sum_{n=1}^{\infty} \frac{(x^3-6x-4) - 2n(x^3-2x^2-12x-8)}{2n(2n-1) \binom{4n}{2n}} \frac{x^{4n-3}}{(1+x)^{2n}} \quad (m=n=2)$$

$$(2.5) \quad \log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{P(n,x)}{3n(3n-1)(3n-2) \binom{4n}{n}} \frac{x^{4n-3}}{(1+x)^{3n}} \quad (m=1, n=3)$$

where $P(n,x) = n^2(64+176x+148x^2+27x^3) - n(48+140x+128x^2+27x^3) + 2(4+12x+12x^2+3x^3)$

$$(2.6) \quad \log(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{P(n,x)}{3n(3n-1)(3n-2) \binom{4n}{n}} \frac{x^{4n-3}}{(1+x)^n} \quad (m=3, n=1)$$

where $P(n,x) = n^2(64+16x-12x^2+9x^3) - n(48+4x-8x^2+9x^3) + 2(4+x^3)$

3 Series for log(2)

We can obtain an endless supply of rapidly converging series for log(2) by specialising these generalised expansions for log(1+x). Here are some typical results:

$$(3.1) \quad \log(2) = \frac{3}{4} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n (2n+1) \frac{(2n)!}{n!^2}}$$

$$(3.2) \quad \log(2) = \frac{3}{16} \sum_{n=0}^{\infty} \frac{14n+11}{(4n+1)(4n+3) \binom{4n}{2n}} \frac{1}{4^n}$$

$$(3.3) \quad \log(2) = \frac{9}{64} \sum_{n=0}^{\infty} (-1)^n \frac{171n^2 + 231n + 74}{(6n+1)(6n+3)(6n+5) \binom{6n}{3n}} \frac{1}{8^n}$$

$$(3.4) \quad \log(2) = \frac{3}{128} \sum_{n=0}^{\infty} \frac{14560n^3 + 27504n^2 + 16466n + 3105}{(8n+1)(8n+3)(8n+5)(8n+7) \binom{8n}{4n}} \frac{1}{16^n}$$

$$(3.5) \quad \log(2) = \frac{1}{3} \sum_{n=0}^{\infty} \frac{5n+4}{(3n+1)(3n+2) \binom{3n}{n}} \frac{1}{2^n}$$

$$(3.6) \quad \log(2) = \frac{1}{12} \sum_{n=0}^{\infty} (-1)^n \frac{28n+17}{(3n+1)(3n+2) \binom{3n}{n}} \frac{1}{4^n}$$

$$(3.7) \quad \log(2) = \frac{1}{8} \sum_{n=0}^{\infty} (-1)^n \frac{77n^2 + 101n + 34}{(4n+1)(4n+2)(4n+3) \binom{4n}{n}} \frac{1}{2^n}$$

$$(3.8) \quad \log(2) = \frac{1}{32} \sum_{n=0}^{\infty} (-1)^n \frac{415n^2 + 487n + 134}{(4n+1)(4n+2)(4n+3) \binom{4n}{n}} \frac{1}{8^n}$$

$$(3.9) \quad \log(2) = \frac{1}{12} \sum_{n=0}^{\infty} (-1)^n \frac{34n+25}{(4n+1)(4n+3)} \binom{4n}{2n} \frac{1}{18^n}$$

$$(3.10) \quad \log(2) = \frac{1}{108} \sum_{n=0}^{\infty} (-1)^n \frac{2800n^2 + 3680n + 1123}{(6n+1)(6n+3)(6n+5)} \frac{(6n)!(2n)!}{(4n)!(3n)!n!} \frac{1}{162^n}$$

$$(3.11) \quad \log(2) = \frac{1}{432} \sum_{n=0}^{\infty} \frac{156128n^3 + 291728n^2 + 171658n + 31441}{(8n+1)(8n+3)(8n+5)(8n+7)} \binom{8n}{4n} \frac{1}{324^n}$$

$$(3.12) \quad \log(2) = \frac{1}{972} \sum_{n=0}^{\infty} (-1)^n \frac{361944n^3 + 672036n^2 + 391770n + 70743}{(8n+1)(8n+3)(8n+5)(8n+7)} \frac{(8n)!(3n)!}{(6n)!(4n)!n!} \frac{1}{1458^n}$$

The first 10 terms of this last series gives a value for $\log(2)$ which is correct in the first 46 decimal places.

4 Series for inverse trigonometric functions

$$(4.1) \quad \tan^{-1} x = \frac{x}{(1+x^2)} \sum_{n=0}^{\infty} \frac{1}{(2n+1) \binom{2n}{n}} \left(\frac{4x^2}{1+x^2} \right)^n \quad (\text{Euler})$$

$$(4.2) \quad \tan^{-1} x = \frac{x}{(1+x^2)^2} \sum_{n=0}^{\infty} \frac{4n(1+2x^2) + 3 + 5x^2}{(4n+1)(4n+3) \binom{4n}{2n}} \left(\frac{4x^2}{1+x^2} \right)^{2n}$$

$$(4.3) \quad \tan^{-1} x = \frac{x}{(1+x^2)^3} \sum_{n=0}^{\infty} \frac{P(n, x)}{(6n+1)(6n+3)(6n+5) \binom{6n}{3n}} \left(\frac{4x^2}{1+x^2} \right)^{3n}$$

where $P(n, x) = 36n^2(1+3x^2+3x^4) + 6n(8+23x^2+21x^4) + 15+40x^2+33x^4$

$$(4.4) \quad \tan^{-1} x = \frac{x}{(1+x^2)^4} \sum_{n=0}^{\infty} \frac{P(n, x)}{(8n+1)(8n+3)(8n+5)(8n+7) \binom{8n}{4n}} \left(\frac{4x^2}{1+x^2} \right)^{4n}$$

where $P(n, x) = 512n^3(1+4x^2+6x^4+4x^6) + 64n^2(15+59x^2+86x^4+54x^6) + 8n(71+272x^2+381x^4+224x^6) + 105+385x^2+511x^4+279x^6$

$$(4.5) \quad \tan^{-1} x = \frac{x}{(1+x^2)} \sum_{n=0}^{\infty} (-1)^n \frac{n(4+2x^2) + 3+2x^2}{(4n+1)(4n+3) \binom{4n}{2n}} \left(\frac{4x^4}{1+x^2} \right)^n$$

$$(4.6) \quad \tan^{-1} x = \frac{x}{(1+x^2)^2} \sum_{n=0}^{\infty} \frac{P(n, x)}{(8n+1)(8n+3)(8n+5)(8n+7) \binom{8n}{4n}} \left(\frac{4x^4}{1+x^2} \right)^{2n}$$

where $P(n, x) = 64n^3(8+12x^2+2x^4-x^6) + 16n^2(60+92x^2+19x^4-7x^6) + 4n(142+225x^2+58x^4-14x^6) + (105+175x^2+56x^4-8x^6)$

Inverse sine

$$(4.7) \quad (\sin^{-1} x)^2 = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n^2 \binom{2n}{n}} \quad (\text{Euler})$$

$$(4.8) \quad \frac{\sin^{-1} x}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty} \frac{(2x)^{2n-1}}{n \binom{2n}{n}}$$

$$(4.9) \quad (\sin^{-1} x)^2 = \sum_{n=1}^{\infty} \frac{8n^2(1+x^2) - 2n(1+4x^2) + 2x^2}{(2n(2n-1))^2 \binom{4n}{2n}} (2x)^{4n-2}$$

$$(4.10) \quad \frac{\sin^{-1} x}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty} \frac{8n(1+x^2) - 2(1+2x^2)}{2n(2n-1) \binom{4n}{2n}} (2x)^{4n-3}$$

$$(4.11) \quad (\sin^{-1} x)^2 = \sum_{n=1}^{\infty} \frac{P(n, x)}{(3n(3n-1)(3n-2))^2 \binom{6n}{3n}} (2x)^{6n-4},$$

$$\text{where } P(n, x) = 648n^4(1+x^2+x^4) - 324n^3(2+3x^2+4x^4) \\ + 18n^2(11+24x^2+52x^4) - 6n(3+8x^2+48x^4) + 32x^4$$

$$(4.12) \quad \frac{\sin^{-1} x}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty} \frac{P(n, x)}{(3n(3n-1)(3n-2)) \binom{6n}{3n}} (2x)^{6n-5},$$

$$\text{where } P(n, x) = 144n^2(1+x^2+x^4) - 24n(4+5x^2+6x^4) + 4(3+4x^2+8x^4)$$

5 A collection of results relating to π

$$\pi \neq \frac{22}{7}$$

$$(5.1) \quad \frac{22}{7} - \pi = 240 \sum_{n=1}^{\infty} \frac{1}{(4n+1)(4n+2)(4n+3)(4n+5)(4n+6)(4n+7)}$$

This is a series companion formula to the integral result of Dalzell,

$$(5.2) \quad \frac{22}{7} - \pi = \int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx.$$

Dalzell, D. P. "On 22/7." *J. London Math. Soc.* 19, 133-134, 1944.

If we expand the integrand in (5.2) into a series and integrate term by term we obtain the alternating series

$$(5.3) \quad \frac{22}{7} - \pi = 24 \sum_{n=2}^{\infty} \frac{(-1)^n}{(2n+1)(2n+2)(2n+3)(2n+4)(2n+5)}.$$

It is also interesting to note that

$$(5.4) \quad \frac{22}{7} + \pi = -240 \sum_{n=-2}^{\infty} \frac{1}{(4n+1)(4n+2)(4n+3)(4n+5)(4n+6)(4n+7)}.$$

A list of series for π derived from the new inverse tangent and inverse sine function expansions

$$(5.5) \quad \pi = \sum_{n=0}^{\infty} \frac{10n+6}{(3n+1)(3n+2)} \frac{1}{\binom{3n}{2n} 2^n}$$

$$(5.6) \quad \pi = 2 \sum_{n=0}^{\infty} \frac{187n^3 + 342n^2 + 201n + 38}{(5n+1)(5n+2)(5n+3)(5n+4)} \frac{(-1)^n}{\binom{5n}{2n} 2^n}$$

$$(5.7) \quad \pi = 2 \sum_{n=0}^{\infty} \frac{820n^3 + 1533n^2 + 902n + 165}{(8n+1)(8n+3)(8n+5)(8n+7)} \frac{(-1)^n}{\binom{8n}{4n} 4^n}$$

$$(5.8) \quad \pi = 2 \sum_{n=0}^{\infty} \frac{6n+5}{(4n+1)(4n+3)} \frac{(-2)^n}{\binom{4n}{2n}}$$

$$(5.9) \quad \pi = \sum_{n=0}^{\infty} \frac{3n+2}{(4n+1)(4n+3)} \frac{4^{n+1}}{\binom{4n}{2n}}$$

$$(5.10) \quad \pi = \frac{8}{3} \sum_{n=0}^{\infty} \frac{7n+6}{(6n+1)(6n+5)} \frac{(6n)!n!}{(3n)!(2n)!(2n)!} (-4)^n$$

$$(5.11) \quad \frac{\pi}{\sqrt{3}} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{14n+11}{(4n+1)(4n+3)} \frac{(-1)^n}{\binom{4n}{2n} 3^n}$$

$$(5.12) \quad \frac{\pi}{\sqrt{3}} = \frac{1}{6} \sum_{n=0}^{\infty} \frac{352n^2 + 488n + 163}{(6n+1)(6n+3)(6n+5)} \frac{(6n)!(2n)!}{(4n)!(3n)!n!} \frac{1}{9^n}$$

$$(5.13) \quad \frac{\pi}{\sqrt{3}} = \frac{1}{12} \sum_{n=0}^{\infty} \frac{10528n^3 + 19984n^2 + 12038n + 2285}{(8n+1)(8n+3)(8n+5)(8n+7)} \frac{1}{\binom{8n}{4n} 9^n}$$

6 Results for $\zeta(2)$

Recall

$$\zeta(2) := \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

Here are a variety of formulas for $\zeta(2)$.

As a limit

$$(6.1) \quad \zeta(2) = \lim_{n \rightarrow \infty} \sum_{k=1}^{n-1} \frac{\ln(n) - \ln(k)}{n-k}$$

$$(6.2) \quad \zeta(2) = \lim_{n \rightarrow \infty} \frac{np(n)}{\sum_{k=1}^n kp(n-k)},$$

where $p(n)$ counts the number of partitions of n - sequence [A000041](#) in Sloane's Online Encyclopedia of Integer Sequences.

As an integral

$$(6.3) \quad \zeta(2) = \int_0^1 (x \wedge -x) \wedge (x \wedge -x) \wedge (x \wedge -x) \dots dx$$

where the notation $a \wedge b \wedge c \wedge \dots$ denotes the tower of powers $a^{b^{c^{\dots}}}$.

An interesting series

$$(6.4) \quad \zeta(2) = 1 + 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2(1+n^2+n^4)}.$$

Maple can evaluate this sum but can't evaluate the companion result for Napier's constant

$$(6.5) \quad e = 2 \sum_{n=1}^{\infty} \frac{1}{n!(1+n^2+n^4)}.$$

Leonhard Euler (1707 - 1783)

In celebration of Euler's tercentenary we offer the amusing

$$(6.6) \quad \zeta(2) = \frac{4}{3} \int_0^{\frac{\pi}{2}} \tan^{-1}(\tan^{1707}(\tan^{-1}(\tan^{1783}(x)))) dx$$

Further series for $\zeta(2)$

Putting $x = \frac{1}{2}$ in the expansion $(\sin^{-1} x)^2 = \frac{1}{2} \sum_{n=1}^{\infty} \frac{(2x)^{2n}}{n^2 \binom{2n}{n}}$ results in the well-known series

$$\zeta(2) = \frac{\pi^2}{6} = 3 \sum_{n=1}^{\infty} \frac{1}{n^2 \binom{2n}{n}}.$$

Using the new representations for $(\sin^{-1} x)^2$ given in this website produces an infinite sequence of faster and faster converging series for $\zeta(2)$, which continues with

$$(6.7) \quad \frac{\pi^2}{6} = 3 \sum_{n=1}^{\infty} \frac{20n^2 - 8n + 1}{(2n(2n-1))^2 \binom{4n}{2n}}, \quad (\text{see Mohammed }^{(1)}, \text{ Example 4})$$

$$(6.8) \quad \frac{\pi^2}{6} = 3 \sum_{n=1}^{\infty} \frac{1701n^4 - 1944n^3 + 729n^2 - 96n + 4}{(3n(3n-1)(3n-2))^2 \binom{6n}{3n}},$$

$$(6.9) \quad \frac{\pi^2}{6} = 12 \sum_{n=1}^{\infty} \frac{87040n^6 - 173568n^5 + 131968n^4 - 47456n^3 + 8084n^2 - 552n + 9}{(4n(4n-1)(4n-2)(4n-3))^2 \binom{8n}{4n}},$$

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- (1) [Mohamud Mohammed](#) *Infinite families of accelerated series for some classical constants by the Markov–WZ Method* Discrete Mathematics and Theoretical Computer Science **7**, 2005, 11-24

7 Series for $\zeta(3)$

Making use of the Taylor series expansion for $(\sin^{-1}(x))^2$ given in (4.7), the integral representation

$$(7.1) \quad \zeta(3) = 10 \int_0^{\frac{1}{2}} \frac{(\sin h^{-1}(t))^2}{t} dt$$

[L.Lewin *Polylogarithms and associated functions*, North-Holland, New York, 1981, Sec. 6.3]

is easily seen to be equivalent to the series

$$(7.2) \quad \zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3 \binom{2n}{n}} \quad (\text{Hjortnaes 1954}).$$

Using the new series expansions for $(\sin^{-1}(x))^2$ provides further results of this type. Examples include

$$(7.3) \quad \zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{24n^3 + 4n^2 - 6n + 1}{(2n(2n-1))^3 \binom{4n}{2n}},$$

$$(7.4) \quad \zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{9477n^6 - 11421n^5 + 5265n^4 - 1701n^3 + 558n^2 - 108n + 8}{(3n(3n-1)(3n-2))^3 \binom{6n}{3n}},$$

and

$$(7.5) \quad \zeta(3) = 20 \sum_{n=1}^{\infty} \frac{P(n)}{(4n(4n-1)(4n-2)(4n-3))^3 \binom{8n}{4n}},$$

where

$$P(n) = 1671168n^9 - 4161536n^8 + 4278272n^7 - 2340864n^6 + 712064n^5 \\ - 98496n^4 - 6360n^3 + 4476n^2 - 594n + 27.$$

8 Series for $\zeta(4)$

Comtet's result

$$(8.1) \quad \zeta(4) = \frac{36}{17} \sum_{n=1}^{\infty} \frac{1}{n^4 \binom{2n}{n}}$$

follows from the identity

$$(8.2) \quad \zeta(4) = \frac{144}{17} \int_0^1 \frac{\sin^{-1}(x/2) \log^2(x)}{\sqrt{1-(x/2)^2}} dx$$

by replacing $\frac{\sin^{-1}(x/2)}{\sqrt{1-(x/2)^2}}$ with its Maclaurin series and integrating term by term. If we

use the equivalent expansions for $\frac{\sin^{-1}(x)}{\sqrt{1-x^2}}$, some of which are listed in section (4), we can extend Comtet's result to

$$(8.3) \quad \zeta(4) = \frac{36}{17} \sum_{n=1}^{\infty} \frac{80n^4 - 48n^3 + 24n^2 - 8n + 1}{(2n(2n-1))^4 \binom{4n}{2n}},$$

$$(8.4) \quad \zeta(4) = \frac{36}{17} \sum_{n=1}^{\infty} \frac{P(n)}{(3n(3n-1)(3n-2))^4 \binom{6n}{3n}}$$

$$\text{where } P(n) = 137781n^8 - 275562n^7 + 240570n^6 - 122472n^5 \\ + 41877n^4 - 10908n^3 + 2232n^2 - 288n + 16,$$

and so on.

9 Series for Catalan's constant

The Dirichlet β function is defined as

$$(9.1) \quad \beta(s) = \frac{1}{1^s} - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \dots \quad \text{where } \operatorname{Re} s \geq 1.$$

It is an example of an L-series. The values $\beta(1)$, $\beta(3)$, $\beta(5)$, ... of the Dirichlet β function at the positive odd integers are rational multiples of powers of π .

Explicitly

$$(9.2) \quad \beta(2n+1) = \frac{E_n^*}{2(2n)!} \left(\frac{\pi}{2}\right)^{2n+1}$$

where E_n^* is an Euler number (secant number). The first few values are

$$\begin{array}{cccccc} n & = & 0 & 1 & 2 & 3 & 4 & 5 \\ E_n^* & = & 1 & 1 & 5 & 61 & 1385 & 50521 \end{array} \quad (\text{Sloane's A000364}).$$

Little is known about the values of $\beta(2n)$ at the positive even integers. $\beta(2)$ is known as Catalan's constant and denoted by G (sometimes K).

$$(9.3) \quad G = 1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \dots = 0.91596\ 55941\ 77219\ 01505\ \dots$$

Unlike $\zeta(3)$, it is not known if G is irrational.

D. M Bradley in [“Representations of Catalan’s Constant”](#) catalogues and proves a large number of infinite series and integral representations for G . In particular we have the integral formula (entry (34) in Bradley)

$$(9.4) \quad G = \frac{1}{8} \pi \log(2 + \sqrt{3}) + \frac{3}{4} \int_0^{\frac{\pi}{6}} \frac{x}{\sin(x)} dx.$$

Make the change of variable $x = \sin^{-1}(y)$ in the integral to give

$$(9.5) \quad G = \frac{1}{8} \pi \log(2 + \sqrt{3}) + \frac{3}{4} \int_0^{\frac{1}{2}} \frac{\sin^{-1}(y)}{y\sqrt{1-y^2}} dy.$$

Now replace $\sin^{-1}(y)/\sqrt{1-y^2}$ by its Taylor series expansion

$$\frac{\sin^{-1} y}{\sqrt{1-y^2}} = \sum_{n=1}^{\infty} \frac{(2y)^{2n-1}}{n \binom{2n}{n}}$$

and integrate term by term to obtain

$$(9.6) \quad G = \frac{1}{8}\pi \log(2+\sqrt{3}) + \frac{3}{8} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 \binom{2n}{n}} ; \text{ (entry (62) in Bradley).}$$

Because of the fast convergence of the series, this representation for G has been used to calculate Catalan's constant to a large number of decimal places.

If in (9.5) we use the new series expansions for $\sin^{-1}(y)/\sqrt{1-y^2}$ given in this website we find new representations for Catalan's constant:

$$(9.7) \quad G = \frac{1}{8}\pi \log(2+\sqrt{3}) + \frac{3}{16} \sum_{n=0}^{\infty} \frac{40n^2 + 54n + 19}{(4n+1)^2 (4n+3)^2 \binom{4n}{2n}}$$

$$(9.8) \quad G = \frac{1}{8}\pi \log(2+\sqrt{3}) + \frac{3}{32} \sum_{n=0}^{\infty} \frac{6804n^4 + 17172n^3 + 15903n^2 + 6405n + 956}{(6n+1)^2 (6n+3)^2 (6n+5)^2 \binom{6n}{3n}}$$

$$(9.9) \quad G = \frac{1}{8}\pi \log(2+\sqrt{3}) + \frac{3}{32} \sum_{n=0}^{\infty} \frac{p(n)}{(8n+1)^2 (8n+3)^2 (8n+5)^2 (8n+7)^2 \binom{8n}{4n}},$$

where $p(n) = 1392640n^6 + 5056512n^5 + 7466752n^4 + 5731040n^3 + 2409488n^2 + 526414n + 46889$,

and so on.

Bradley (entry (4) in the above reference) also gives the integral representation

$$\begin{aligned}
 (9.10) \quad G &= \frac{1}{2} \int_0^{\frac{\pi}{2}} \frac{x}{\sin(x)} dx \\
 &= \frac{1}{2} \int_0^1 \frac{\sin^{-1}(x)}{x\sqrt{1-x^2}} dx
 \end{aligned}$$

which produces the poorly converging series

$$(9.11) \quad 2G = \sum_{n=0}^{\infty} \frac{4^n}{(2n+1)^2 \binom{2n}{n}} ; \text{ entry (61) in Bradley.}$$

Using the expansion (4.10)

$$\frac{\sin^{-1} x}{\sqrt{1-x^2}} = \sum_{n=1}^{\infty} \frac{8n(1+x^2) - 2(1+2x^2)}{2n(2n-1) \binom{4n}{2n}} (2x)^{4n-3}$$

in (9.10) leads to the representation

$$(9.12) \quad 2G = \sum_{n=0}^{\infty} \frac{(32n^2 + 36n + 11) 16^n}{(4n+1)^2 (4n+3)^2 \binom{4n}{2n}}.$$

Again the convergence is slow.